Interference Alignment

(for Interference Channels)

Bobak Nazer, Boston University

Wireless Information Theory Summer School
University of Oulu

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I. K-User Interference Channels

II. Alignment via Linear Precoding

III. Ergodic Alignment

IV. Lattice Alignment for Fixed Channels

K-User Interference Channel

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- K transmitter-receiver pairs share a common wireless channel.

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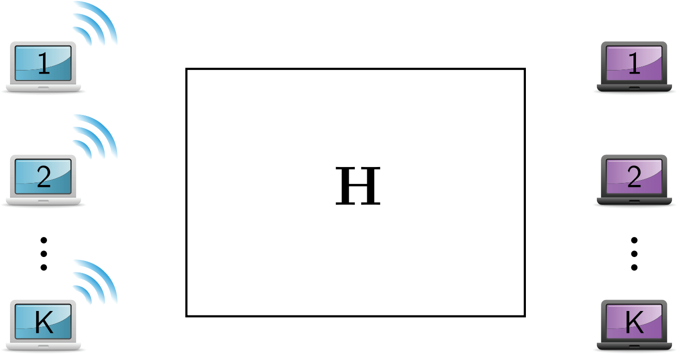

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- Receivers observe noisy linear combinations of transmitted signals:

$$Y_k = \sum_{\ell=1}^{K} h_{k\ell} X_\ell + Z_k$$

K-User Interference Channel

$$\begin{matrix} h_{11} & h_{12} & \cdots & h_{1K} \\ h_{21} & h_{22} & \cdots & h_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ h_{K1} & h_{K2} & \cdots & h_{KK} \end{matrix}$$

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  (**Carleial '75, Sato '81, Han-Kobayashi '81, Sankar-Erkip-Poor '08**)

Weak Interference

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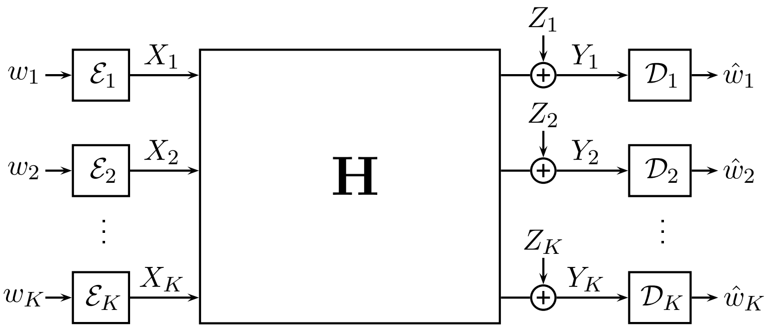
K-User Interference Channel – Problem Statement



- Transmitter ℓ maps message $w_\ell \in \{1, 2, \ldots, 2^{nR_\ell}\}$ into complex-valued codeword $X_\ell^n = (X_\ell[1], \ldots, X_\ell[n])$ obeying power constraint $\sum_{i=1}^n |X_\ell[i]|^2 \leq nP$.
- Receiver k observes $Y_k[i] = \sum_{\ell=1}^K h_{k\ell} X_\ell[i] + Z_k[i]$. Noise $Z_k[i]$ is i.i.d. $\mathcal{CN}(0, N)$.
- What rates $R_1, \ldots, R_K$ are sustainable with vanishing probability of error $\mathbb{P}\big(\{\hat{w}_1 \neq w_1\} \cup \cdots \cup \{\hat{w}_K \neq w_K\}\big)$?

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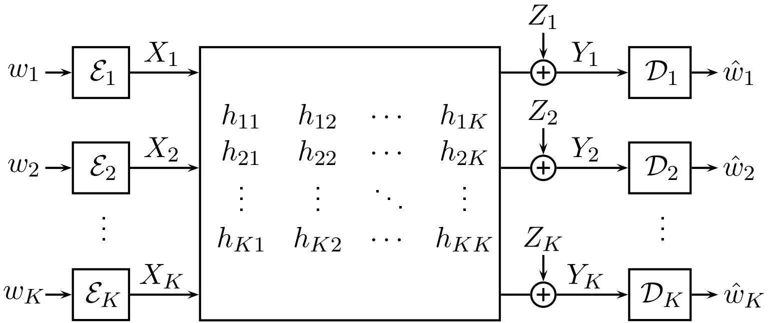

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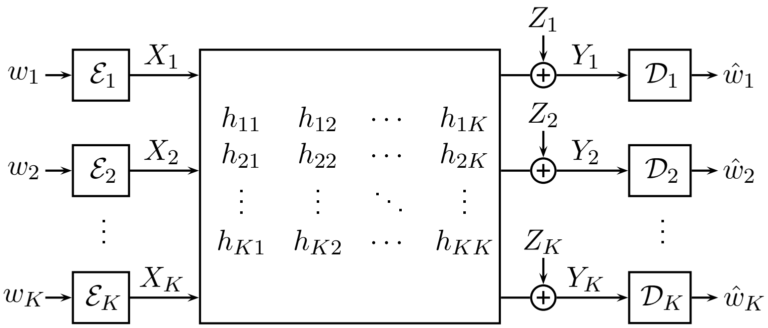
K-User Interference Channel – Symmetric Case



- Equal rates $R_1 = \cdots = R_K = R$.
- Direct gains $h_{kk} = 1$ and cross-gains $h_{k\ell} = \beta$.
- Very Strong Case: β is large enough so that $R = \log(1 + \frac{P}{N})$.
- Weak Case: β is small enough so that receivers can treat interference as noise.
- How do these thresholds on β scale with K?

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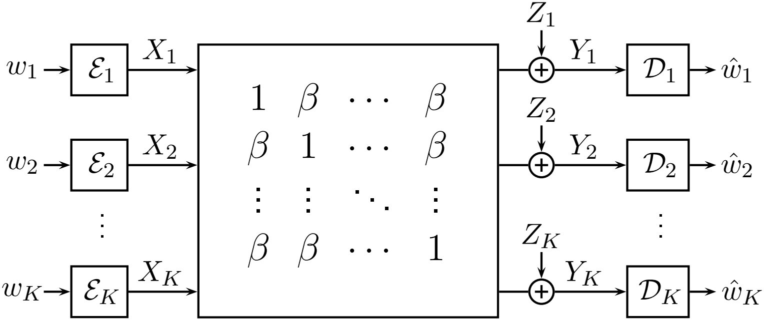

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Symmetric Very Strong Case

- Receiver k must cancel out interference before decoding w_k.
- Simple approach: Each receiver decodes all $K-1$ undesired messages and removes them.
- Multiple-access capacity region requires that:

$$R \leq \frac{1}{K-1} \log \left(1 + \frac{\beta^2 (K-1) P}{N+P} \right)$$

- Set equal to $R = \log(1 + \frac{P}{N})$ and solve for β^2:

$$\beta^2 \geq \frac{\left((1 + \frac{P}{N})^{K-1} - 1 \right) (N+P)}{(K-1)P}$$

- β threshold increases exponentially with K.

Symmetric Weak Case

- Receiver treats all of the interference as noise. Resulting rate is

$$R = \log\left(1 + \frac{P}{N + (K-1)\beta^2 P}\right).$$

- Genie-aided bounds show this is the capacity if

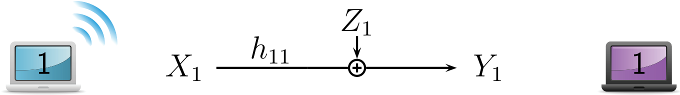
$$P \leq \frac{\sqrt{\frac{K-1}{\beta^2}} - 2(K-1)}{2(K-1)^2\beta^2}$$

- Implies that β threshold must fall with K:

$$\beta^2 \leq \frac{1}{4(K-1)}$$

- See **Shang-Kramer-Chen '07, Motahari-Khandani '07, Annapureddy-Veeravalli '08** for more general results.

Interference-Free Capacity



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Time Division

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$$R_k^{\mathsf{TDMA}} = \frac{1}{K} \log\left(1 + \frac{K|h_{kk}|^2 P}{N}\right)$$

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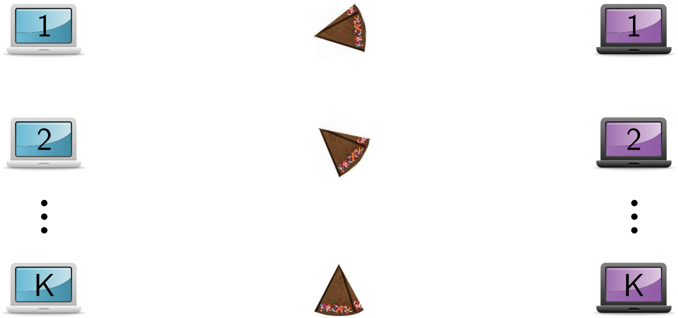

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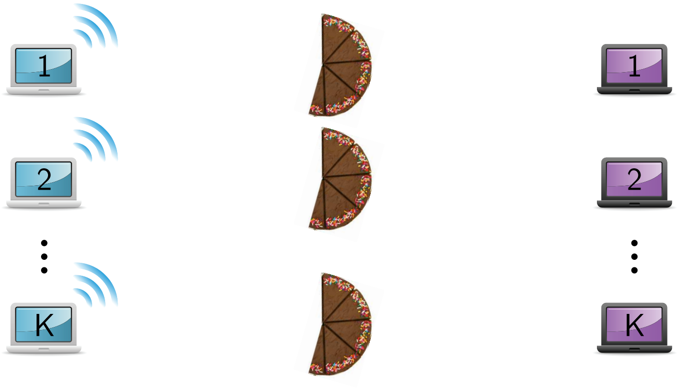
Can each user get half the cake?

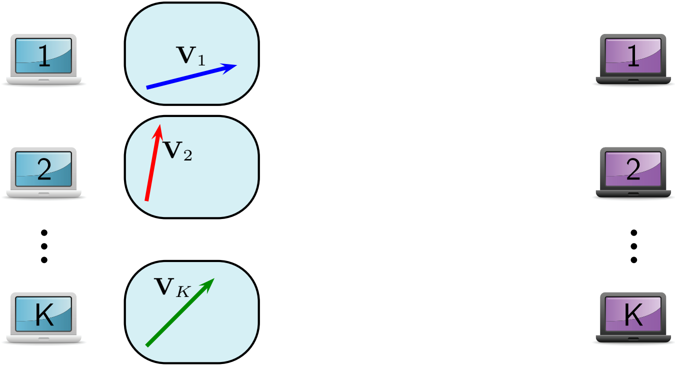


- Is it possible for each user to communicate as if there is only one other user?

$$R_k^{\mathsf{HALF}} = \frac{1}{2} \log \left(1 + \frac{2|h_{kk}|^2 P}{N} \right)$$

- **Maddah-Ali - Motahari - Khandani '08:** Proposed interference alignment for the MIMO X channel.
- **Cadambe-Jafar '08:** Alignment can get "half the cake" for the interference channel as the SNR $\to \infty$:

$$\lim_{P \to \infty} \frac{R_k^{IA}}{\log(1+P)} = \frac{1}{2}$$



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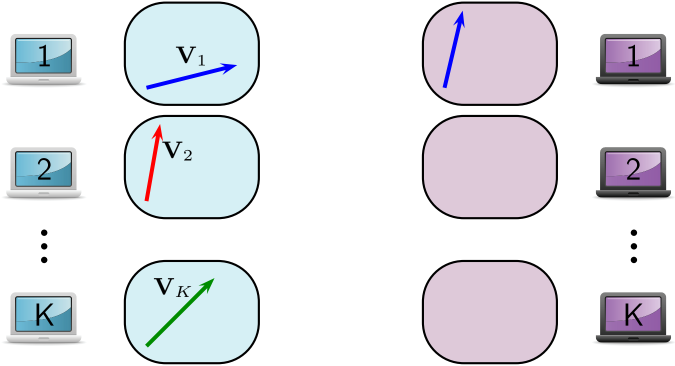

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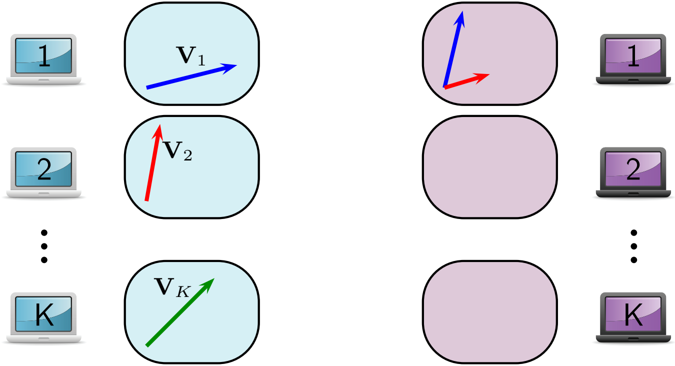

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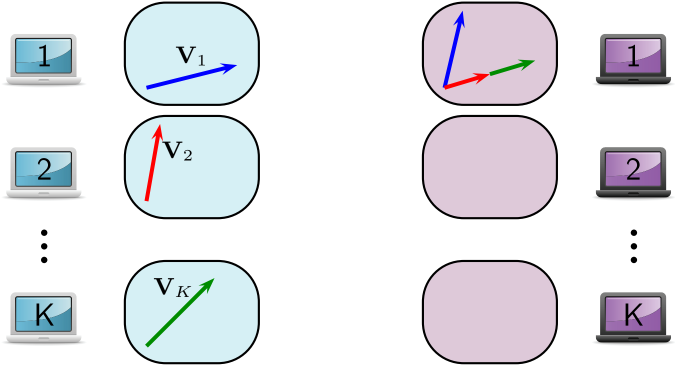

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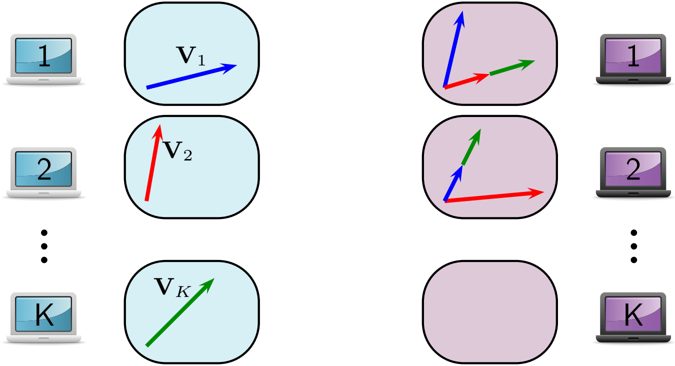
Interference Alignment



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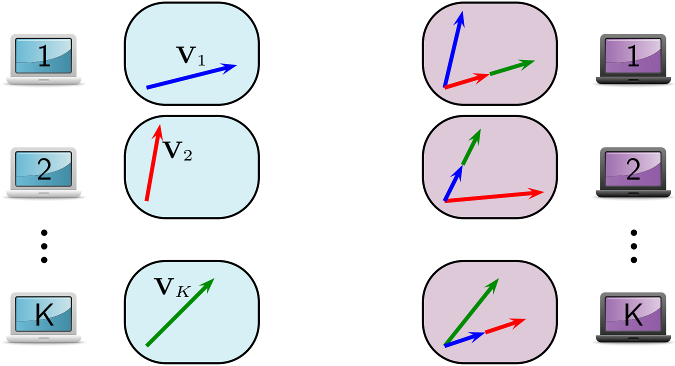

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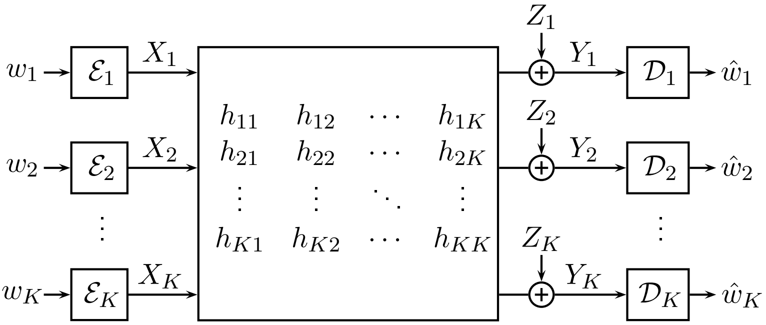
Interference Alignment – Fixed Channels



- Receiver see statistically equivalent channels: $Y_k = \sum_{\ell=1}^{K} X_\ell + Z_k$
- Interference channel capacity depends only on marginal channels.
  $\Longrightarrow$ If one user can decode w_ℓ, they all can.
- Multiple-access capacity: $R = \dfrac{1}{K} \log\left(1 + \dfrac{KP}{N}\right)$

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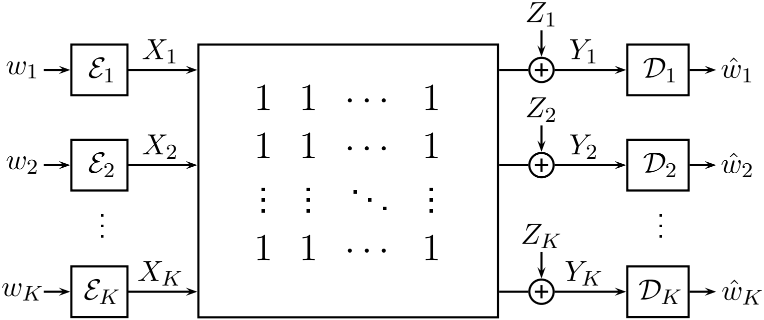

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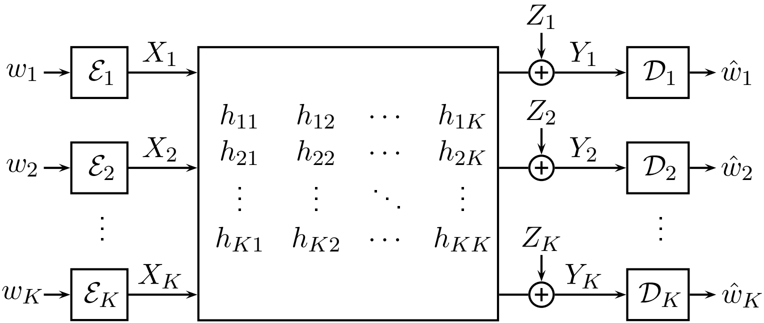
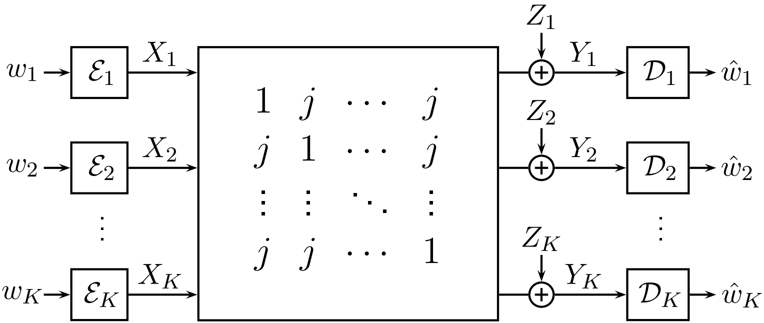


- Receivers see $Y_k = X_k + j \sum_{\ell \neq k}^{K} X_\ell + Z_k$
- Transmitters send only real-valued signals.
- Receivers ignore imaginary part of observed signal to get:

$$R = \frac{1}{2} \log \left(1 + \frac{2P}{N} \right)$$



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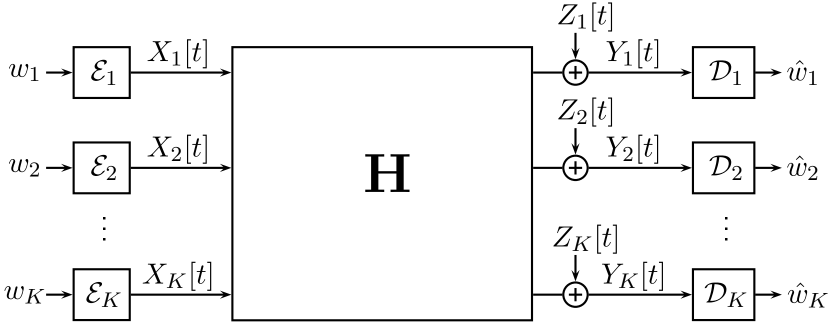
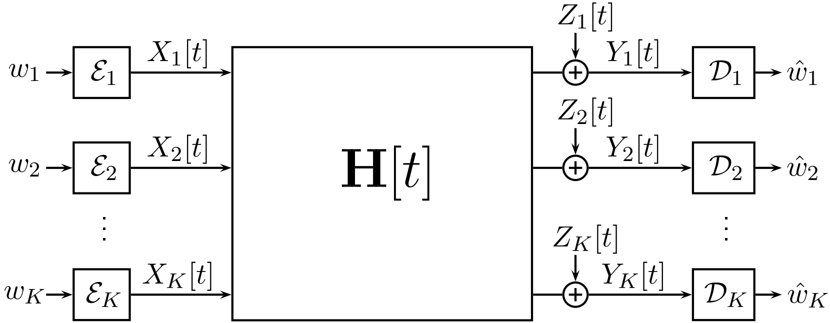


- What about general fixed $\mathbf{H}$? Only partially understood (e.g. high SNR, special cases).
- Much more is known for time-varying channels.
- Assume every transmitter and receiver knows $\mathbf{H}[t]$ causally (i.e. knows $\mathbf{H}[t]$ before time t).



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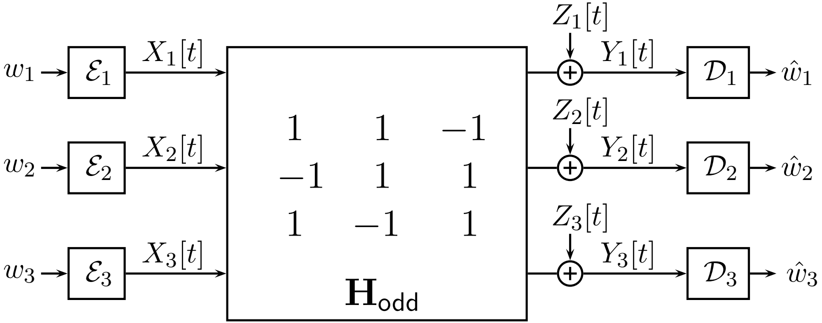
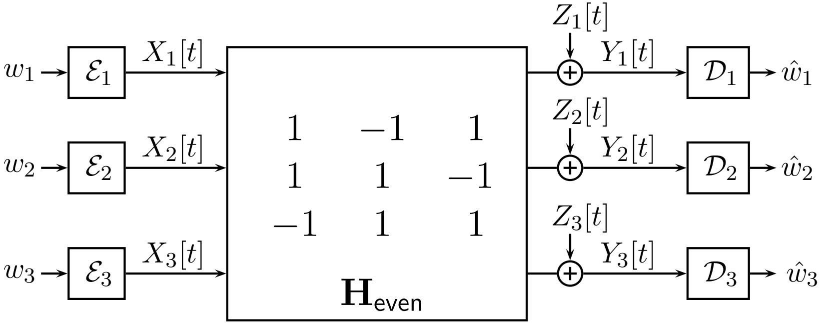


- Example from **Cadambe-Jafar '09**.
- Separate coding over $\mathbf{H}_{\text{odd}}$ and $\mathbf{H}_{\text{even}}$: $R = \frac{1}{3}\log\left(1 + \frac{3P}{N}\right)$
- Joint coding over $\mathbf{H}_{\text{odd}}$ and $\mathbf{H}_{\text{even}}$: $R = \frac{1}{2}\log\left(1 + \frac{2P}{N}\right)$

Interference Alignment – Time-Varying Channels



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Joint Coding

- t odd: $Y_1[t] = X_1[t] + X_2[t] - X_3[t] + Z_1[t]$
- t even: $Y_1[t] = X_1[t] - X_2[t] + X_3[t] + Z_1[t]$
- Joint Coding: Send new symbol every odd time.
  Repeat symbols on even times.
- Decoding: $Y_1[t-1] + Y_1[t] = 2X_1[t-1] + Z_1[t-1] + Z_1[t]$
- Effective SNR: $4P/2N = 2P/N$.
- Two channel uses per symbol: $R = \dfrac{1}{2}\log\left(1 + \dfrac{2P}{N}\right)$
- Same strategy works for users 2 and 3.

I. **K-User Interference Channels**

II. **Alignment via Linear Precoding**

III. **Ergodic Alignment**

IV. **Lattice Alignment for Fixed Channels**

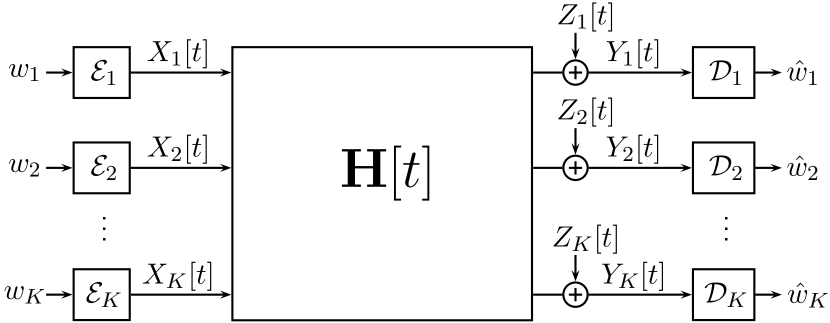
This section is almost entirely drawn from:

- V. Cadambe and S. A. Jafar, *Interference Alignment and Degrees of Freedom of the K-User Interference Channel*. IEEE Transactions on Information Theory, vol. 54, no. 8, pp. 3425-3441, August 2008.

For a comprehensive overview of interference alignment, see:

- Syed A. Jafar, *Interference Alignment: A New Look at Signal Dimensions in a Communication Network*, Foundations and Trends in Communications and Information Theory, Vol. 7, No. 1, pages: 1-136.

Interference Alignment – Time-Varying Channels



- At each time t, each channel gain $h_{k\ell}[t]$ is drawn according to an independent distribution with uniform phase.

- Milder assumptions possible.

- Every transmitter and receiver knows $\mathbf{H}[t]$ causally (i.e. knows $\mathbf{H}[t]$ before time t).

Symbol Extension

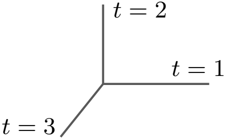
- Joint coding over m time slots:

$$\begin{array}{c} \mathbf{H}[1] = \{h_{k\ell}[1]\} \\ \mathbf{H}[2] = \{h_{k\ell}[2]\} \\ \vdots \\ \mathbf{H}[m] = \{h_{k\ell}[m]\} \end{array} \qquad \mathbf{x}_\ell \triangleq \begin{bmatrix} X_\ell[1] \\ X_\ell[2] \\ \vdots \\ X_\ell[m] \end{bmatrix} \qquad \mathbf{y}_k \triangleq \begin{bmatrix} Y_k[1] \\ Y_k[2] \\ \vdots \\ Y_k[m] \end{bmatrix}$$

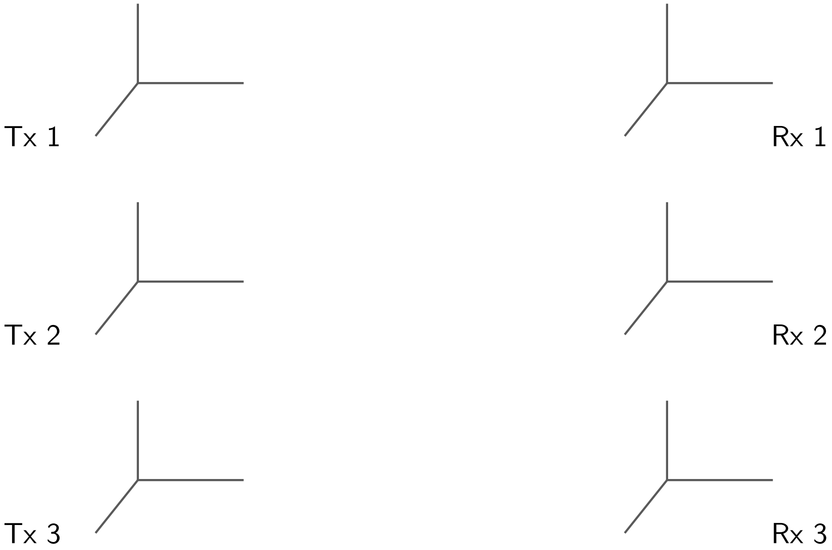
- Convenient to represent this problem with diagonal matrices:

$$\mathbf{D}_{k\ell} \triangleq \begin{bmatrix} h_{k\ell}[1] & 0 & \cdots & 0 \\ 0 & h_{k\ell}[2] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{k\ell}[m] \end{bmatrix} \qquad \mathbf{y}_k = \sum_{\ell=1}^{K} \mathbf{D}_{k\ell}\mathbf{x}_\ell + \mathbf{z}_k$$

- Can visualize $m = 3$ in 3D:



Non-Aligned Signaling over 3 Time Slots



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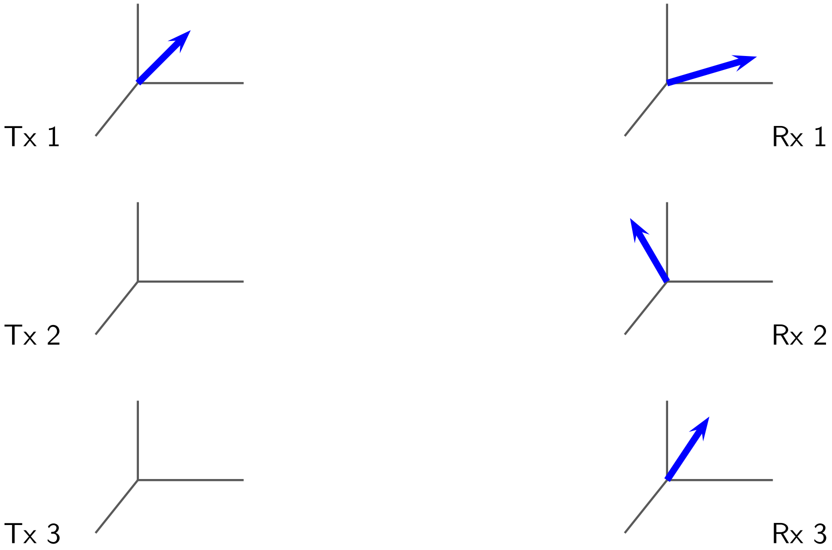

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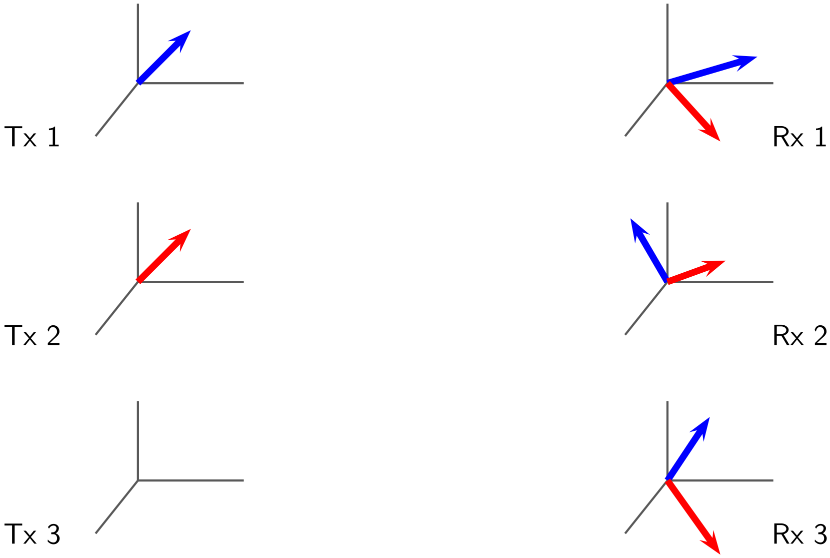

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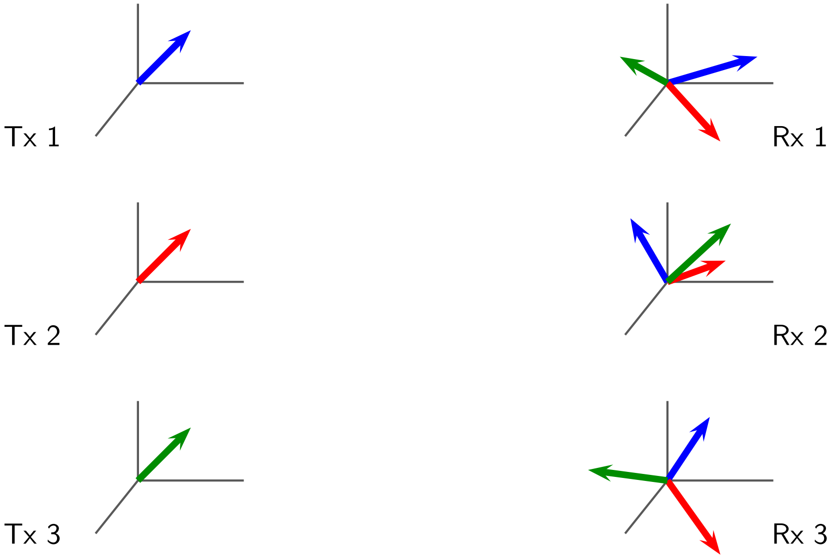

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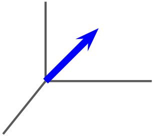
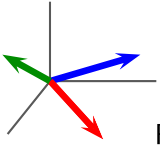


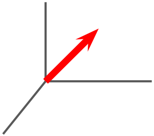
Tx 1



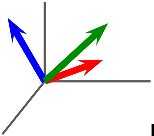
Rx 1

Total Degrees of Freedom

$$\text{DoF} = \frac{3 \text{ vectors}}{3 \text{ channel uses}}$$
$$= 1$$

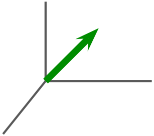


Tx 2

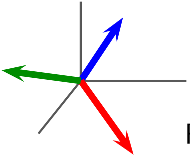


Rx 2

Each user gets $1/3$ the cake.

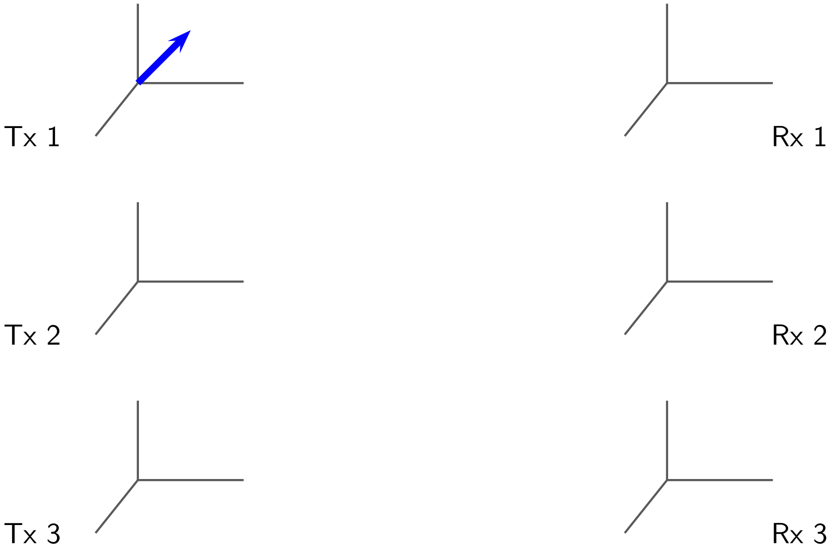


Tx 3



Rx 3

Aligned Signaling over 3 Time Slots



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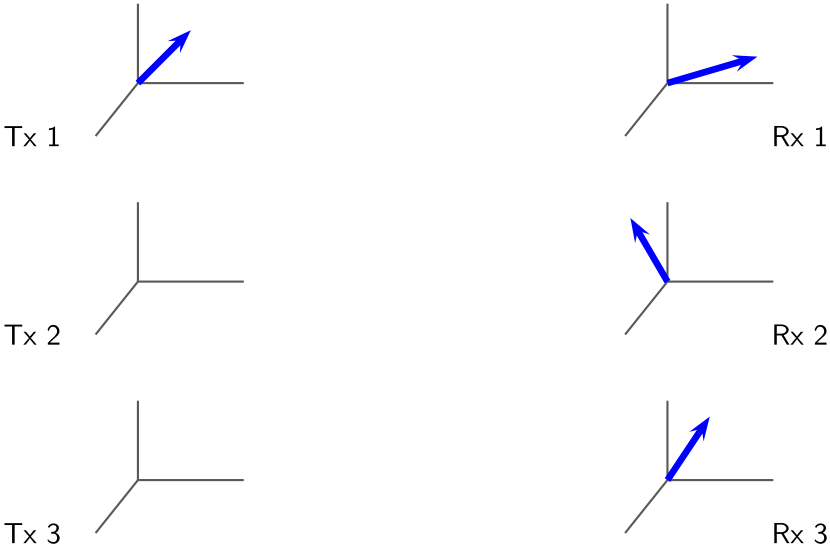

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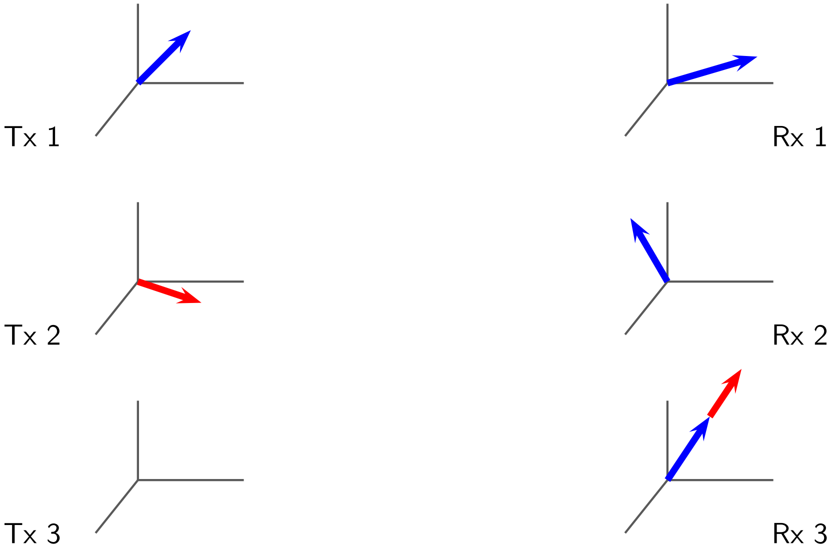

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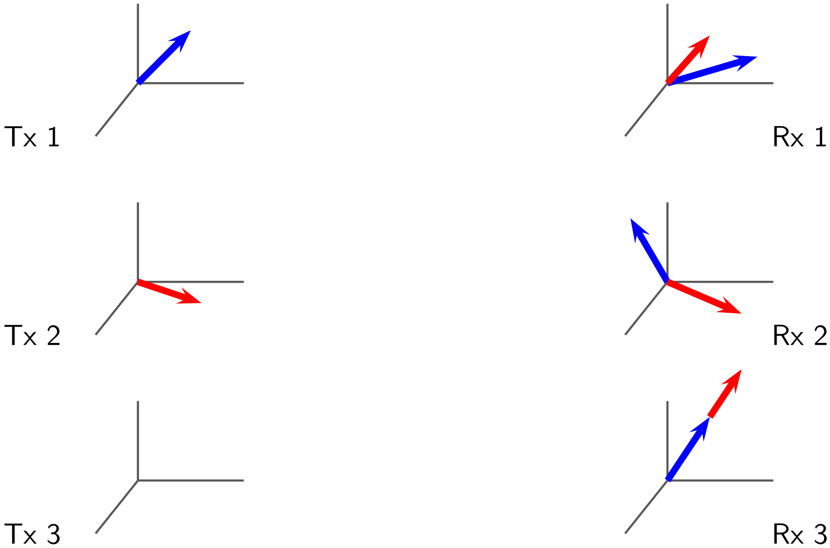

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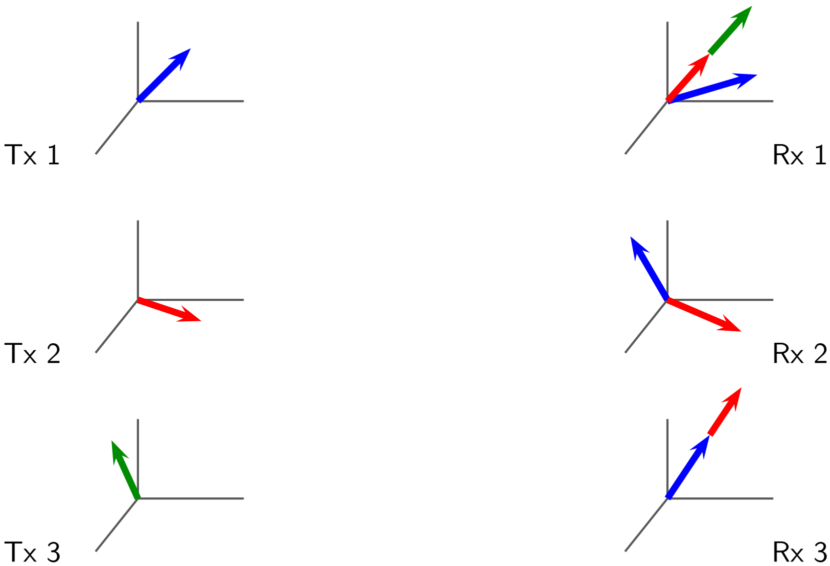

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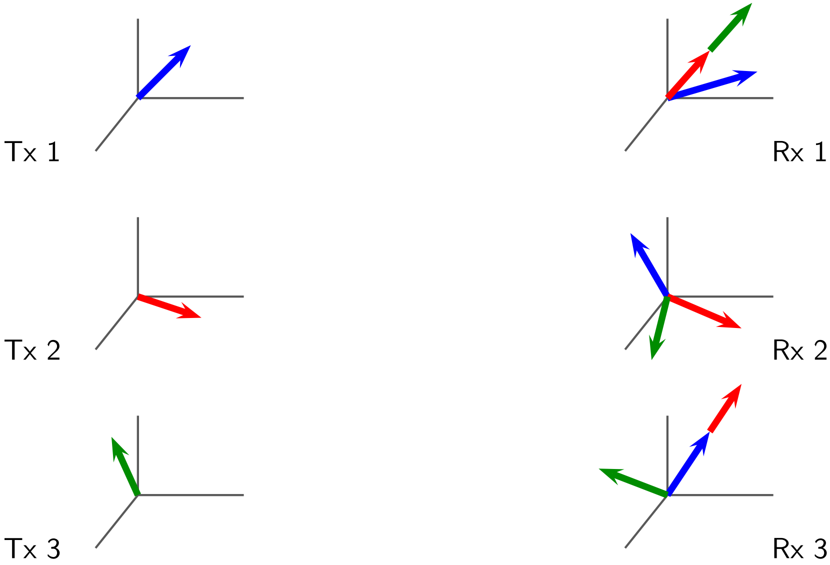

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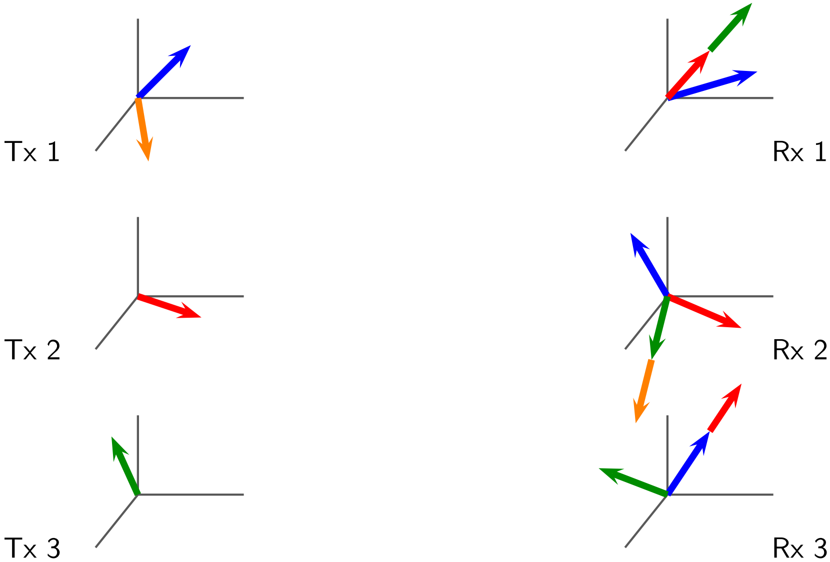

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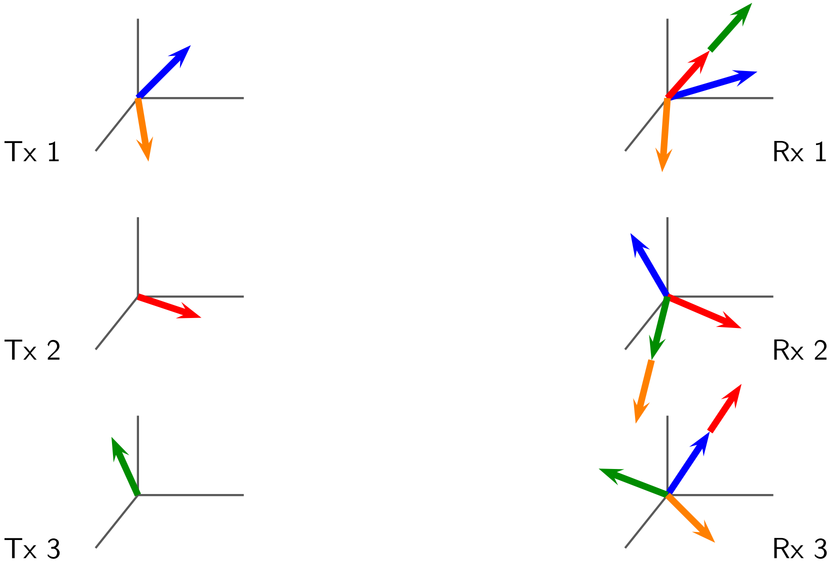

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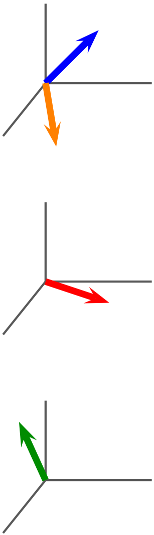


Tx 1

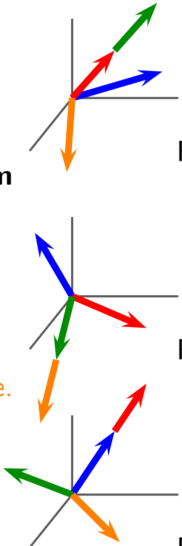
Tx 2

Tx 3

Total Degrees of Freedom

$$\text{DoF} = \frac{4 \text{ vectors}}{3 \text{ channel uses}} = \frac{4}{3}$$

Each user gets $4/9$ the cake.



Rx 1

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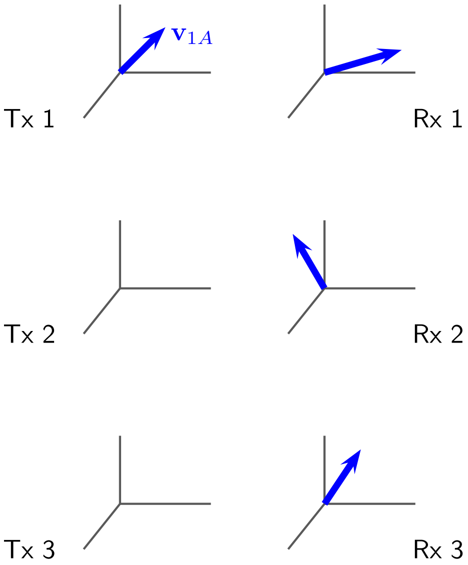
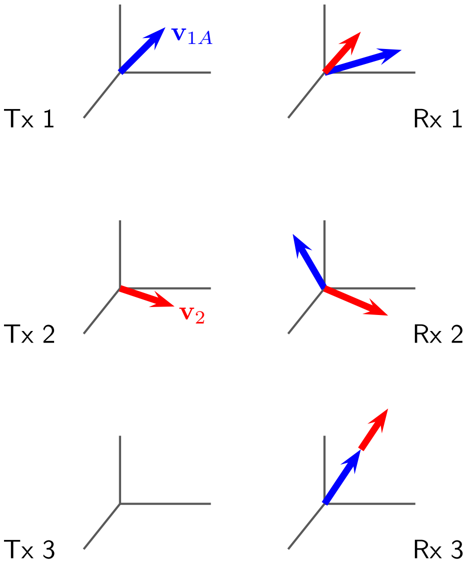
Rx 3

Aligned Signaling over 3 Time Slots



- Set $\mathbf{v}_{1A} = \frac{1}{\sqrt{3}} \left[\ 1\ 1\ 1\ \right]^T$

Aligned Signaling over 3 Time Slots

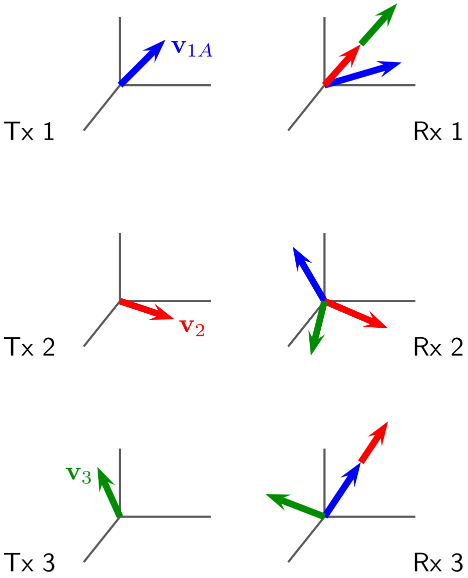


- Set $\mathbf{v}_{1A} = \frac{1}{\sqrt{3}} \left[\ 1\ \ 1\ \ 1\ \right]^T$
- $\mathbf{v}_2$ aligns with $\mathbf{v}_{1A}$ at Rx 3 :

$$\mathbf{D}_{32}\mathbf{v}_2 = \mathbf{D}_{31}\mathbf{v}_{1A}$$

$$\mathbf{v}_2 = \mathbf{D}_{32}^{-1}\mathbf{D}_{31}\mathbf{v}_{1A}$$

Aligned Signaling over 3 Time Slots



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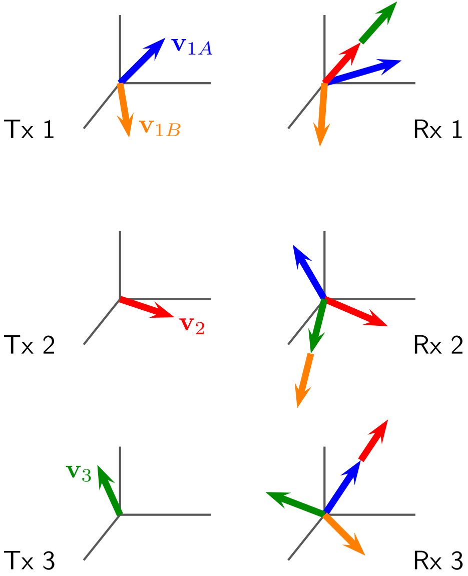
Aligned Signaling over 3 Time Slots



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- $\mathbf{v}_{1B}$ aligns with $\mathbf{v}_3$ at Rx 2 :

$$\mathbf{D}_{21}\mathbf{v}_{1B} = \mathbf{D}_{23}\mathbf{v}_3$$
$$\mathbf{v}_{1B} = \mathbf{D}_{21}^{-1}\mathbf{D}_{23}\mathbf{v}_3$$

- Collect signaling vectors into matrices $\mathbf{V}_\ell = [\mathbf{v}_{\ell 1} \; \mathbf{v}_{\ell 2} \; \cdots \; \mathbf{v}_{\ell m}]$.
- Receiver k allocates subspace $\mathcal{I}_k$ as interference space.
- Alignment conditions:

$$\begin{array}{cccc}
\underline{\text{Receiver 1}} & \underline{\text{Receiver 2}} & & \underline{\text{Receiver K}} \\
\mathbf{D}_{11}\mathbf{V}_1 \cap \mathcal{I}_1 = \emptyset & \mathbf{D}_{21}\mathbf{V}_1 \subseteq \mathcal{I}_2 & & \mathbf{D}_{K1}\mathbf{V}_1 \subseteq \mathcal{I}_K \\
\mathbf{D}_{12}\mathbf{V}_2 \subseteq \mathcal{I}_1 & \mathbf{D}_{22}\mathbf{V}_2 \cap \mathcal{I}_2 = \emptyset & \ldots & \mathbf{D}_{K2}\mathbf{V}_2 \subseteq \mathcal{I}_K \\
\vdots & \vdots & & \vdots \\
\mathbf{D}_{1K}\mathbf{V}_K \subseteq \mathcal{I}_1 & \mathbf{D}_{2K}\mathbf{V}_K \subseteq \mathcal{I}_2 & & \mathbf{D}_{KK}\mathbf{V}_K \cap \mathcal{I}_K = \emptyset
\end{array}$$

- Want $m/2$ dimensions for $\mathbf{V}_k$ and $\mathcal{I}_k$.
- Not feasible in general.

- Enumerate all $S \triangleq K(K-1)$ cross-channels with a single index:

$$\mathcal{T} = \left\{\mathbf{T}_i\right\} = \left\{\mathbf{D}_{k\ell} : k \neq \ell\right\}$$

- Use the same signaling vectors $\mathcal{V}^{(m)}$ at every transmitter.
- Define signaling vectors recursively:

$$\begin{aligned} \mathcal{V}^{(0)} &= \{\mathbf{1}\} \\ \mathcal{V}^{(m)} &= \left\{\mathbf{v}_i,\ \mathbf{T}_1\mathbf{v}_i,\ \ldots\ ,\ \mathbf{T}_S\mathbf{v}_i\ :\ \mathbf{v}_i \in \mathcal{V}^{(m-1)}\right\} \\ &= \left\{\mathbf{T}_1^{\alpha_1}\mathbf{T}_2^{\alpha_2}\cdots\mathbf{T}_S^{\alpha_S}\mathbf{1} : \alpha_1 + \alpha_2 + \cdots + \alpha_S \leq m\right\} \end{aligned}$$

- Size of signaling space:

$$\left|\mathcal{V}^{(m)}\right| = \binom{m+S}{m}$$

Getting Half the Cake – Asymptotic Alignment

- Interference space at receiver k is $\mathcal{I}_k = \bigcup_{\ell \neq k} \mathbf{D}_{k\ell} \mathcal{V}^{(m)} \subset \mathcal{V}^{(m+1)}$
- Desired signal space at receiver k is $\mathbf{D}_{kk} \mathcal{V}^{(m)}$.
- $\mathcal{V}^{(m)}$ only contains products of cross-channels so there is **no overlap** between desired signal space and interference space.
- Number of vectors is nearly the same for large m:

$$\frac{\left|\mathcal{V}^{(m)}\right|}{\left|\mathcal{V}^{(m+1)}\right|} = \frac{\binom{m+S}{m}}{\binom{m+1+S}{m+1}} = \frac{m+1}{m+1+S} \xrightarrow{m\to\infty} 1$$

- Desired signal space asymptotically gets half the dimensions:

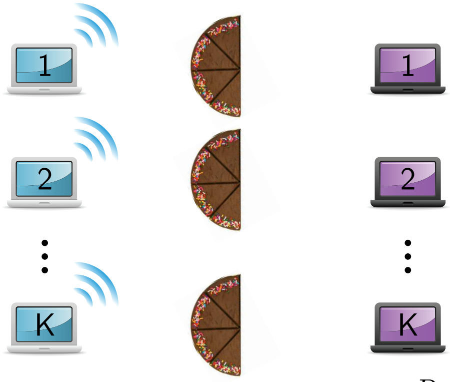
$$\frac{\left|\mathbf{D}_{kk}\mathcal{V}^{(m)}\right|}{\left|\mathbf{D}_{kk}\mathcal{V}^{(m)}\right| + \left|\mathcal{I}_k\right|} \xrightarrow{m\to\infty} \frac{1}{2}$$

K-User Interference Channel – Degrees-of-Freedom Region

- Everyone can get half the cake $d_k = \lim_{P\to\infty} \frac{R_k}{\log(1+P)} = \frac{1}{2}$
- If all but user k quiet: $R_k = \log(1 + |h_{kk}|^2 P)$
- Time share to get degrees-of-freedom region:

$$d_k + d_\ell \leq 1, \qquad \forall k \neq \ell.$$

K-User Interference Channel – Degrees-of-Freedom Region



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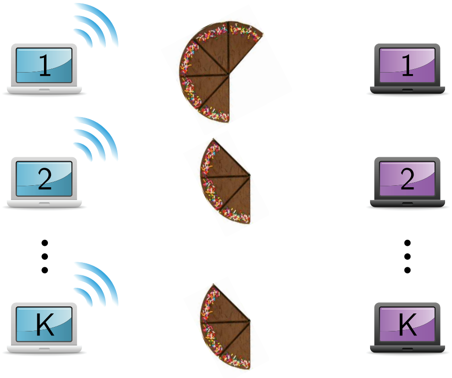

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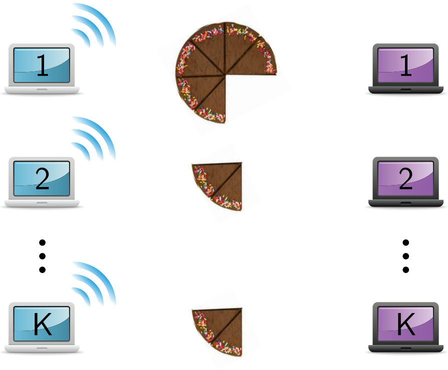

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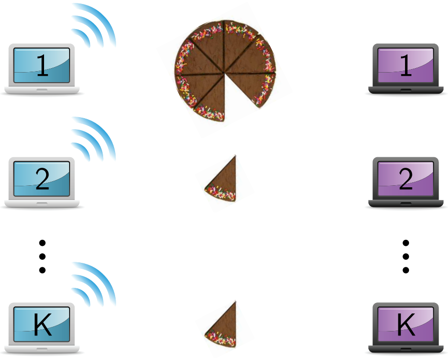

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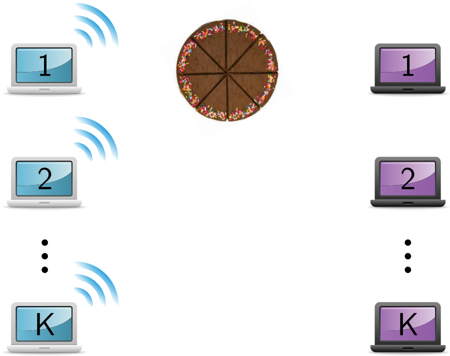

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I. K-User Interference Channels

II. Alignment via Linear Precoding

III. Ergodic Alignment

IV. Lattice Alignment for Fixed Channels

This section is almost entirely drawn from:

- B. Nazer, M. Gastpar, S. A. Jafar, and S. Vishwanath, *Ergodic Interference Alignment*,

- We can get (slightly more than) half the interference-free rate at any SNR!

$$R_k^{\mathsf{EIA}} = \frac{1}{2}\mathbb{E}_{\mathbf{H}}\left[\log\left(1 + \frac{2|h_{kk}|^2 P}{N}\right)\right]$$

Ergodic Interference Alignment

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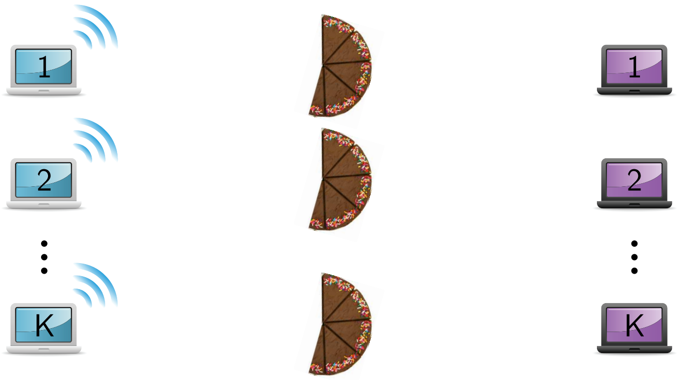

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Key Idea

1. At time t with channel $\mathbf{H}$, user k transmits signal X_k.

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1K} \\ h_{21} & h_{22} & \cdots & h_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ h_{K1} & h_{K2} & \cdots & h_{KK} \end{bmatrix}$$

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2. When complementary matrix $\mathbf{H}_C$ occurs, retransmit signal X_k.

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3. Otherwise, transmit new signals and wait for their $\mathbf{H}_C$.

Ergodic Alignment at the Receivers

$$\begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_K(t) \end{bmatrix}$$

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$$+$$

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$+$

$$\begin{bmatrix} Y_1(t) + Y_1(t_C) \\ Y_2(t) + Y_2(t_C) \\ \vdots \\ Y_K(t) + Y_K(t_C \end{bmatrix}$$

Ergodic Alignment at the Receivers

$$
\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1K} \\ h_{21} & h_{22} & \cdots & h_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ h_{K1} & h_{K2} & \cdots & h_{KK} \end{bmatrix} \qquad \mathbf{X} \qquad + \qquad \mathbf{Z}(t)
$$

$$
+ \quad \left(\begin{bmatrix} h_{11} & -h_{12} & \cdots & -h_{1K} \\ -h_{21} & h_{22} & \cdots & -h_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -h_{K1} & -h_{K2} & \cdots & h_{KK} \end{bmatrix} \pm \delta \right) \mathbf{X} \qquad + \qquad \mathbf{Z}(t_C)
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$$\left(\begin{bmatrix} 2h_{11} & 0 & \cdots & 0 \\ 0 & 2h_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2h_{KK} \end{bmatrix} \pm \delta \right) \mathbf{X} \quad + \quad \mathbf{Z}(t) + \mathbf{Z}(t_C)$$

Sum of channel observations is (nearly) interference-free:

$$\mathbf{H} + \mathbf{H}_C = \begin{bmatrix} 2h_{11} & & 0 \\ & \ddots & \\ 0 & & 2h_{KK} \end{bmatrix} \pm \delta$$

Worst case SINR:

$$\frac{2P\big(|h_{kk}|^2 - 2\delta(\mathsf{Re}(h_{kk}) + \mathsf{Im}(h_{kk})) + \delta^2\big)}{1 + 4\delta^2(K-1)P}$$

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- We need to match up almost every matrix with its complement.

Pairing Up Channels

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- Choose δ, h_{MAX} to get desired rate gap.
- Since phase is i.i.d. uniform, $\mathbb{P}(\mathbf{H}) = \mathbb{P}(\mathbf{H}_C)$.

Convergence in Type

Sequence of quantized channel matrices $\hat{\mathbf{H}}^n$ is ϵ-typical if:

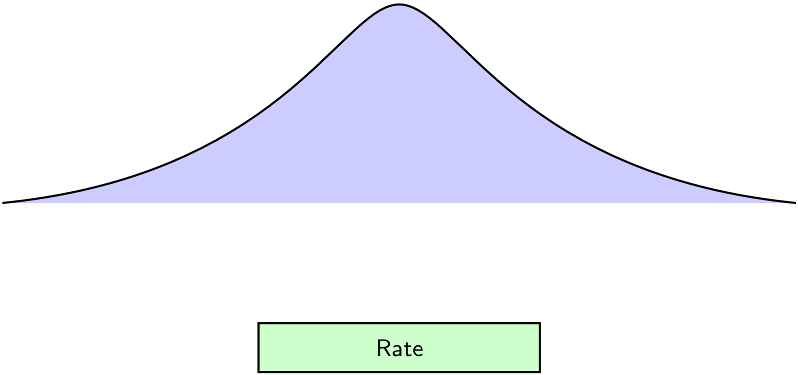
$$\left|\frac{1}{n}\#(\hat{\mathsf{H}}|\hat{\mathbf{H}}^n) - P(\hat{\mathsf{H}})\right| \leq \epsilon \quad \forall \hat{\mathsf{H}} \in \hat{\mathcal{H}}$$

Lemma (Csiszar-Körner 2.12)

For any i.i.d. sequence of quantized channel matrices, $\hat{\mathbf{H}}^n$, the probability of the set of all ϵ-typical sequences, A_ϵ^n, is lower bounded by:

$$\mathbb{P}(A_\epsilon^n) \geq 1 - \frac{|\hat{\mathcal{H}}|}{4n\epsilon^2}$$

Convergence in Type



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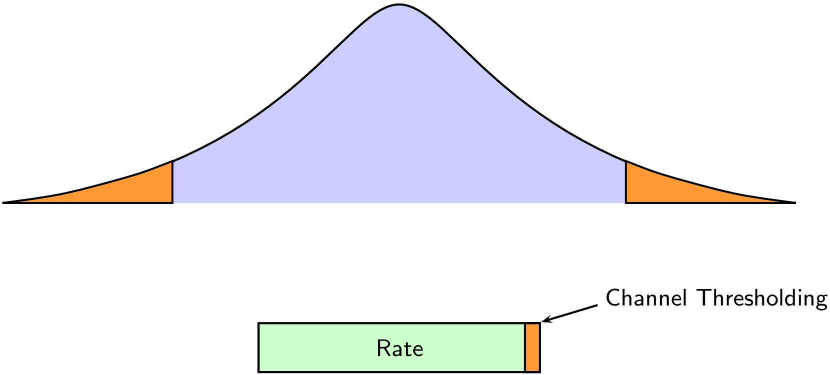

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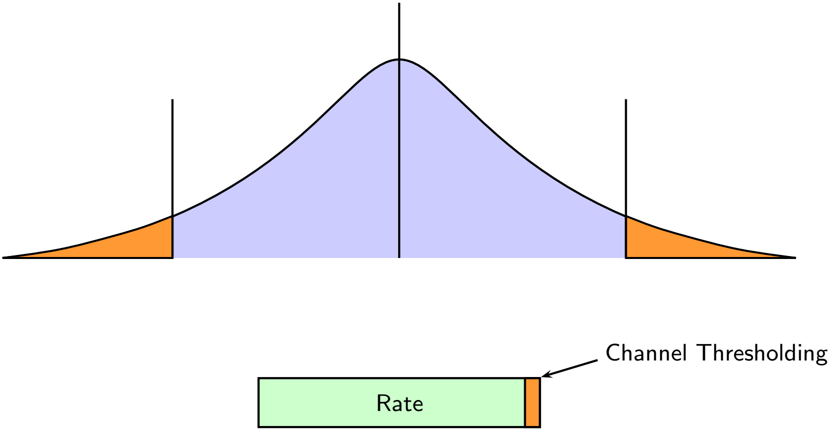

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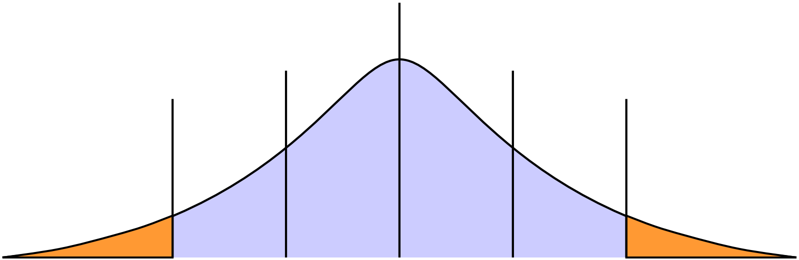

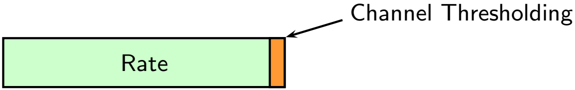

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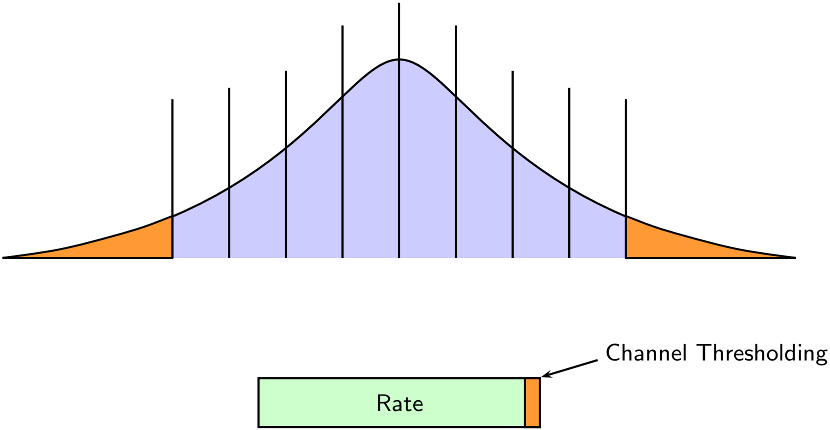

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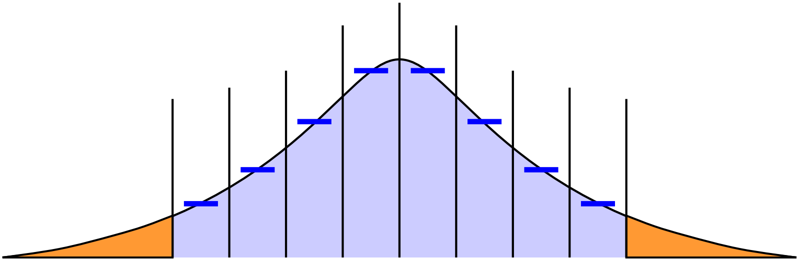

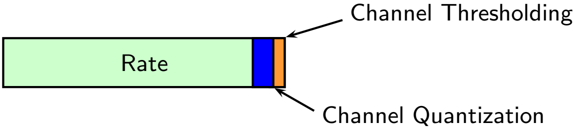

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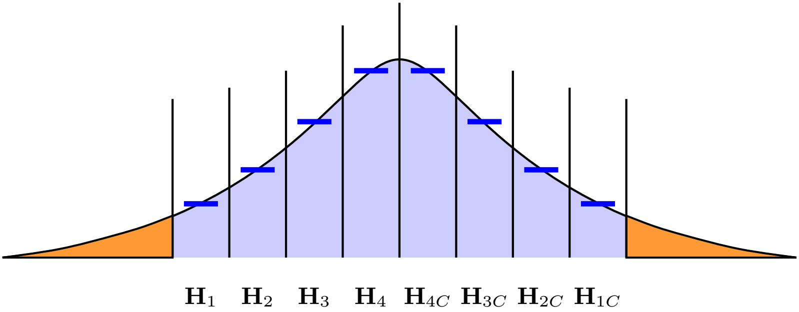

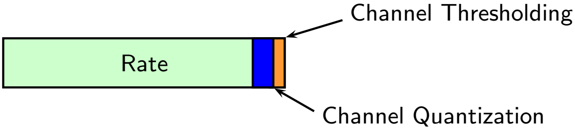

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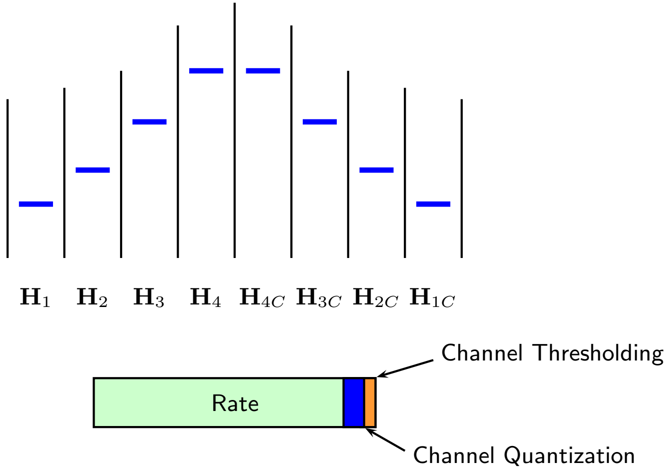

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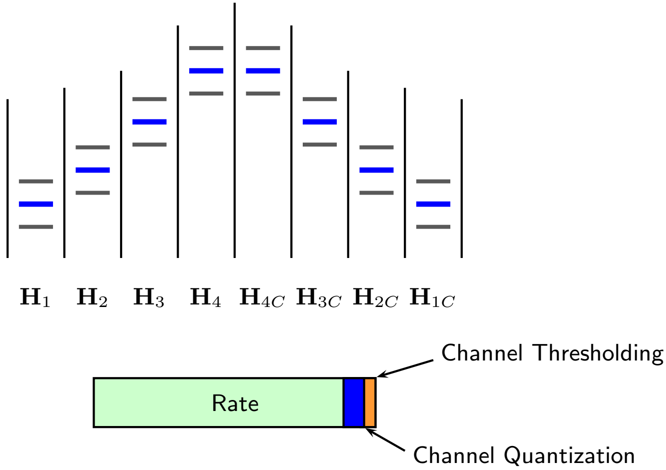

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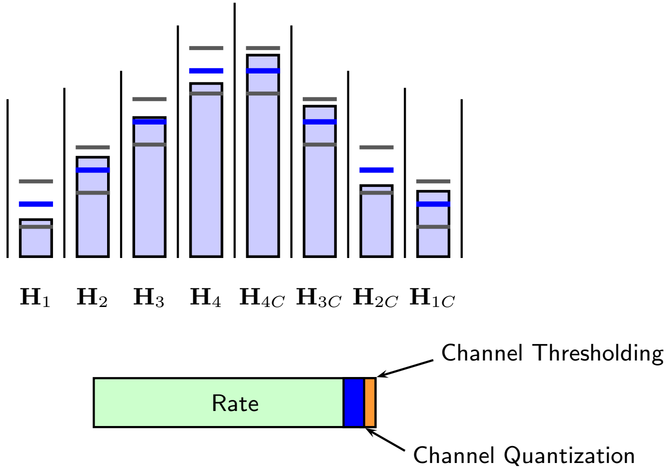

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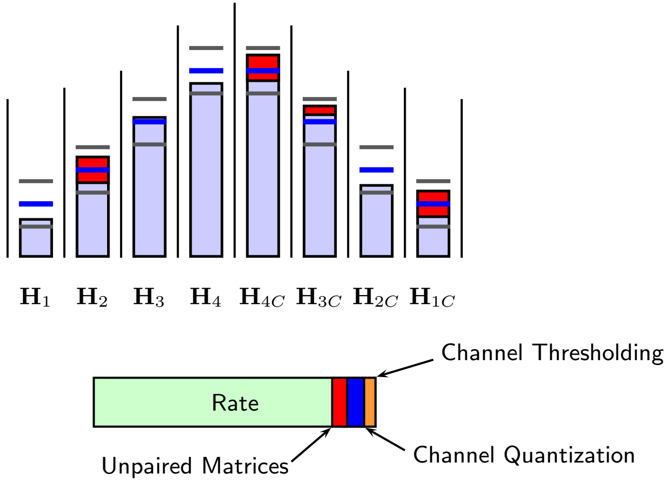


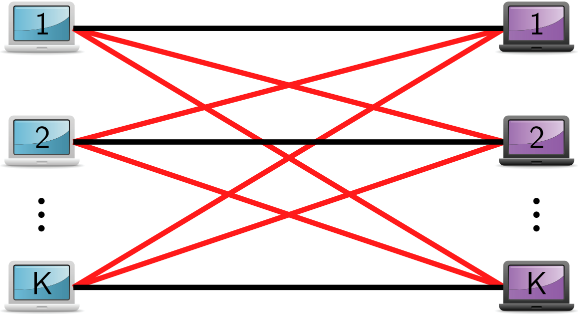
Achievable Rate

Theorem

Each user can achieve at least half its interference-free capacity at **any signal-to-noise ratio***:*

$$R_k = \frac{1}{2} E\left[\log\left(1 + 2|h_{kk}|^2 P_k\right)\right] > \frac{1}{2} R_k^{\mathit{FREE}}$$

Network Transformation



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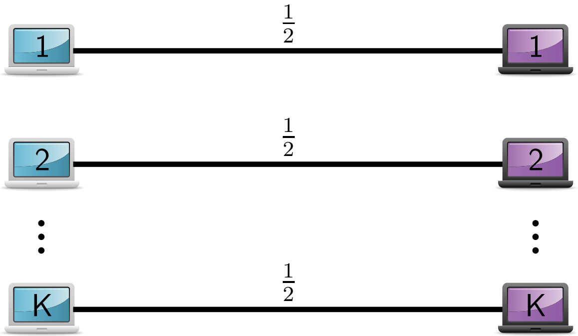
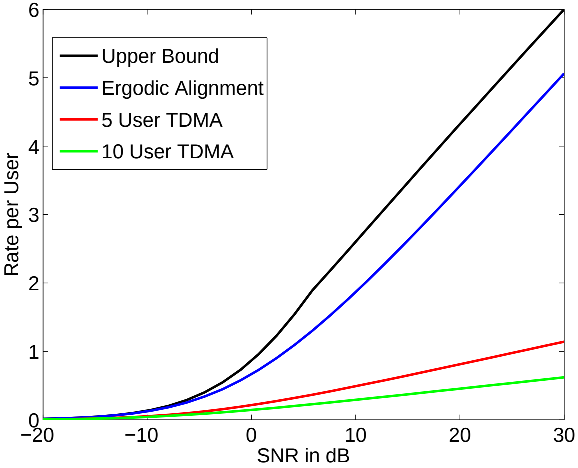


Rayleigh Fading



- Channel coefficients i.i.d. Rayleigh. Equal transmit power per user.

When does ergodic alignment reach capacity?

- If all channel gains have fixed, equal magnitudes (and time-varying i.i.d. uniform phase), ergodic alignment reaches capacity:

$$C = \frac{1}{2}\log\left(1 + \frac{2P}{N}\right) \qquad \text{Symmetric Case}$$

- In general, we should waterfill power allocation over channel states.

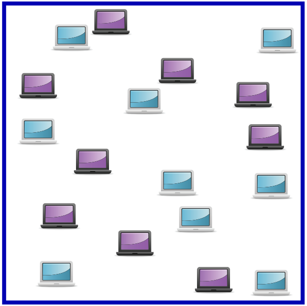
- Is this enough?

When does ergodic alignment reach capacity?

- For Rayleigh fading, we get a very weak interference channel with some constant probability $\rho > 0$.
- Ignore all interference in weak interference case. Get R_k^{WEAK}.
- Otherwise, use ergodic alignment to get R_k^{EA}.
- Each user gets $R_k = \rho R_k^{\mathsf{WEAK}} + (1-\rho) R_k^{\mathsf{EA}} > R_k^{\mathsf{EA}}$
- We need to mix between decoding, ignoring, and aligning interference.
- **Open Question:** Does this come to within a constant gap of the capacity region?

When does ergodic alignment reach capacity?

- **Jafar '09:** Whenever the channel is in a bottleneck state, ergodic alignment achieves the capacity.
- Example: K transmitter-receiver pairs randomly placed in a square. Signal strength governed by distance. As $K \to \infty$, ergodic alignment achieves capacity.



I. **K-User Interference Channels**

II. **Alignment via Linear Precoding**

III. **Ergodic Alignment**

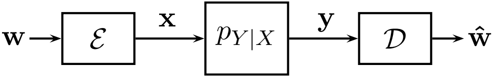
IV. **Lattice Alignment for Fixed Channels**

Nested lattice framework in this section is almost entirely drawn from:

- U. Erez and R. Zamir, *Achieving* $\frac{1}{2}\log(1+\mathsf{SNR})$ *on the AWGN channel with lattice encoding and decoding*, IEEE Transactions on Information Theory, vol. 50, pp. 2293-2314, October 2004.
- U. Erez, S. Litsyn, and R. Zamir, *Lattices which are good for (almost) everything*, IEEE Transactions on Information Theory, vol. 51, pp. 3401-3416, October 2005.
- R. Zamir, *Lattices are everywhere*, in Proceedings of the 4th Annual Workshop on Information Theory and its Applications, La Jolla, CA, February 2009.

See Ram Zamir's lattice tutorials (`http://www.eng.tau.ac.il/~zamir/`) or my ISIT 2011 tutorial (`http://iss.bu.edu/bobak/tutorial_isit11.pdf`) for more information.

Point-to-Point Channels



The Usual Suspects:

- Message $\mathbf{w} \in \{0,1\}^k$
- Encoder $\mathcal{E} : \{0,1\}^k \to \mathcal{X}^n$
- Input $\mathbf{x} \in \mathcal{X}^n$
- Estimate $\hat{\mathbf{w}} \in \{0,1\}^k$
- Decoder $\mathcal{D} : \mathcal{Y}^n \to \{0,1\}^k$
- Output $\mathbf{y} \in \mathcal{Y}^n$

- Memoryless Channel $p(\mathbf{y}|\mathbf{x}) = \displaystyle\prod_{i=1}^{n} p(y_i|x_i)$
- Rate $R = \dfrac{k}{n}$.
- (Average) Probability of Error: $\mathbb{P}\{\hat{\mathbf{w}} \neq \mathbf{w}\} \to 0$ as $n \to \infty$. Assume $\mathbf{w}$ is uniform over $\{0,1\}^k$.

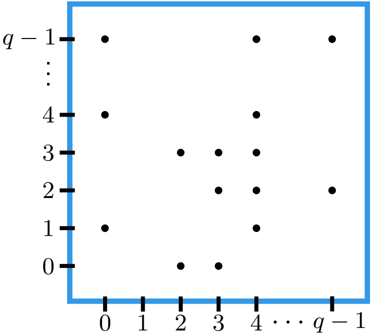
i.i.d. Random Codes

- Generate 2^{nR} codewords $\mathbf{x} = [X_1 \ X_2 \ \cdots \ X_n]$ independently and elementwise i.i.d. according to some distribution p_X

$$p(\mathbf{x}) = \prod_{i=1}^{n} p_X(x_i)$$

- Bound the average error probability for a random codebook.

- If the average performance over codebooks is good, there must exist at least one good fixed codebook.



Decoder looks for a codeword that is jointly typical with the received sequence $\mathbf{y}$

Error Events

1. Transmitted codeword $\mathbf{x}$ is not jointly typical with $\mathbf{y}$.
   $\Longrightarrow$ Low probability by the Weak Law of Large Numbers.
2. Another codeword $\tilde{\mathbf{x}}$ is jointly typical with $\mathbf{y}$.

Cuckoo's Egg Lemma

Let $\tilde{\mathbf{x}}$ be an i.i.d. sequence that is independent from the received sequence $\mathbf{y}$.

$$\mathbb{P}\Big\{(\tilde{\mathbf{x}}, \mathbf{y}) \ \text{ is jointly typical}\Big\} \leq 2^{-n(I(X;Y)-3\epsilon)}$$

See **Cover and Thomas**.

- We can upper bound the probability of error via the union bound:

$$\mathbb{P}\{\hat{\mathbf{w}} \neq \mathbf{w}\} \leq \sum_{\tilde{\mathbf{w}} \neq \mathbf{w}} \mathbb{P}\Big\{(\mathbf{x}(\tilde{\mathbf{w}}), \mathbf{y}) \text{ is jointly typical.}\Big\}$$
$$\leq 2^{-n(I(X;Y)-R-3\epsilon)} \qquad \leftarrow \text{Cuckoo's Egg Lemma}$$

- If $R < I(X;Y)$, then the probability of error can be driven to zero as the blocklength increases.

Theorem (Shannon '48)

The capacity of a point-to-point channel is $C = \max_{p_X} I(X;Y)$.

- Linear Codebook: A linear map between messages and codewords (instead of a lookup table).

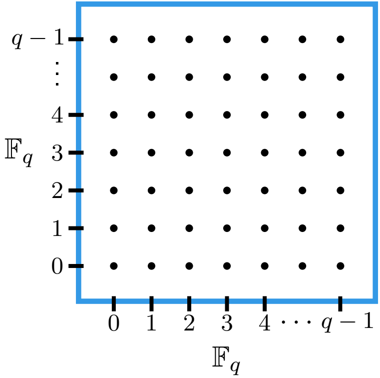
q-ary Linear Codes

- Represent message $\mathbf{w}$ as a length-k vector over $\mathbb{F}_q$.
- Codewords $\mathbf{x}$ are length-n vectors over $\mathbb{F}_q$.
- Encoding process is just a matrix multiplication, $\mathbf{x} = \mathbf{G}\mathbf{w}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1k} \\ g_{21} & g_{22} & \cdots & g_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nk} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}$$

- Recall that, for prime q, operations over $\mathbb{F}_q$ are just $\mod q$ operations over the reals.
- Rate $R = \dfrac{k}{n} \log q$

- Linear code looks like a regular subsampling of the elements of $\mathbb{F}_q^n$.
- **Random linear code:** Generate each element g_{ij} of the generator matrix $\mathbf{G}$ elementwise i.i.d. according to a uniform distribution over $\{0, 1, 2, \ldots, q-1\}$.
- How are the codewords distributed?



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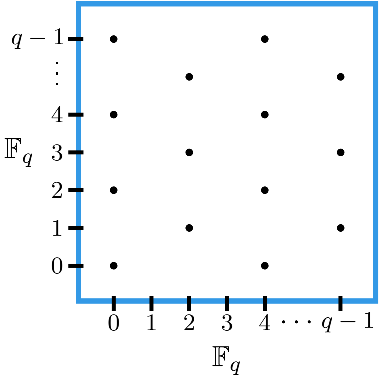


It is convenient to instead analyze the shifted ensemble $\bar{\mathbf{x}} = \mathbf{G}\mathbf{w} \oplus \mathbf{v}$ where $\mathbf{v}$ is an i.i.d. uniform sequence. (See **Gallager**.)

Shifted Codeword Properties

1. Marginally uniform over $\mathbb{F}_q^n$. For a given message $\mathbf{w}$, the codeword $\bar{\mathbf{x}}$ looks like an i.i.d. uniform sequence.

$$\mathbb{P}\{\bar{\mathbf{x}} = \mathsf{x}\} = \frac{1}{q^n} \quad \text{for all } \mathsf{x} \in \mathbb{F}_q^n$$

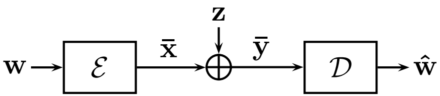
2. Pairwise independent. For $\mathbf{w}_1 \neq \mathbf{w}_2$, codewords $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2$ are independent.

$$\mathbb{P}\{\bar{\mathbf{x}}_{\mathbf{1}} = \mathsf{x}_1, \bar{\mathbf{x}}_{\mathbf{2}} = \mathsf{x}_2\} = \frac{1}{q^{2n}} = \mathbb{P}\{\bar{\mathbf{x}}_1 = \mathsf{x}_1\}\mathbb{P}\{\bar{\mathbf{x}}_2 = \mathsf{x}_2\}$$

- Cuckoo's Egg Lemma only requires independence between the true codeword $\mathbf{x}(\mathbf{w})$ and the other codeword $\mathbf{x}(\tilde{\mathbf{w}})$. From the union bound:

$$\begin{aligned}\mathbb{P}\{\hat{\mathbf{w}} \neq \mathbf{w}\} &\leq \sum_{\tilde{\mathbf{w}} \neq \mathbf{w}} \mathbb{P}\Big\{(\mathbf{x}(\tilde{\mathbf{w}}), \mathbf{y}) \ \text{ is jointly typical.}\Big\} \\ &\leq 2^{-n(I(X;Y)-R-3\epsilon)}\end{aligned}$$

- This is exactly what we get from pairwise independence.
- Thus, there exists a good fixed generator matrix $\mathbf{G}$ and shift $\mathbf{v}$ for any rate $R < I(X;Y)$ where X is uniform.



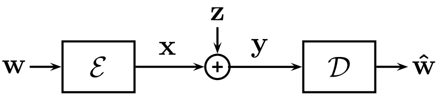
- For a binary symmetric channel (BSC), the output can be written as the modulo sum of the input plus i.i.d. Bernoulli(p) noise,

$$\bar{\mathbf{y}} = \bar{\mathbf{x}} \oplus \mathbf{z}$$
$$\bar{\mathbf{y}} = \mathbf{G}\mathbf{w} \oplus \mathbf{v} \oplus \mathbf{z}$$

- Due to this symmetry, the probability of error depends *only* on the realization of the noise vector $\mathbf{z}$. For a BSC, $\mathbf{x} = \mathbf{G}\mathbf{w}$ is a good code as well.
- We can now assume the existence of good generator matrices for channel coding.

- Codewords must satisfy power constraint:

$$\|\mathbf{x}\|^2 \leq nP \ .$$



- i.i.d. Gaussian noise with variance N:

$$\mathbf{z} \sim \mathcal{N}(\mathbf{0}, N\mathbf{I}) \ .$$

- **Shannon '48**: Channel capacity:

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N} \right)$$

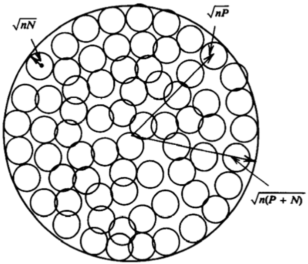


Figure 10.2. Sphere packing for the Gaussian channel.

(Cover and Thomas, *Elements of Information Theory*)

- In high dimensions, noise starts to look spherical.

Lattices

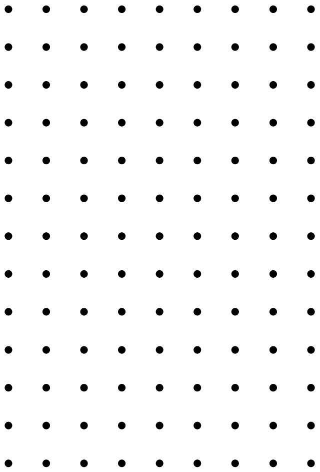
- A lattice Λ is a discrete subgroup of $\mathbb{R}^n$.
- Can write a lattice as a linear transformation of the integer vectors,

$$\Lambda = \{\mathbf{B}\mathbf{s} : \mathbf{s} \in \mathbb{Z}^n\} \ ,$$

for some $\mathbf{B} \in \mathbb{R}^{n \times n}$.

Lattice Properties

- Closed under addition: $\lambda_1, \lambda_2 \in \Lambda \implies \lambda_1 + \lambda_2 \in \Lambda$.
- Symmetric: $\lambda \in \Lambda \implies -\lambda \in \Lambda$



$\mathbb{Z}^n$ is a simple lattice.

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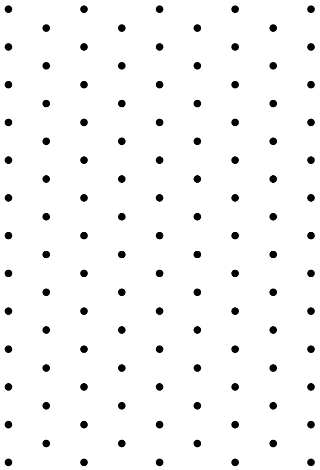


$\mathbf{B}\mathbb{Z}^n$

Voronoi Regions

- Nearest neighbor quantizer:

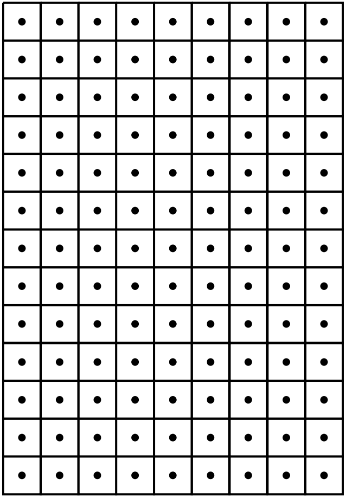
$$Q_\Lambda(\mathbf{x}) = \arg\min_{\lambda \in \Lambda} \|\mathbf{x} - \lambda\|_2$$

- The Voronoi region of a lattice point is the set of all points that quantize to that lattice point.

- Fundamental Voronoi region $\mathcal{V}$: points that quantize to the origin,

$$\mathcal{V} = \{\mathbf{x} : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$$

- All Voronoi regions are just shifts of $\mathcal{V}$



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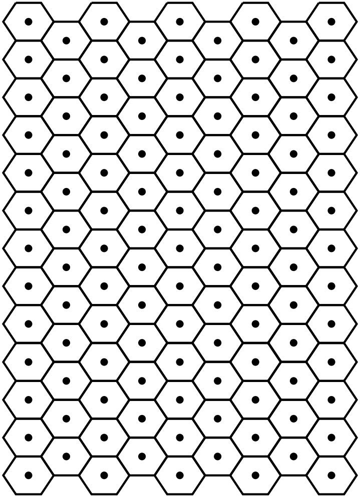


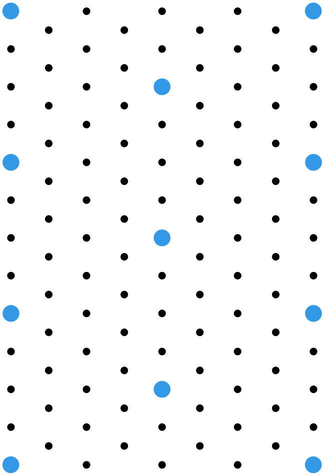
Nested Lattices

- Two lattices Λ and Λ_{FINE} are nested if $\Lambda \subset \Lambda_{\mathsf{FINE}}$

- **Nested Lattice Code:** All lattice points from Λ_{FINE} that fall in the fundamental Voronoi region $\mathcal{V}$ of Λ.

- $\mathcal{V}$ acts like a power constraint

$$\text{Rate} = \frac{1}{n} \log \left(\frac{\mathsf{Vol}(\mathcal{V})}{\mathsf{Vol}(\mathcal{V}_{\mathsf{FINE}})} \right)$$



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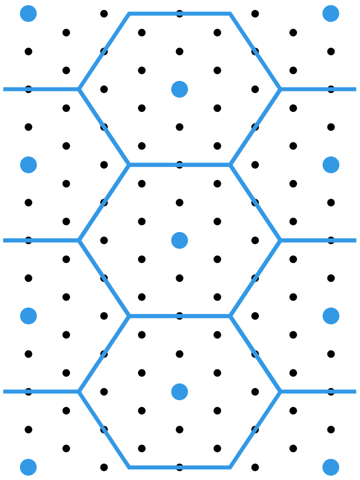

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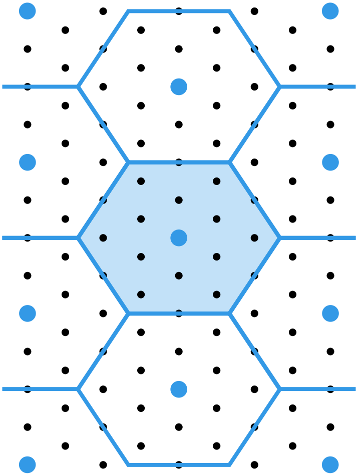

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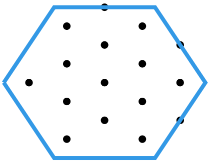

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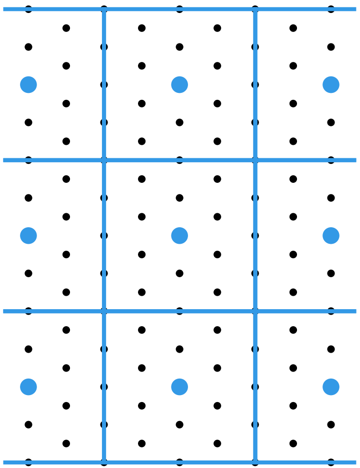
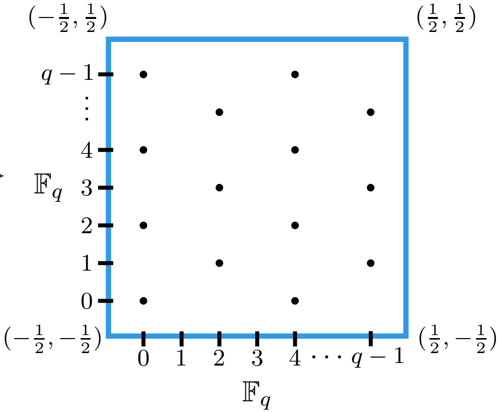


Nested Lattice Codes from q-ary Linear Codes

- Choose an $n \times k$ generator matrix $\mathbf{G} \in \mathbb{F}_q^{n \times k}$ for q-ary code.
- Integers serve as coarse lattice, $\Lambda = \mathbb{Z}^n$.
- Map elements $\{0, 1, 2, \ldots, q-1\}$ to equally spaced points between $-1/2$ and $1/2$.
- Place codewords $\mathbf{x} = \mathbf{G}\mathbf{w}$ into the fundamental Voronoi region $\mathcal{V} = [-1/2, 1/2)^n$



Modulo Operation

- Modulo operation with respect to lattice Λ is just the residual quantization error,

$$[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}) \ .$$

- Mimics the role of $\bmod q$ in q-ary alphabet.

- Distributive Law:

$$\Big[\mathbf{x}_1 + [\mathbf{x}_2] \bmod \Lambda\Big] \bmod \Lambda$$
$$= [\mathbf{x}_1 + \mathbf{x}_2] \bmod \Lambda$$

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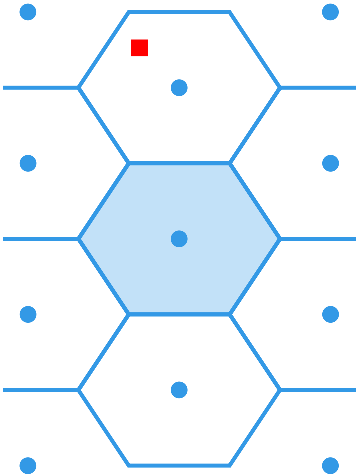

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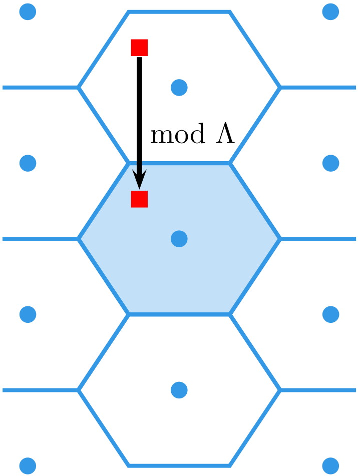
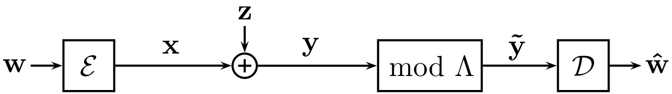
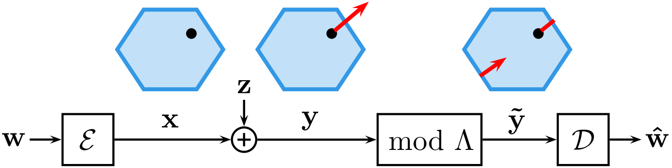


$\bmod \Lambda$ *AWGN Channel*



- Codebook lives on Voronoi region $\mathcal{V}$ of coarse lattice Λ.
- Take $\bmod \Lambda$ of received signal prior to decoding.
- What is the capacity of the $\bmod \Lambda$ channel?

Using random codes: $$C = \frac{1}{n} \max_{p(\mathbf{x})} I(\mathbf{x}; \tilde{\mathbf{y}})$$



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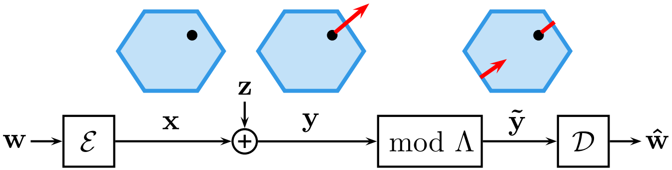
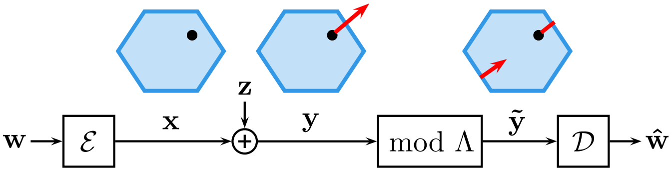
mod Λ *AWGN Channel Capacity*



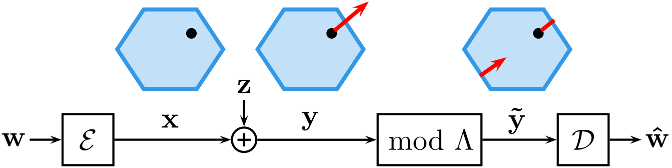
$$
\begin{aligned}
nC &= \max_{p(\mathbf{x})} I(\mathbf{x}; \tilde{\mathbf{y}}) \\
&= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\tilde{\mathbf{y}}|\mathbf{x}) \Big)
\end{aligned}
$$

mod Λ *AWGN Channel Capacity*



$$\begin{aligned} nC &= \max_{p(\mathbf{x})} I(\mathbf{x};\tilde{\mathbf{y}}) \\ &= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\tilde{\mathbf{y}}|\mathbf{x}) \Big) \\ &= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h\Big([\mathbf{z}] \bmod \Lambda \Big) \Big) \end{aligned}$$

Distributive Law



$$
\begin{aligned}
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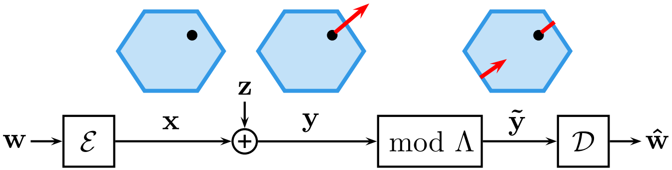
&= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\tilde{\mathbf{y}}|\mathbf{x}) \Big) \\
&= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h\Big([\mathbf{z}] \bmod \Lambda \Big) \Big) && \text{Distributive Law} \\
&\geq \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\mathbf{z}) \Big) && \text{Point Symmetry of Voronoi Region}
\end{aligned}
$$

mod Λ *AWGN Channel Capacity*



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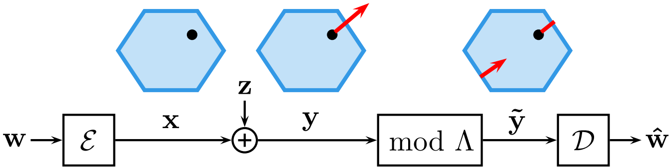
&= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h\Big([\mathbf{z}] \bmod \Lambda \Big) \Big) && \text{Distributive Law} \\
&\geq \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\mathbf{z}) \Big) && \text{Point Symmetry of Voronoi Region} \\
&= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - \frac{n}{2}\log(2\pi e N) \Big) && \text{Entropy of Gaussian Noise}
\end{aligned}
$$

mod Λ *AWGN Channel Capacity*



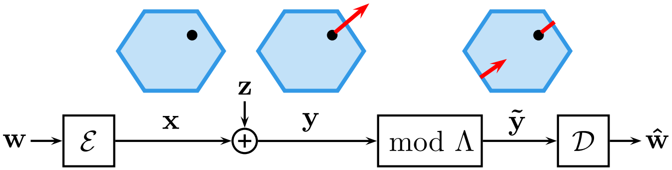
- Channel output entropy upper bounded by the logarithm of the Voronoi region volume:

$$h(\tilde{\mathbf{y}}) \leq \log(\mathsf{Vol}(\mathcal{V})) \qquad \text{with equality if } \tilde{\mathbf{y}} \sim \mathsf{Unif}(\mathcal{V})$$

- $\tilde{\mathbf{y}} = [\mathbf{x} + \mathbf{z}] \bmod \Lambda$ is uniform over $\mathcal{V}$ if $\mathbf{x}$ is uniform over $\mathcal{V}$.
- Random coding over the Voronoi region $\mathcal{V}$ can achieve:

$$C = \frac{1}{n}\log(\mathsf{Vol}(\mathcal{V})) - \frac{1}{2}\log(2\pi e N)$$

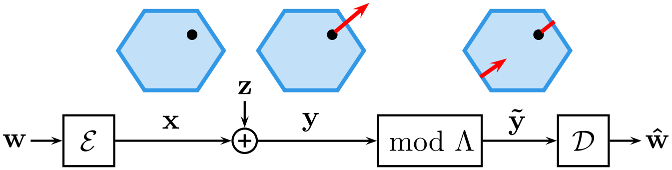
Power Constraints and Second Moments



- Must scale lattice Λ so that the uniform distribution over the Voronoi region $\mathcal{V}$ meets the power constraint P.
- Set second moment $\sigma_\Lambda^2 = \dfrac{1}{n\mathsf{Vol}(\mathcal{V})} \displaystyle\int_{\mathcal{V}} \|\mathbf{x}\|^2 d\mathbf{x}$ equal to P.

Normalized Second Moment: $\quad G(\Lambda) = \dfrac{\sigma_\Lambda^2}{(\mathsf{Vol}(\mathcal{V}))^{2/n}}$

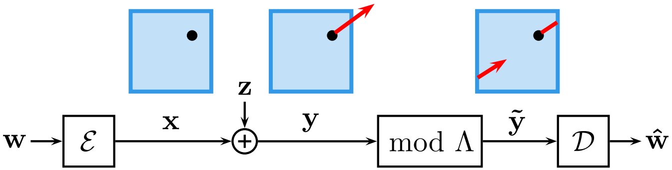
$$\implies \frac{1}{n}\log(\mathsf{Vol}(\mathcal{V})) = \frac{1}{2}\log\left(\frac{\sigma_\Lambda^2}{G(\Lambda)}\right) = \frac{1}{2}\log\left(\frac{P}{G(\Lambda)}\right)$$



- Usual i.i.d. random coding over $\mathcal{V}$ combined with the union bound:

$$
\begin{aligned}
C &\geq \frac{1}{n}\log(\mathsf{Vol}(\mathcal{V})) - \frac{1}{2}\log(2\pi e N) \\
&= \frac{1}{2}\log\left(\frac{P}{G(\Lambda)}\right) - \frac{1}{2}\log(2\pi e N) \\
&= \frac{1}{2}\log\left(\frac{P}{N}\right) - \frac{1}{2}\log(2\pi e G(\Lambda))
\end{aligned}
$$

What is $G(\Lambda)$?



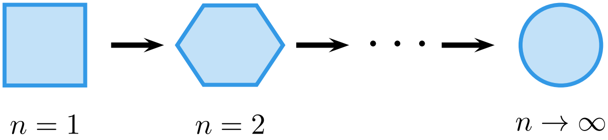
- The normalized second moment $G(\Lambda)$ is a dimensionless quantity that captures the shaping gain.
- Integer lattice is not so bad, $G(\mathbb{Z}^n) = 1/12$.
- Capacity under $\bmod \mathbb{Z}^n$ is at least

$$\begin{aligned} C &\geq \frac{1}{2}\log\left(\frac{P}{N}\right) - \frac{1}{2}\log\left(\frac{2\pi e}{12}\right) \\ &\approx \frac{1}{2}\log\left(\frac{P}{N}\right) - 0.255 \end{aligned}$$

Asymptotically Good $G(\Lambda)$

Theorem (Zamir-Feder-Poltyrev '94)

There exists a sequence of lattices $\Lambda^{(n)}$ *such that* $\lim_{n\to\infty} G(\Lambda^{(n)}) = \frac{1}{2\pi e}$.

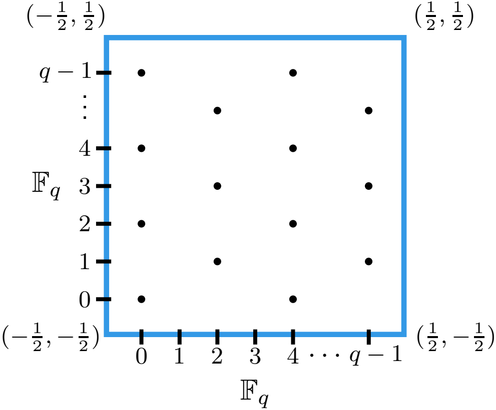


- Best possible normalized second moment is that of a sphere.
- Using a sequence $\Lambda^{(n)}$ with an asymptotically good $G(\Lambda^{(N)})$ allows to approach

$$\begin{aligned} R &= \frac{1}{2}\log\left(\frac{P}{N}\right) - \frac{1}{2}\log\left(\frac{2\pi e}{2\pi e}\right) \\ &= \frac{1}{2}\log\left(\frac{P}{N}\right) \end{aligned}$$

Linear Codes for $\bmod \Lambda$ *Channels*

- Instead of an "inner" random codes, we can use a q-ary linear code.
- This is exactly a nested lattice.
- Each codeword has a uniform marginal distribution over the grid.
- Rate loss due to finite constellation which goes to 0 as $q \to \infty$.
- Codewords are pairwise independent so we can apply the union bound.



MMSE Scaling

- **Erez-Zamir '04:** Prior to taking $\bmod \Lambda$, scale by α.

$$\begin{aligned}\tilde{\mathbf{y}} &= [\alpha \mathbf{y}] \bmod \Lambda \\ &= [\alpha \mathbf{x} + \alpha \mathbf{z}] \bmod \Lambda \\ &= [\mathbf{x} + \underbrace{\alpha \mathbf{z} - (1-\alpha)\mathbf{x}}_{\text{Effective Noise}}] \bmod \Lambda\end{aligned}$$

- For now, ignore that the effective noise is not independent of the codeword. Effective noise variance $N_{\mathsf{EFFEC}} = \alpha^2 N + (1-\alpha)^2 P$.
- Optimal choice of α is the MMSE coefficient $\alpha_{\mathsf{MMSE}} = \dfrac{P}{N+P}$.

$$N_{\mathsf{EFFEC}} = \alpha_{\mathsf{MMSE}}^2 N + (1-\alpha_{\mathsf{MMSE}})^2 P = \frac{PN}{N+P}$$

$$C = \frac{1}{2}\log\left(\frac{P}{N_{\mathsf{EFFEC}}}\right) = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

Dithering

- Now the noise is dependent on the codeword.
- Dithering can solve this problem (just as in the discrete case).
- Map message to a codeword $\mathbf{t}$.
- Generate a random dither vector $\mathbf{d}$ uniformly over $\mathcal{V}$.
- Transmitter sends a dithered codeword:

$$\mathbf{x} = [\mathbf{t} + \mathbf{d}] \bmod \Lambda$$

- $\mathbf{x}$ is now independent of the codeword $\mathbf{t}$.

Decoding – Remove Dither First

- Transmitter sends dithered codeword $\mathbf{x} = [\mathbf{t} + \mathbf{d}] \bmod \Lambda$.
- After scaling the channel output $\mathbf{y}$ by α, the decoder subtracts the dither $\mathbf{d}$.

$$\begin{aligned}
\tilde{\mathbf{y}} &= [\alpha\mathbf{y} - \mathbf{d}] \bmod \Lambda \\
&= [\alpha\mathbf{x} + \alpha\mathbf{z} - \mathbf{d}] \bmod \Lambda \\
&= [\mathbf{x} - \mathbf{d} + \alpha\mathbf{z} - (1-\alpha)\mathbf{x}] \bmod \Lambda \\
&= \Big[[\mathbf{t} + \mathbf{d}] \bmod \Lambda - \mathbf{d} + \alpha\mathbf{z} - (1-\alpha)\mathbf{x}\Big] \bmod \Lambda \\
&= [\mathbf{t} + \alpha\mathbf{z} - (1-\alpha)\mathbf{x}] \bmod \Lambda \qquad \text{Distributive Law}
\end{aligned}$$

- Effective noise is now independent from the codeword $\mathbf{t}$.
- By the probabilistic method, (at least) one good fixed dither exists. No common randomness necessary.

Summary

- Linear code embedded in the integer lattice:

$$R = \frac{1}{2}\log\left(\frac{P}{N}\right) - \frac{1}{2}\log\left(\frac{2\pi e}{12}\right)$$

- Linear code embedded in the integer lattice, MMSE scaling:

$$R = \frac{1}{2}\log\left(1+\frac{P}{N}\right) - \frac{1}{2}\log\left(\frac{2\pi e}{12}\right)$$

- Linear code embedded in a good shaping lattice, MMSE scaling:

$$R = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

Theorem (Erez-Zamir '04)

Nested lattice codes can achieve the AWGN capacity.

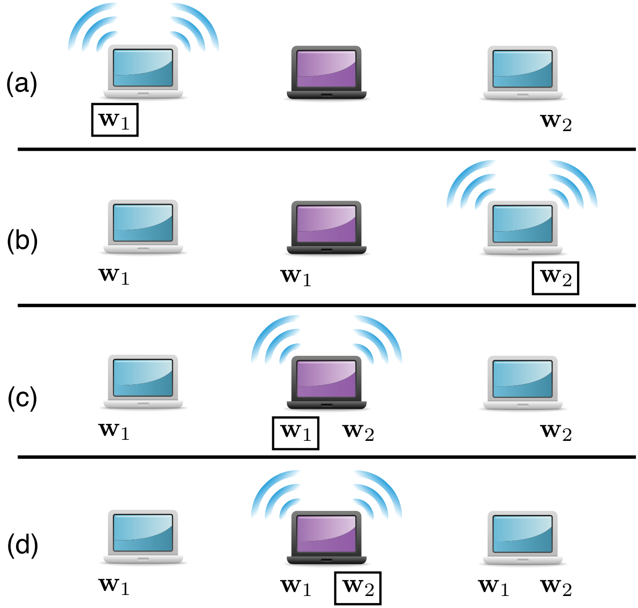
Two-Way Relay Channel

Has $\mathbf{w}_1$
Wants $\mathbf{w}_2$

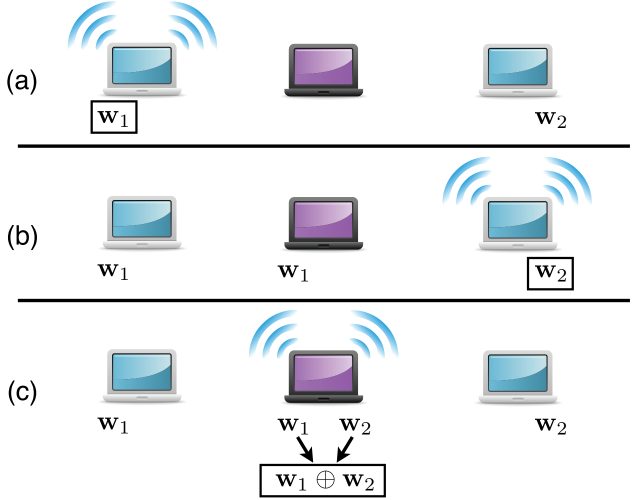
Relay

Has $\mathbf{w}_2$
Wants $\mathbf{w}_1$

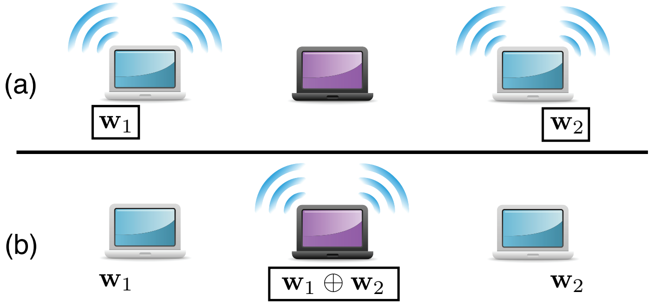
Two-Way Relay Channel – Time-Division



Two-Way Relay Channel – Network Coding



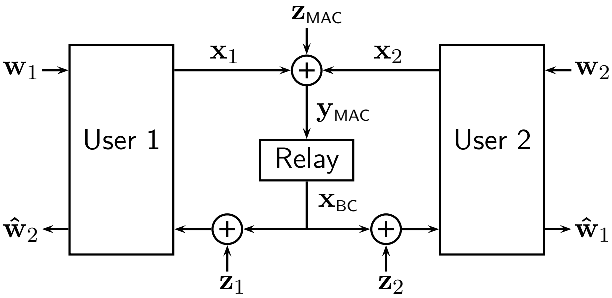
Two-Way Relay Channel – Physical-Layer Network Coding



AWGN Two-Way Relay Channel – Symmetric Rates

- Upper Bound:
  $R \leq \frac{1}{2} \log\left(1 + \frac{P}{N}\right)$
- Decode-and-Forward: Relay decodes $\mathbf{w}_1, \mathbf{w}_2$ and transmits $\mathbf{w}_1 \oplus \mathbf{w}_2$.
  $R = \frac{1}{4} \log\left(1 + \frac{2P}{N}\right)$
- Compress-and-Forward: Relay transmits quantized $\mathbf{y}$.
  $R = \frac{1}{2} \log\left(1 + \frac{P}{N} \frac{P}{3P + N}\right)$

AWGN Two-Way Relay Channel – Symmetric Rates



- Equal power constraints P.
- Equal noise variances N.
- Equal rates R.

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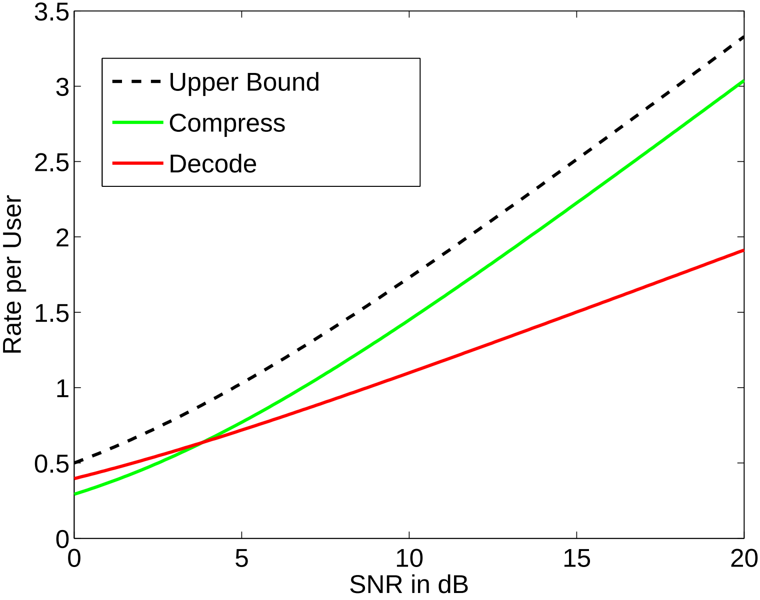
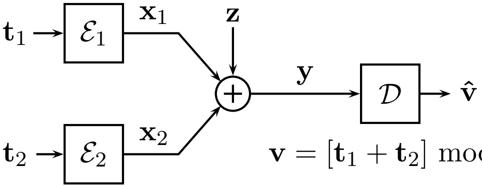
AWGN Two-Way Relay Channel – Symmetric Rates



Decoding the Sum of Lattice Codewords

Encoders use the same nested lattice codebook.



Transmit dithered codewords:

$$\mathbf{x}_1 = \mathbf{t}_1$$
$$\mathbf{x}_2 = \mathbf{t}_2$$

Decoder recovers modulo sum.

$$\begin{aligned}
&[\mathbf{y}] \bmod \Lambda \\
&= [\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}] \bmod \Lambda \\
&= [\mathbf{t}_1 + \mathbf{t}_2 + \mathbf{z}] \bmod \Lambda \\
&= \Big[[\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda + \mathbf{z}\Big] \bmod \Lambda \qquad \text{Distributive Law} \\
&= [\mathbf{v} + \mathbf{z}] \bmod \Lambda
\end{aligned}$$

$$R = \frac{1}{2} \log\left(\frac{P}{N}\right)$$

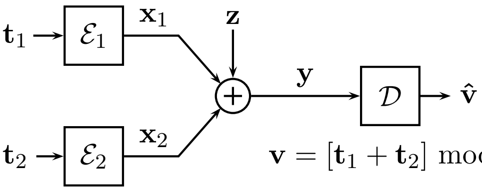
Decoding the Sum of Lattice Codewords – MMSE Scaling

Encoders use the same nested lattice codebook.

Transmit dithered codewords:

$$\mathbf{x}_1 = [\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda$$

$$\mathbf{x}_2 = [\mathbf{t}_2 + \mathbf{d}_2] \bmod \Lambda$$



$$\mathbf{v} = [\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda$$

Decoder scales by α, removes dithers, recovers modulo sum.

$$
\begin{aligned}
&[\alpha\mathbf{y} - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= [\alpha(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}) - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= [\mathbf{x}_1 + \mathbf{x}_2 - (1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{z} - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= \Big[[\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda - (1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{z}\Big] \bmod \Lambda \\
&= [\mathbf{v} - \underbrace{(1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{z}}_{\text{Effective Noise}}] \bmod \Lambda
\end{aligned}
$$

$$N_{\mathsf{EFFEC}} = (1-\alpha)^2 2P + \alpha^2 N$$

- Effective noise after scaling is $N_{\mathsf{EFFEC}} = (1-\alpha)^2 2P + \alpha^2 N$.
- Minimized by setting α to be the MMSE coefficient:

$$\alpha_{\mathsf{MMSE}} = \frac{2P}{N+2P}$$

- Plugging in, get

$$N_{\mathsf{EFFEC}} = \frac{2NP}{N+2P}$$

- Resulting rate is

$$R = \frac{1}{2}\log\left(\frac{P}{N_{\mathsf{EFFEC}}}\right) = \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{N}\right)$$

- Getting the full "one plus" term is an open challenge. Does not seem possible with nested lattices.

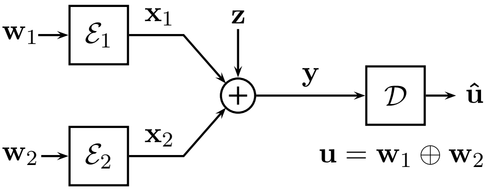
Finite Field Computation over a Gaussian MAC

Map messages to lattice points:

$$\mathbf{t}_1 = \phi(\mathbf{w}_1)$$
$$\mathbf{t}_2 = \phi(\mathbf{w}_2)$$

Transmit dithered codewords:

$$\mathbf{x}_1 = [\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda$$
$$\mathbf{x}_2 = [\mathbf{t}_2 + \mathbf{d}_2] \bmod \Lambda$$



- Integer coarse lattice $\Lambda = \mathbb{Z}^n$, $\phi(\mathbf{w}) = [\gamma \mathbf{G} \mathbf{w}] \bmod \mathbb{Z}^n$ where γ is a scalar and $\mathbf{G}$ is the generator matrix for the q-ary code.

- General coarse lattice $\Lambda = \mathbf{B}\mathbb{Z}^n$, $\phi(\mathbf{w}) = [\mathbf{B}\gamma \mathbf{G} \mathbf{w}] \bmod \Lambda$

- Mapping between finite field messages and lattice codewords preserves linearity:

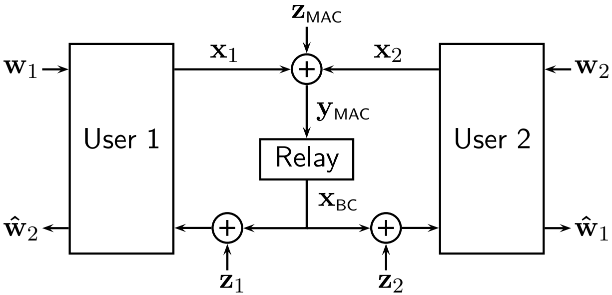
$$\phi^{-1}\Big([\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda\Big) = \mathbf{w}_1 \oplus \mathbf{w}_2$$

AWGN Two-Way Relay Channel – Symmetric Rates

- Equal power constraints P.
- Equal noise variances N.
- Equal rates R.

- Upper Bound:
  $$R \leq \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$
- Compute-and-Forward: Relay decodes $\mathbf{w}_1 \oplus \mathbf{w}_2$ and retransmits.
  $$R = \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{N}\right)$$
- See **Wilson-Narayanan-Pfister-Sprintson '10**.

AWGN Two-Way Relay Channel – Symmetric Rates



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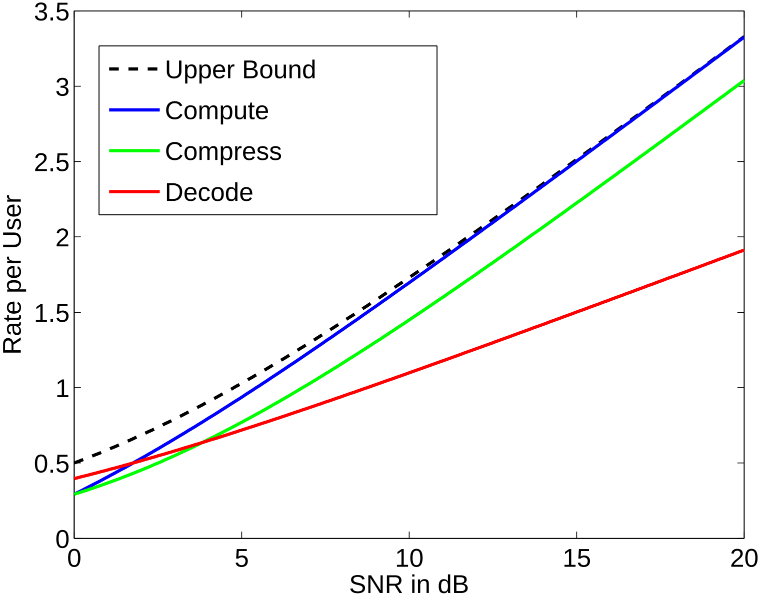
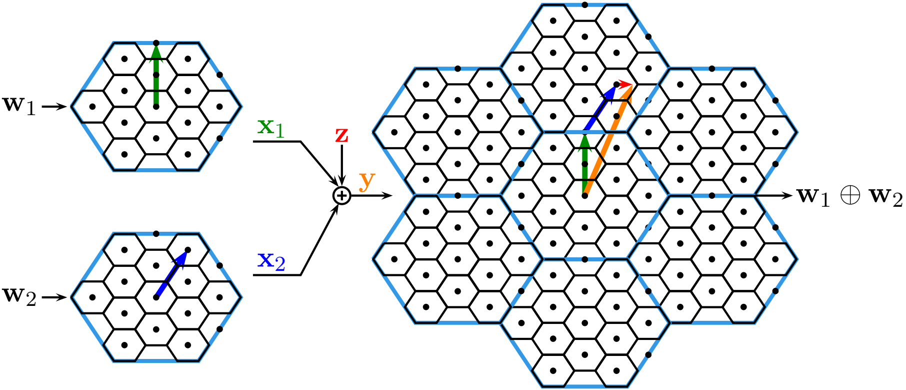
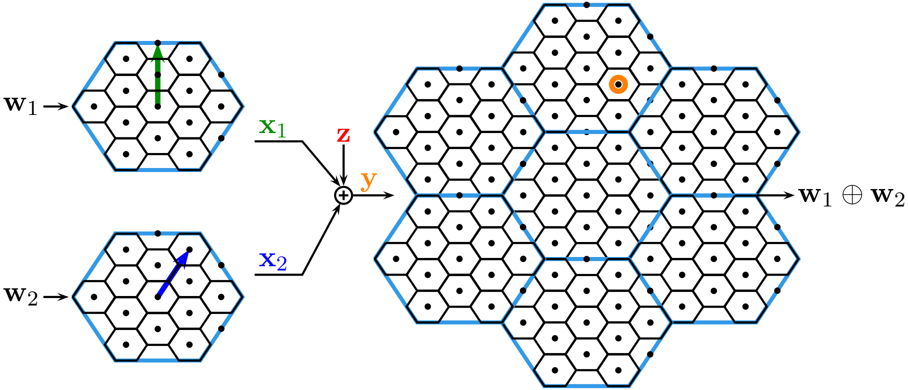
AWGN Two-Way Relay Channel – Symmetric Rates



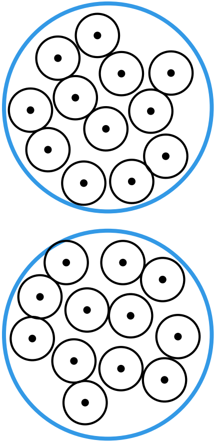
Compute-and-Forward Illustration



Compute-and-Forward Illustration



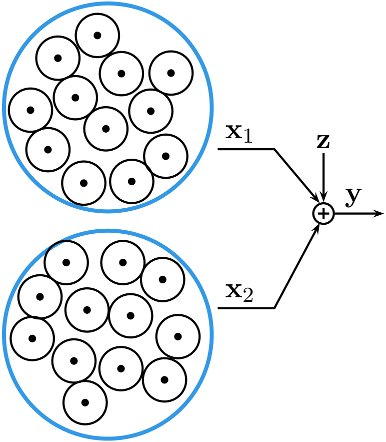
Random i.i.d. codes are not good for computation



2^{nR} codewords each.

2^{n2R} possible sums of codewords.

Random i.i.d. codes are not good for computation



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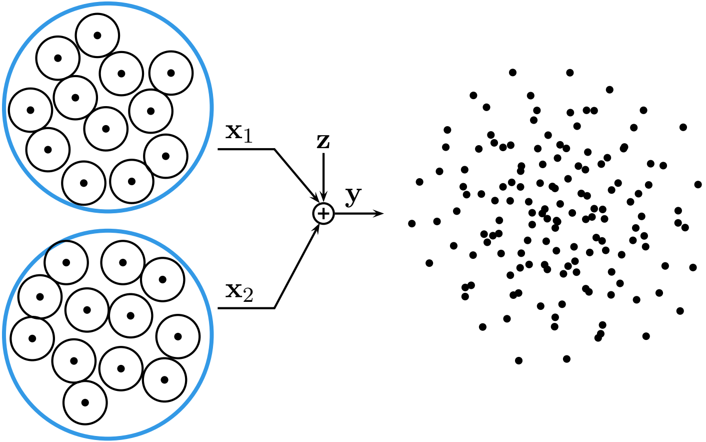

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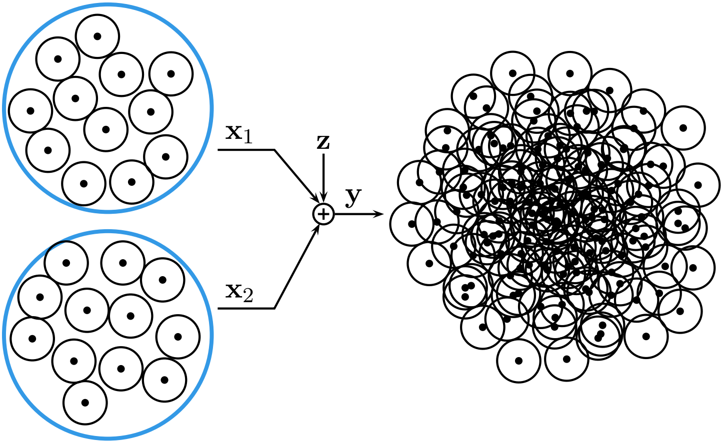
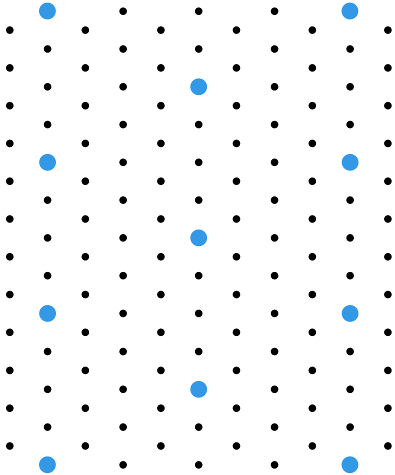


2^{nR} codewords each.

2^{n2R} possible sums of codewords.

Unequal Power Constraints – Double Nesting

- What if the power constraints are not equal?
- Idea from **Nam-Chung-Lee '10**:
- Draw the codewords from the same fine lattice Λ_{FINE}.
- Use two nested coarse lattices Λ_1 and Λ_2 to enforce the power constraints P_1 and P_2.



Unequal Power Constraints – Double Nesting

- What if the power constraints are not equal?
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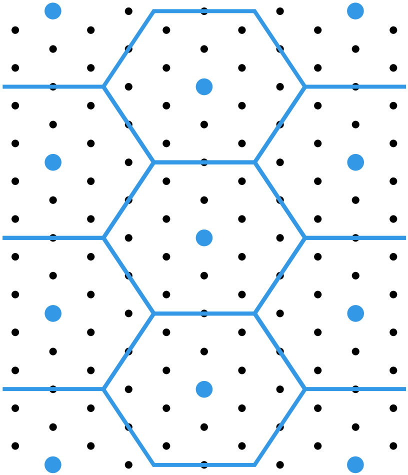

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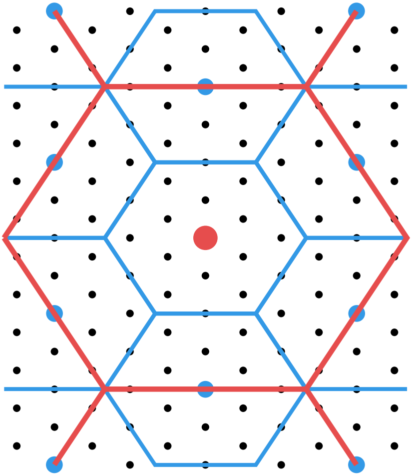

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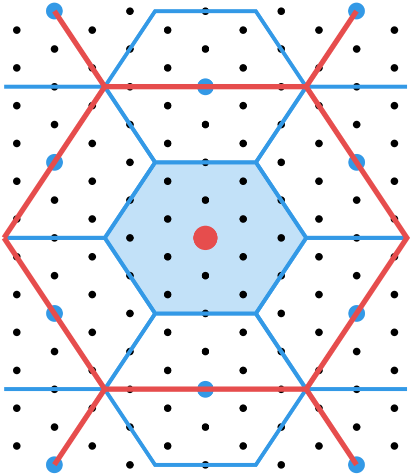

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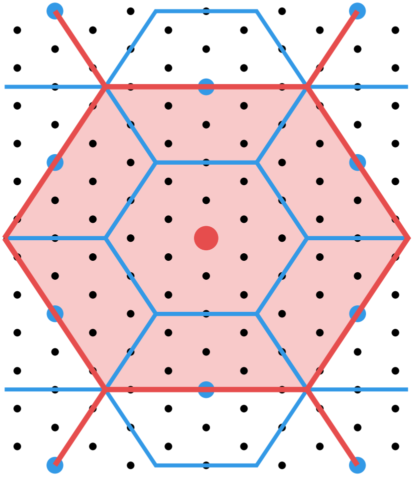

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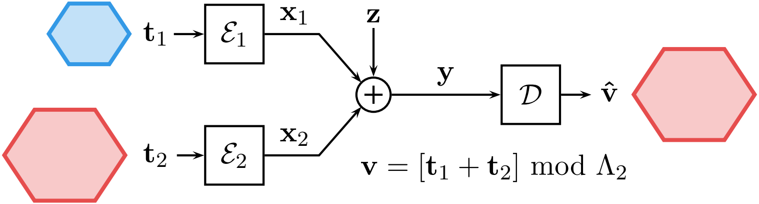
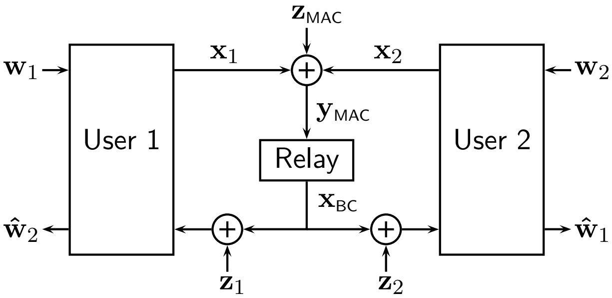


- Encoder 1 sends $\mathbf{x}_1 = [\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda_1$. Coarse lattice Λ_1 has second moment P_1.
- Encoder 2 sends $\mathbf{x}_2 = [\mathbf{t}_2 + \mathbf{d}_2] \bmod \Lambda_2$. Coarse lattice Λ_2 has second moment $P_2 > P_1$.
- Decoder performs MMSE scaling, remove dithers, recovers $\bmod \Lambda_2$ sum.

$$R_1 = \frac{1}{2}\log\left(\frac{P_1}{P_1 + P_2} + \frac{P_1}{N}\right) \qquad R_2 = \frac{1}{2}\log\left(\frac{P_2}{P_1 + P_2} + \frac{P_2}{N}\right)$$

AWGN Two-Way Relay Channel



- User powers P_1, P_2.
- MAC noise variance N_{MAC}.
- Relay power P_{BC}.
- Broadcast noise variances N_1, N_2.

Theorem (Nam-Chung-Lee '10)

Rate region is within $1/2$ *bit of:*

$$R_1 \leq \min\left(\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2} + \frac{P_1}{N_{MAC}}\right),\ \frac{1}{2}\log\left(1 + \frac{P_{BC}}{N_2}\right)\right)$$
$$R_2 \leq \min\left(\frac{1}{2}\log\left(\frac{P_2}{P_1+P_2} + \frac{P_2}{N_{MAC}}\right),\ \frac{1}{2}\log\left(1 + \frac{P_{BC}}{N_1}\right)\right)$$

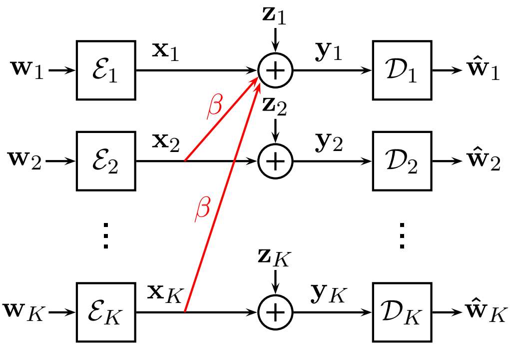
Moreover, "constant gap" goes to zero as powers increase.

Many-to-One Interference Channel – Symmetric Very Strong Case

- Equal rates R.
- Only receiver 1 sees interference:

$$\mathbf{y}_1 = \mathbf{x}_1 + \beta \sum_{\ell=2}^{K} \mathbf{x}_\ell + \mathbf{z}_1$$

- How big does β have to be to achieve $R = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$? (i.e. "very strong" case)



- Scheme A: Decode $\mathbf{w}_2, \ldots, \mathbf{w}_K$ at receiver 1 and remove prior to decoding $\mathbf{w}_1$.

$$R \leq \frac{1}{2(K-1)} \log\left(1 + \frac{\beta^2 (K-1) P}{N+P}\right)$$

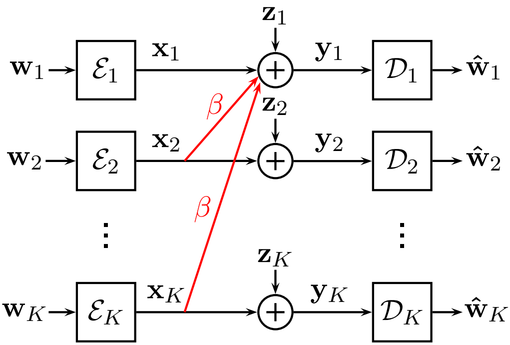
- Scheme B: Decode $\mathbf{w}_2 \oplus \cdots \oplus \mathbf{w}_K$ at receiver 1 and remove prior to decoding $\mathbf{w}_1$.

Many-to-One Interference Channel – Symmetric Very Strong Case

Encoders use the same nested lattice codebook.

Transmit dithered codewords:

$$\mathbf{x}_\ell = [\mathbf{t}_\ell + \mathbf{d}_\ell] \bmod \Lambda$$



Decoder scales by β^{-1}, removes dithers, recovers modulo sum.

$$\left[\beta^{-1}\mathbf{y}_1 - \sum_{\ell=2}^{K} \mathbf{d}_\ell\right] \bmod \Lambda = \left[\sum_{\ell=2}^{K}(\mathbf{x}_\ell - \mathbf{d}_\ell) + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right] \bmod \Lambda$$

$$= \left[\left[\sum_{\ell=2}^{K} \mathbf{t}_\ell\right] \bmod \Lambda + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right] \bmod \Lambda$$

$$\left[\beta^{-1}\mathbf{y}_1 - \sum_{\ell=2}^{K}\mathbf{d}_\ell\right] \bmod \Lambda = \left[\left[\sum_{\ell=2}^{K}\mathbf{t}_\ell\right] \bmod \Lambda + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right] \bmod \Lambda$$

- Effective noise variance $N_{\mathsf{EFFEC}} = \beta^{-2}(P+N)$.
- Can decode $\bmod \Lambda$ sum of lattice points at rate $R = \frac{1}{2}\log\left(\frac{\beta^2 P}{P+N}\right)$.
- Setting equal to "very strong" condition $R = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$ we get

$$\beta^2 = \frac{(P+N)^2}{PN}$$

- How can we recover $\mathbf{w}_1$?
- We need to first subtract the real sum of the codewords. So far, we only have the modulo-sum.

Successive Cancellation of Sums

- First, add back in dithers to get modulo sum of codewords:

$$\left[\left[\sum_{\ell=2}^{K}\mathbf{t}_\ell\right] \bmod \Lambda + \left[\sum_{\ell=2}^{K}\mathbf{d}_\ell\right] \bmod \Lambda\right] \bmod \Lambda = \left[\sum_{\ell=2}^{K}\mathbf{x}_\ell\right] \bmod \Lambda$$

- Subtract from $\mathbf{y}_1$ to expose the coarse lattice point nearest to the real sum $\sum_{\ell=2}^{K}\mathbf{x}_\ell$:

$$\beta^{-1}\mathbf{y}_1 - \left[\sum_{\ell=2}^{K}\mathbf{x}_\ell\right] \bmod \Lambda = Q_\Lambda\Big(\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big) + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)$$

- Coarse lattice point easier to decode than fine lattice point:

$$Q_\Lambda\left(Q_\Lambda\Big(\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big) + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right) = Q_\Lambda\Big(\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big) \quad \text{w.h.p.}$$

- Finally, get back the real sum

$$\left[\sum_{\ell=2}^{K}\mathbf{x}_\ell\right] \bmod \Lambda + Q_\Lambda\Big(\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big) = \sum_{\ell=2}^{K}\mathbf{x}_\ell$$

Successive Cancellation of Sums

- We now have the sum of interfering codewords and can cancel its effects:

$$\mathbf{y}_1 - \beta \sum_{\ell=2}^{K} \mathbf{x}_\ell = \mathbf{x}_1 + \mathbf{z}_1$$

- Can apply standard MMSE lattice decoding to recover lattice point $\mathbf{t}_1$ and then map back to $\mathbf{w}_1$.
- Overall, structured coding permits

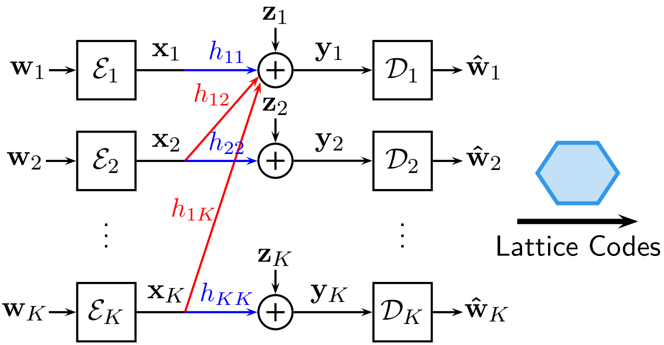
$$\beta^2 \geq \frac{(P+N)^2}{PN}$$

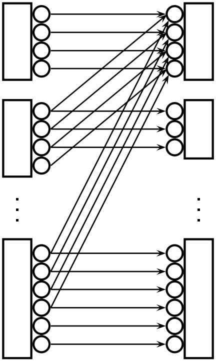
- Compare to decoding interfering codewords in their entirety:

$$\beta^2 \geq \frac{\left((1+\frac{P}{N})^{K-1} - 1\right)(N+P)}{(K-1)P}$$

- Originally shown in **Sridharan-Jafarian-Vishwanath-Jafar '08** using spherical shaping region. Nested lattice scheme is new.

Many-to-One Interference Channel – Approximate Capacity



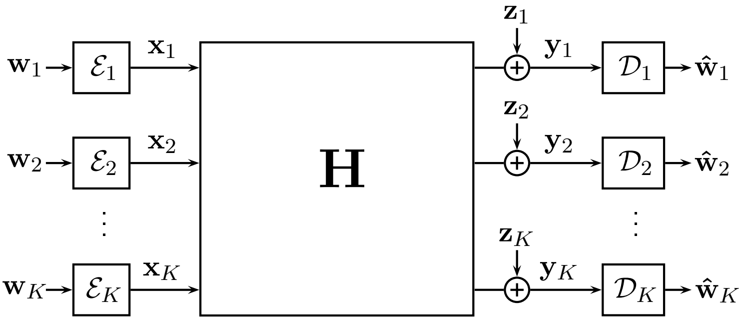


- Deterministic model by **Avestimehr-Diggavi-Tse ’11** shows how to decompose by signal scale.

Theorem (Bresler-Parekh-Tse ’10)

Lattices codes combined with the deterministic model can approach the capacity region to within $(3K+3)(1+\log(K+1))$ *bits per user.*

Interference Channel – Symmetric Very Strong Case



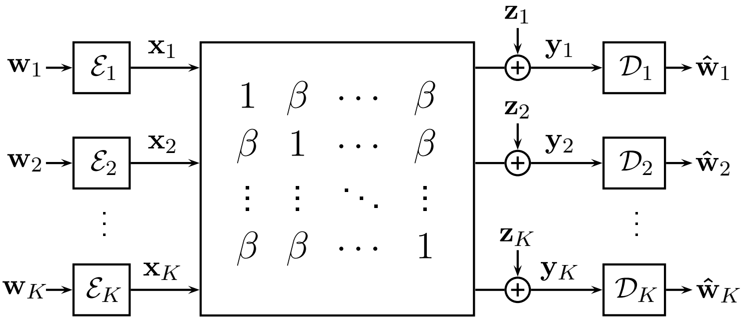
- Equal rates R. How big does β have to be to achieve $R = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$? (i.e. "very strong" case)
- Can use the many-to-one decoder at every receiver to get

$$\beta^2 \geq \frac{(P+N)^2}{PN}$$

Does not depend on K.

- What about asymmetric interference channels?

Interference Channel – Symmetric Very Strong Case



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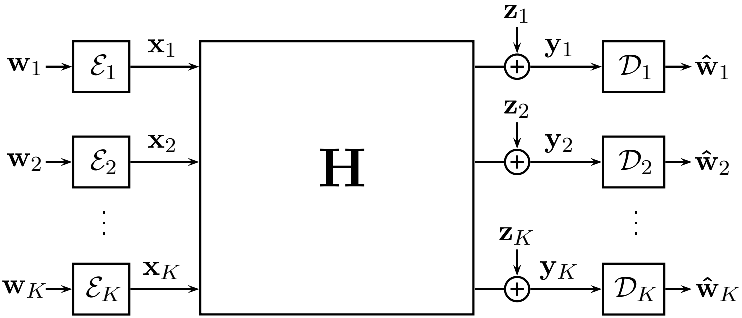

- Not clear how to map to a deterministic model using lattices.
- "Real" interference alignment scheme of **Motahari et al. '08** uses a lattice structure to get $K/2$ DoF (up to a set of measure one)
- Some special cases at finite SNR: **Jafarian-Viswanath '09,'10, Ordentlich-Erez '11**

- Interference alignment can lead to dramatically higher rates for interference channels.
- Many unanswered questions: delay, channel state information, etc.
- Many other applications: secrecy (see work of Ulukus and Yener), distributed storage, etc.

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