Compute-and-Forward:

An Explicit Link between Finite Field and Gaussian Interference Networks

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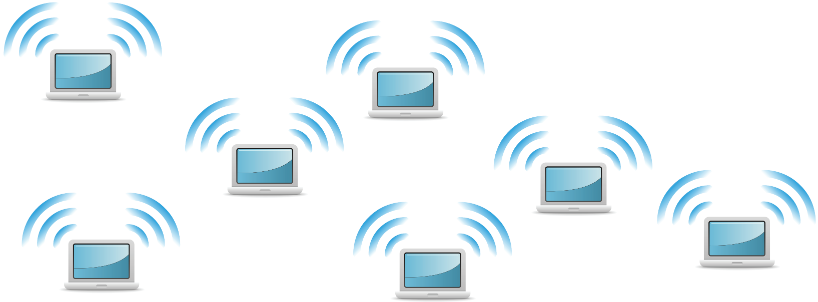
Boston University

European Information Theory School
April 14, 2014

Multi-User Wireless Networks

- Must cope with interference, fading, and noise.

Multi-User Wireless Networks



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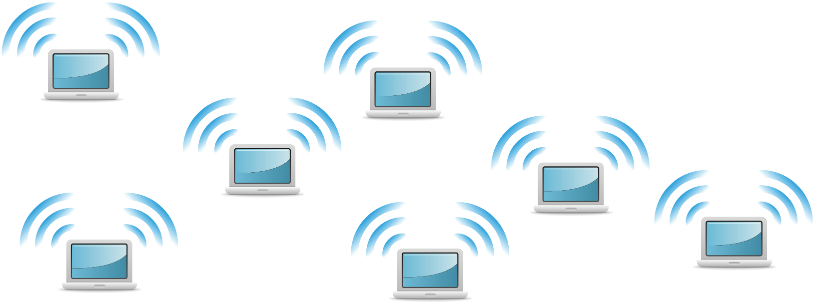

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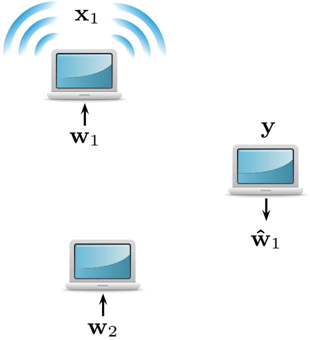
- How should we deal with interference?



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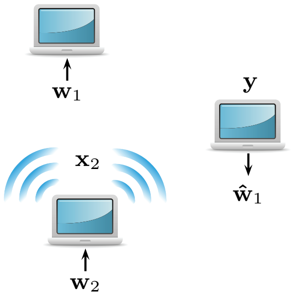

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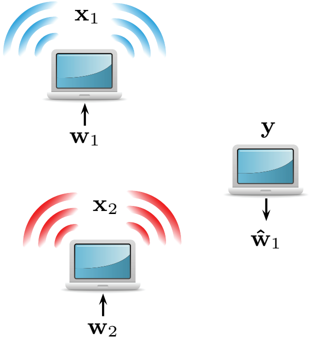

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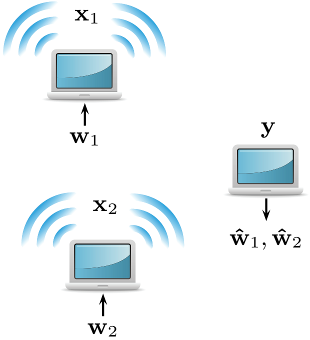

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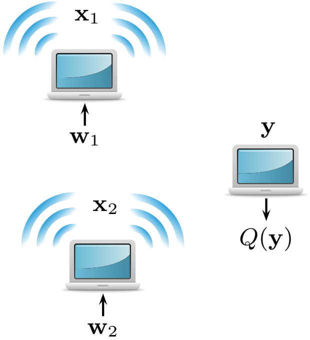

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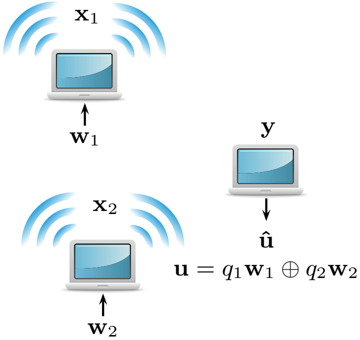

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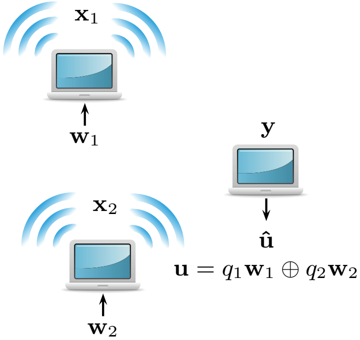

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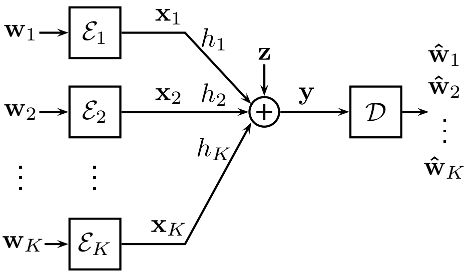

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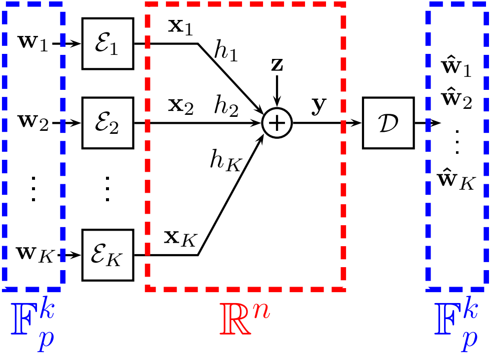

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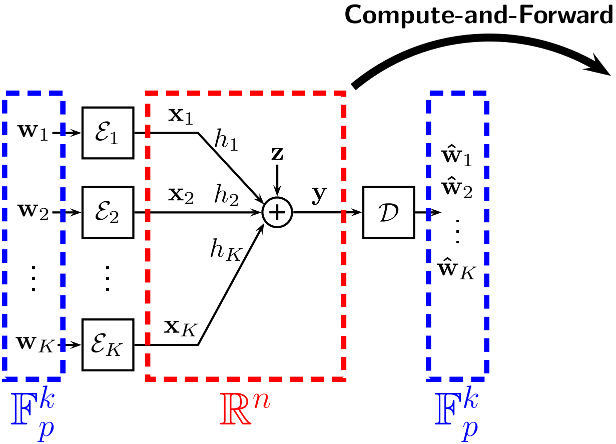

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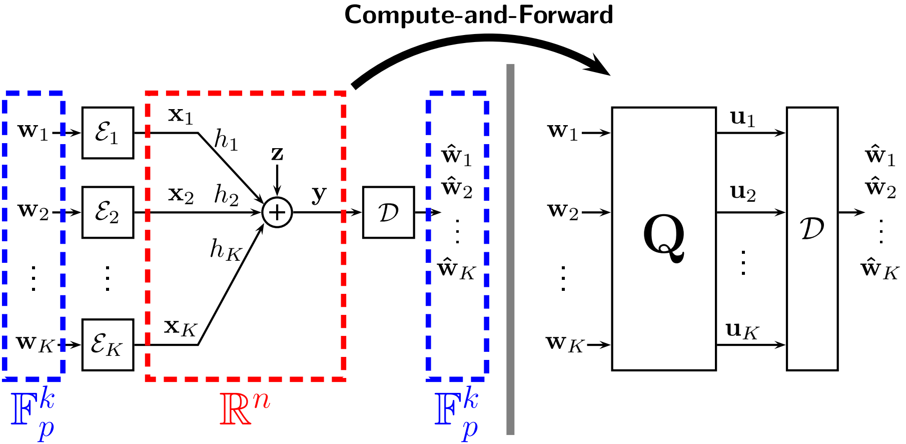

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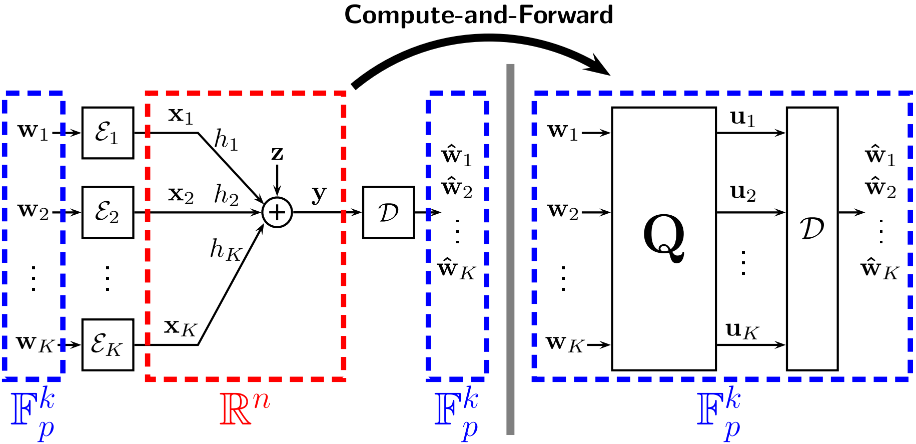

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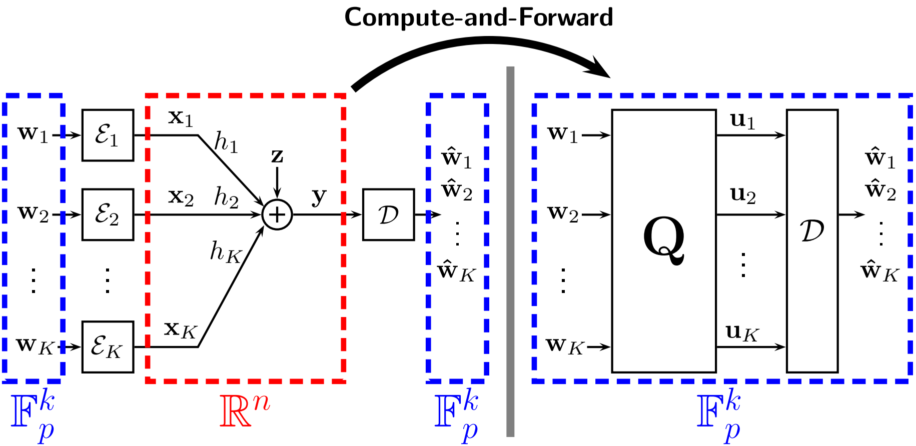

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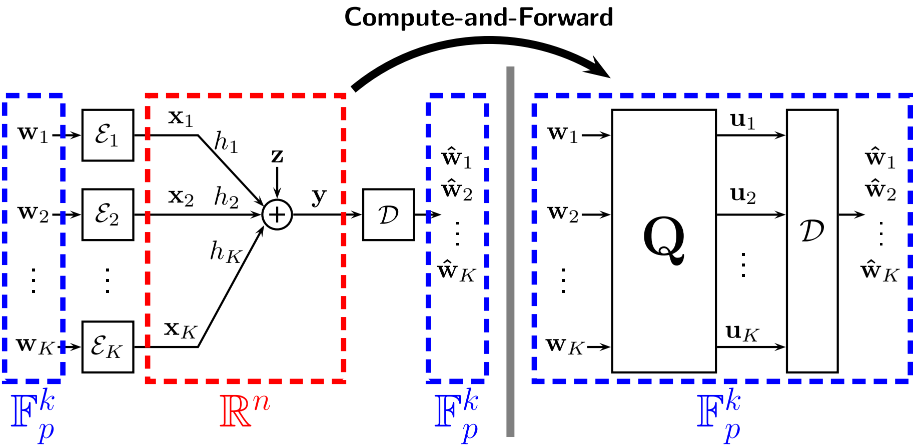

- Which linear combinations can be sent over a given channel?
- Where can this help us?

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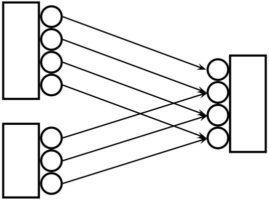

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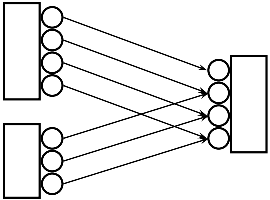

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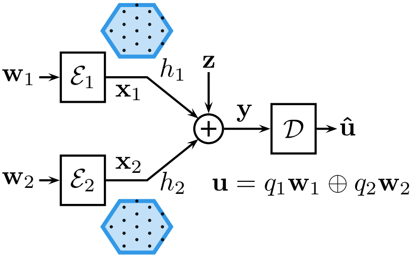

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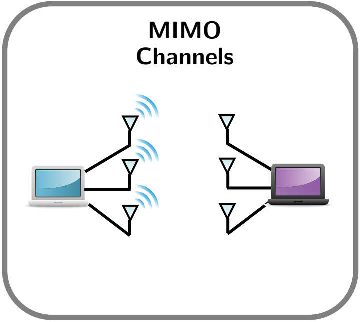

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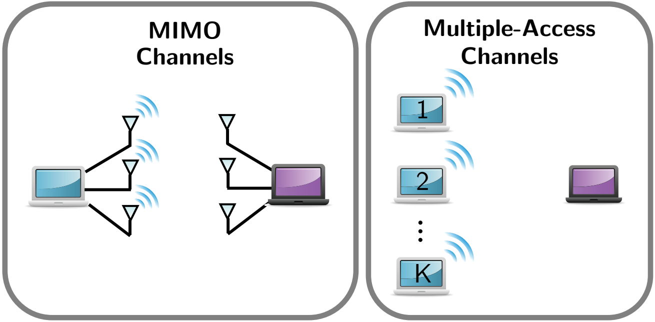

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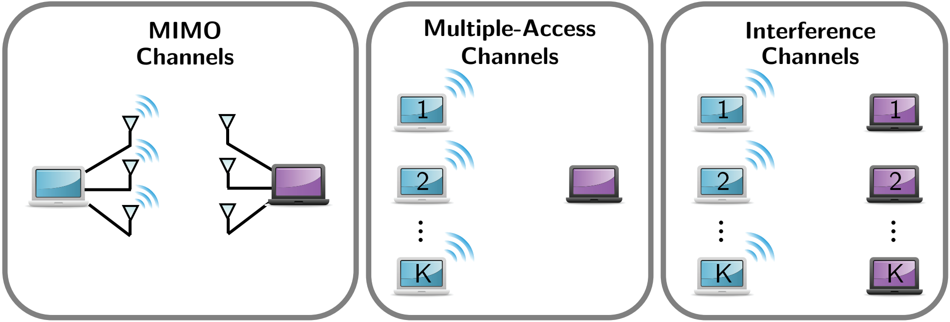

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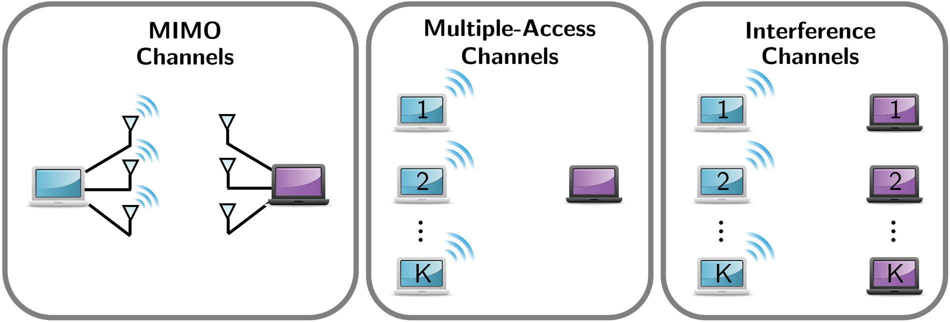

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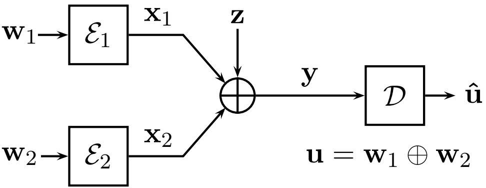

Computation over Finite Field Multiple-Access Channels

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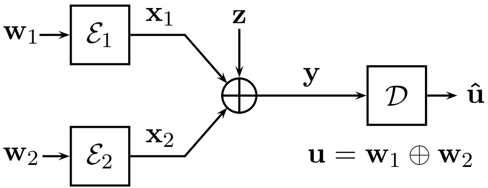

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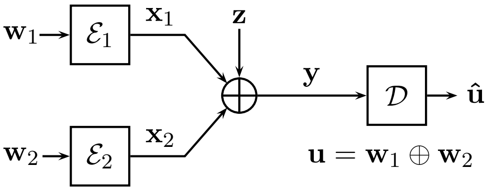

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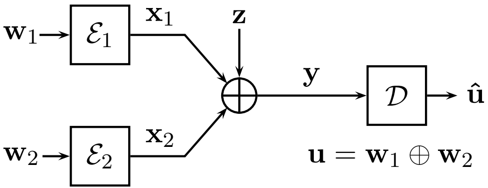

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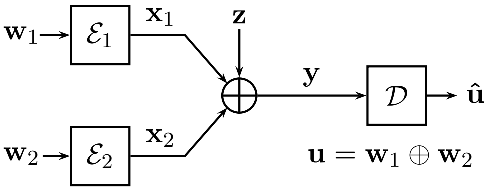

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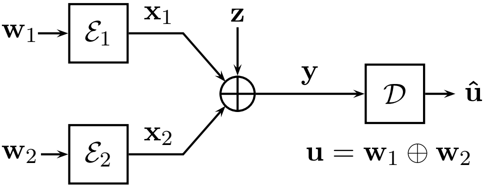

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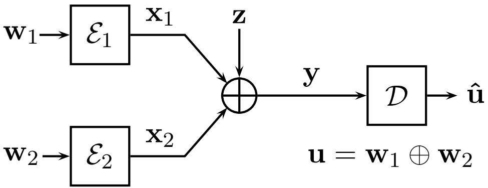

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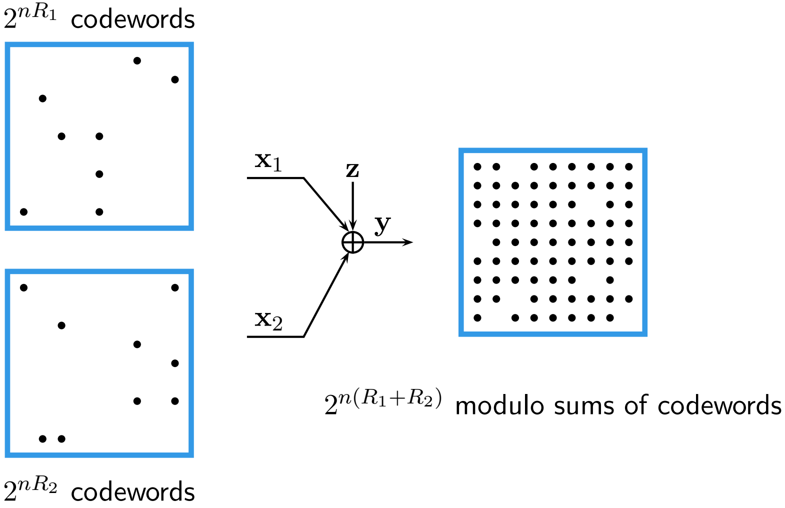
$$\begin{aligned} R_1 + R_2 &\leq I(X_1, X_2; Y) \\ &= H(Y) - H(Y|X_1, X_2) \\ &= \log p - H(Z) \end{aligned}$$

Random i.i.d. codes are not good for computation



- Linear Codebook: A linear map between messages and codewords (instead of a lookup table).

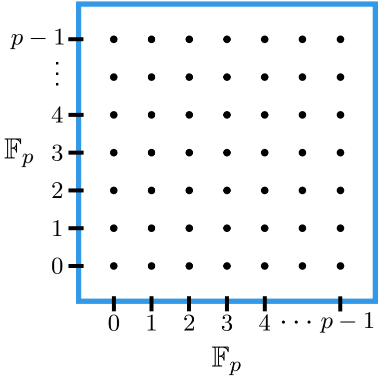
<u>p-**ary Linear Codes**</u>

- Message $\mathbf{w}$ is a length-k vector over $\mathbb{F}_p$.
- Codeword $\mathbf{x}$ is a length-n vector over $\mathbb{F}_p$.
- Encoding process is just a matrix multiplication, $\mathbf{x} = \mathbf{G}\mathbf{w}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1k} \\ g_{21} & g_{22} & \cdots & g_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nk} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}$$

- Recall that, for prime p, operations over $\mathbb{F}_p$ are just $\mod p$ operations over the reals.
- Rate $R = \dfrac{k}{n} \log p$ (in bits)

- Linear code looks like a regular subsampling of the elements of $\mathbb{F}_p^n$.
- **Random linear code:** Generate each element g_{ij} of the generator matrix $\mathbf{G}$ elementwise i.i.d. according to a uniform distribution over $\mathbb{F}_p$.
- How are the codewords distributed?



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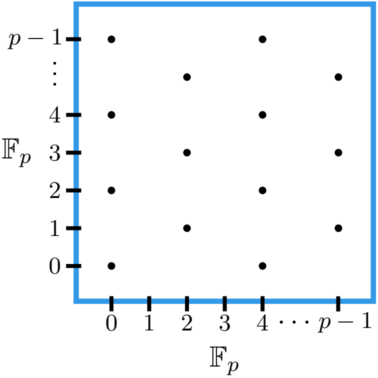

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Shifted Codeword Properties

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2. Pairwise independent. For $\mathbf{w}_1 \neq \mathbf{w}_2$, the associated codewords $\mathbf{x}_1 = \mathbf{G}\mathbf{w}_1 \oplus \mathbf{v}$ and $\mathbf{x}_2 = \mathbf{G}\mathbf{w}_2 \oplus \mathbf{v}$ are independent.

$$\mathbb{P}(\mathbf{x_1} = \mathsf{x}_1, \mathbf{x_2} = \mathsf{x}_2) = \frac{1}{p^{2n}} = \mathbb{P}(\mathbf{x}_1 = \mathsf{x}_1)\mathbb{P}(\mathbf{x}_2 = \mathsf{x}_2)$$

Achievable Rates

- Transmitter sends: $\mathbf{x} = \mathbf{G}\mathbf{w} \oplus \mathbf{v}$.
- Point-to-point channel: $\mathbf{y} = \mathbf{x} \oplus \mathbf{z}$. Noise is i.i.d.
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- Shift $\mathbf{v}$ is unnecessary for additive noise channels.

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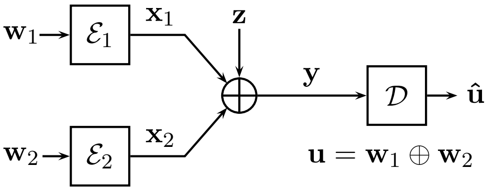
- Want the sum $\mathbf{u} = \mathbf{w}_1 \oplus \mathbf{w}_2$ with vanishing prob. of error $\mathbb{P}(\hat{\mathbf{u}} \neq \mathbf{u}) \to 0$



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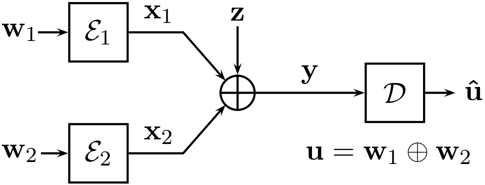

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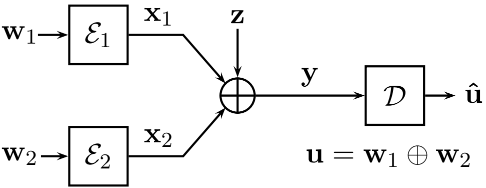

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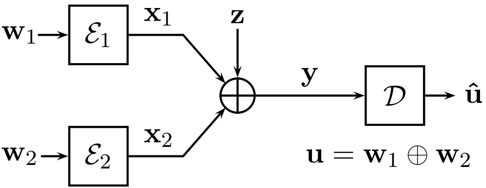

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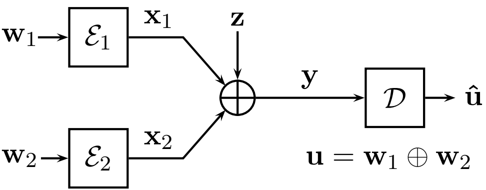

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Computation over Finite Field Multiple-Access Channels

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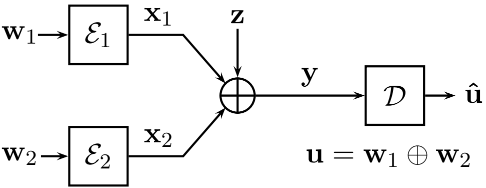

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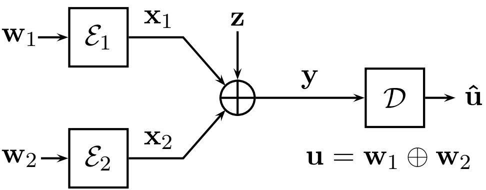

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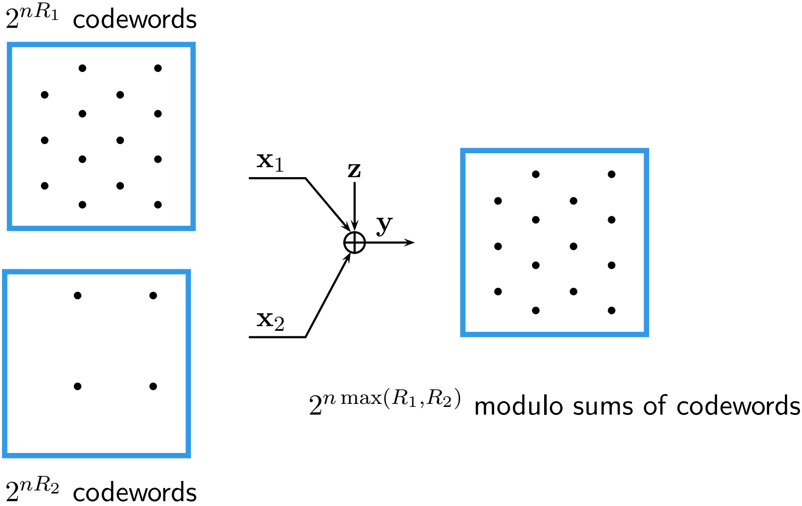
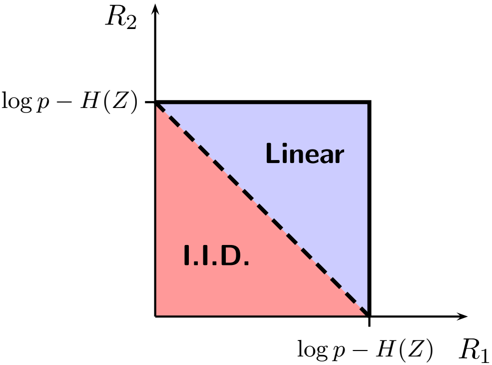
- **Nazer-Gastpar '07:** Decoding succeeds w.h.p. if

$$\begin{aligned}\max(R_1, R_2) \leq I(X_1 \oplus X_2; Y) &= H(Y) - H(Y|X_1 \oplus X_2) \\ &= \log p - H(Z)\ .\end{aligned}$$

Random linear codes are good for computation

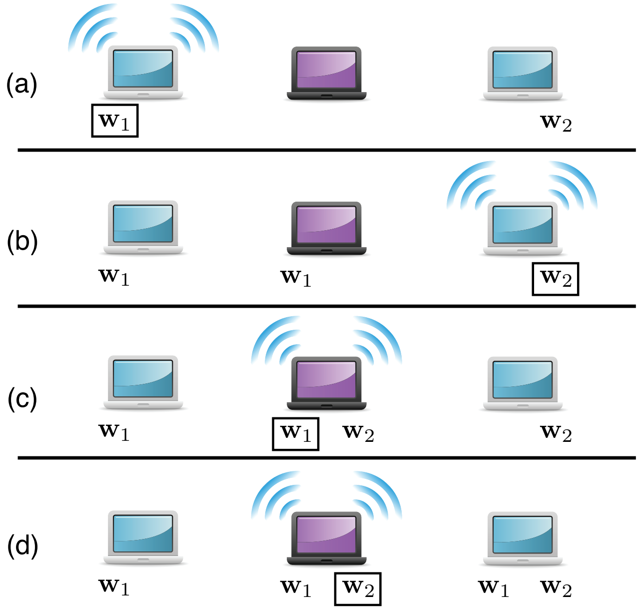




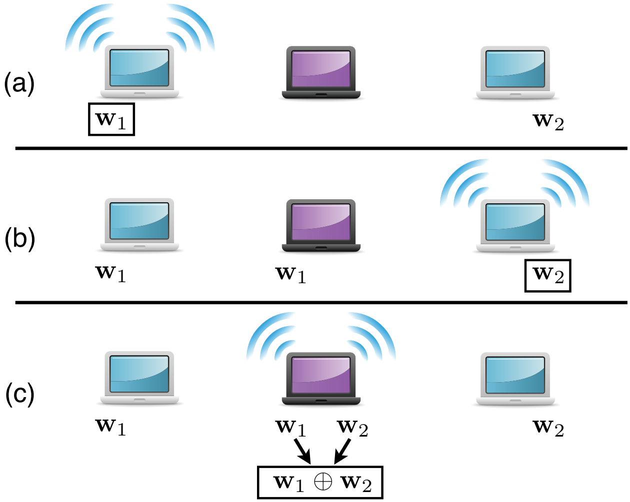
- **I.I.D. Random Coding**: $R_1 + R_2 \leq \log p - H(Z)$
- **Random Linear Coding**: $\max(R_1, R_2) \leq \log p - H(Z)$
- Linear codes *double the sum rate.*
- Are they also useful for *sending messages* (rather than functions thereof)?

- Elegant example proposed by **Wu-Chou-Kung '04.**
- Closely related to butterfly network from **Ahlswede-Cai-Li-Yeung '00.**

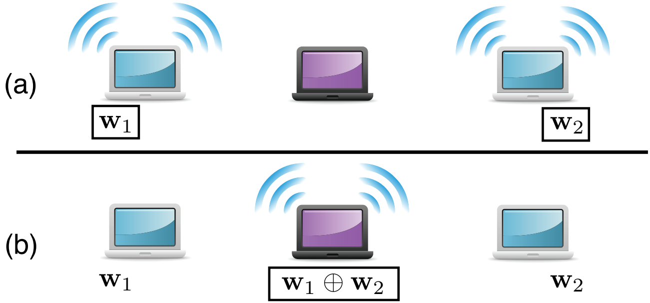
Two-Way Relay Channel – Time-Division

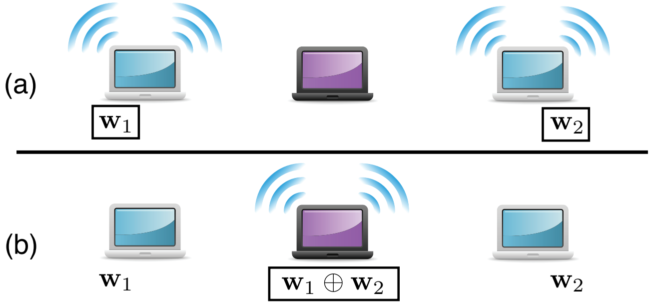


Two-Way Relay Channel – Network Coding



Two-Way Relay Channel – Physical-Layer Network Coding





- Physical-layer network coding: exploiting the wireless medium for network coding. Independently and concurrently proposed by **Zhang-Liew-Lam '06, Popovski-Yomo '06, Nazer-Gastpar '06.**
- Sometimes referred to as Analog Network Coding **Katti-Gollakota-Katabi '07.**
- Some recent surveys **Liew-Zhang-Lu '11, Nazer-Gastpar '11.**

(Finite Field) Two-Way Relay Channel

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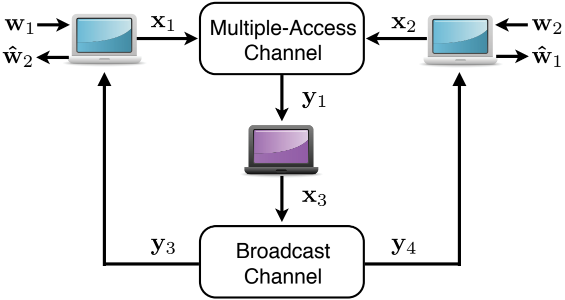

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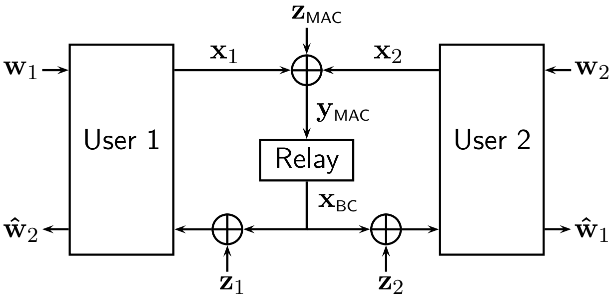


- i.i.d. noise sequences with entropy $H(Z)$.
- Rates R_1 and R_2.

- Cut-Set Upper Bound:

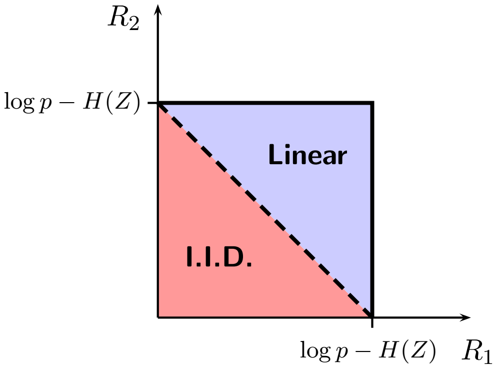
$$\max(R_1, R_2) \leq \log p - H(Z)$$

- I.I.D. Random Coding: Relay decodes $\mathbf{w}_1, \mathbf{w}_2$, transmits $\mathbf{w}_1 \oplus \mathbf{w}_2$.

$$R_1 + R_2 \leq \log p - H(Z)$$

- Random Linear Coding: Relay decodes and retransmits $\mathbf{w}_1 \oplus \mathbf{w}_2$.

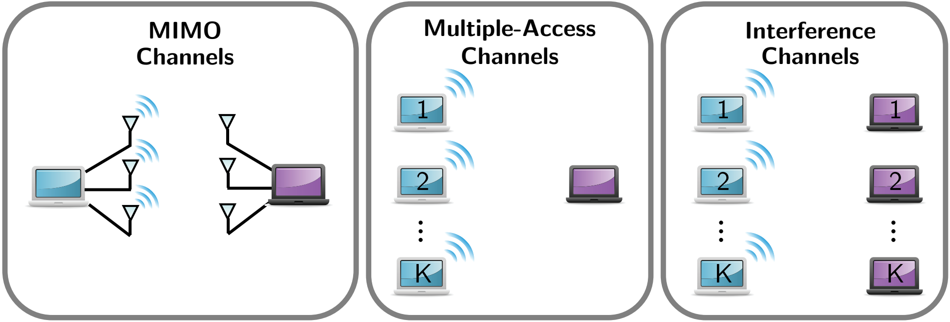
$$\max(R_1, R_2) \leq \log p - H(Z)$$



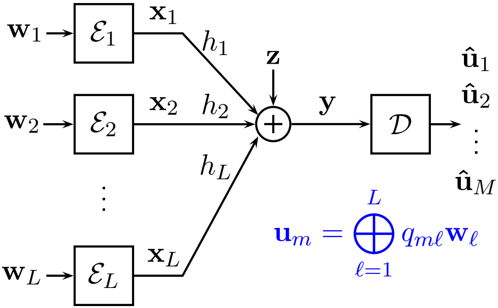
- Linear codes can double the sum rate *for exchanging messages.*

Road Map

- Warm-up: Compute-and-forward over finite field channels.
- Compute-and-forward over Gaussian channels.
- Applications to communication across single-hop Gaussian networks.



Compute-and-Forward: Problem Statement



- Messages are finite field vectors, $\mathbf{w}_\ell \in \mathbb{F}_p^k$.
- Real-valued inputs and outputs, $\mathbf{x}_\ell, \mathbf{y} \in \mathbb{R}^n$.
- Power constraint, $\frac{1}{n}\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq P$.
- Gaussian noise, $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.
- Equal rates: $R = \dfrac{k}{n}\log_2 p$

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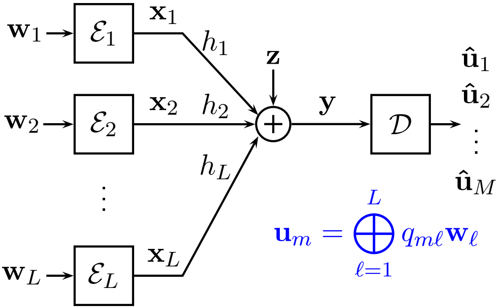

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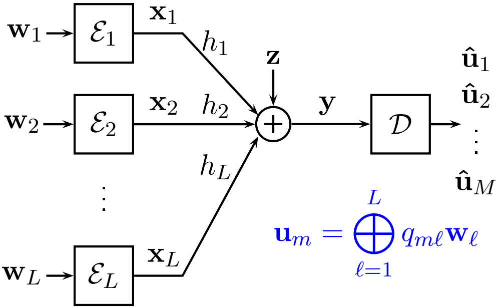

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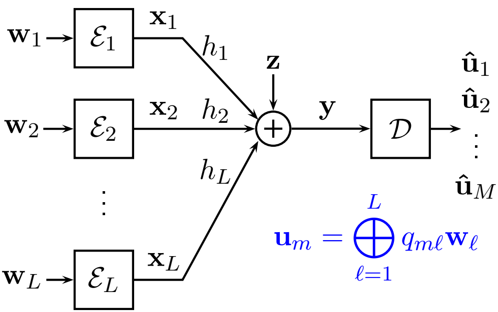

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- What rates are achievable as a function of h_ℓ and $q_{m\ell}$?

Computation Rate

- Want to characterize achievable rates as a function of h_ℓ and $q_{m\ell}$.

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- The linear combination with integer coefficient vector $\mathbf{a}_m = [a_{m1}\ a_{m2}\ \cdots\ a_{mL}]^\mathsf{T} \in \mathbb{Z}^L$ corresponds to

$$\mathbf{u}_m = \bigoplus_{\ell=1}^{L} q_{m\ell}\mathbf{w}_\ell \qquad \text{where } q_{m\ell} = [a_{m\ell}] \bmod p$$

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  (where we assume an implicit mapping between $\mathbb{F}_p$ and $\mathbb{Z}_p$).
- **Key Definition:** The computation rate region described by $R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a})$ is *achievable* if, for any $\epsilon > 0$ and n, p large enough, a receiver can decode any linear combinations with integer coefficient vectors $\mathbf{a}_1, \dots, \mathbf{a}_M \in \mathbb{Z}^L$ for which the message rate R satisfies

$$R < \min_m R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a}_m)$$

$$
\begin{aligned}
\mathbf{y} &= \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z} \\
&= \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \sum_{\ell=1}^{L} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}
\end{aligned}
$$

Desired Codebook:

- Closed under integer linear combinations $\implies$ lattice codebook.

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- Independent effective noise $\Longrightarrow$ dithering.

Compute-and-Forward: Effective Noise

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$$= \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{L} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}}_{\text{Effective Noise}} \xrightarrow{\text{Decode}} \bigoplus_{\ell=1}^{L} q_\ell \mathbf{w}_\ell$$

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- Closed under integer linear combinations $\Longrightarrow$ lattice codebook.
- Independent effective noise $\Longrightarrow$ dithering.
- Isomorphic to $\mathbb{F}_p^k$ $\Longrightarrow$ nested lattice codebook.

Lattices

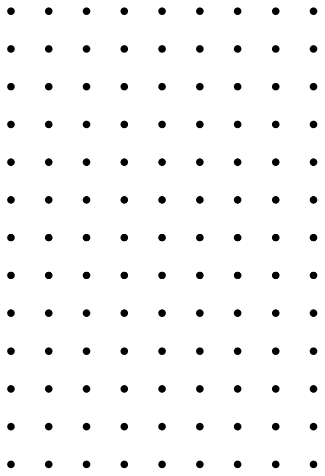
- A lattice Λ is a discrete subgroup of $\mathbb{R}^n$.
- Can express as a linear transformation of the integer vectors,

$$\Lambda = \mathbf{B}\mathbb{Z}^n ,$$

for some (non-unique) $\mathbf{B} \in \mathbb{R}^{n \times n}$.

Lattice Properties

- Closed under addition: $\lambda_1, \lambda_2 \in \Lambda \implies \lambda_1 + \lambda_2 \in \Lambda$.
- Symmetric: $\lambda \in \Lambda \implies -\lambda \in \Lambda$



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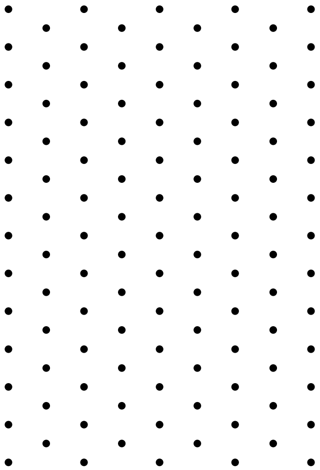

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Voronoi Regions

- Nearest neighbor quantizer:

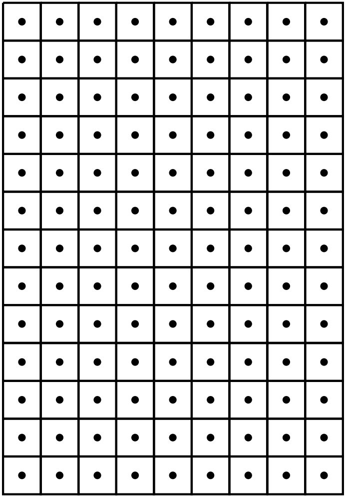
$$Q_\Lambda(\mathbf{x}) = \arg\min_{\lambda \in \Lambda} \|\mathbf{x} - \lambda\|_2$$

- The Voronoi region of a lattice point is the set of all points that quantize to that lattice point.

- Fundamental Voronoi region $\mathcal{V}$: points that quantize to the origin,

$$\mathcal{V} = \left\{\mathbf{x} \in \mathbb{R}^n : Q_\Lambda(\mathbf{x}) = \mathbf{0}\right\}$$

- Each Voronoi region is just a shift of the fundamental Voronoi region $\mathcal{V}$



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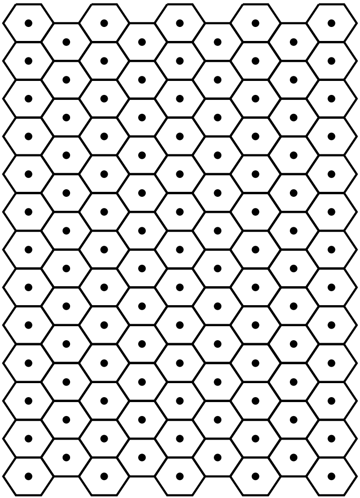
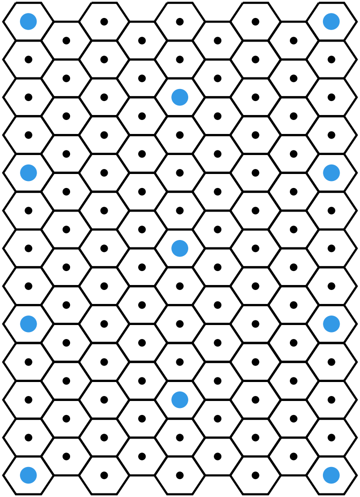


Nested Lattices

- Two lattices Λ_C and Λ_F are nested if $\Lambda_C \subset \Lambda_F$



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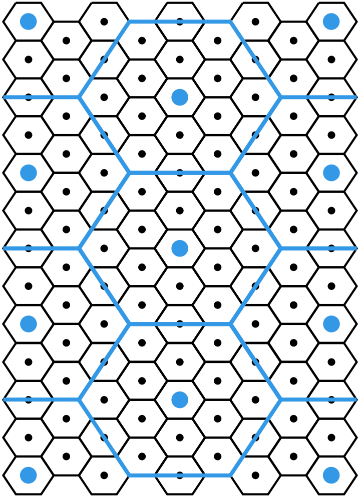

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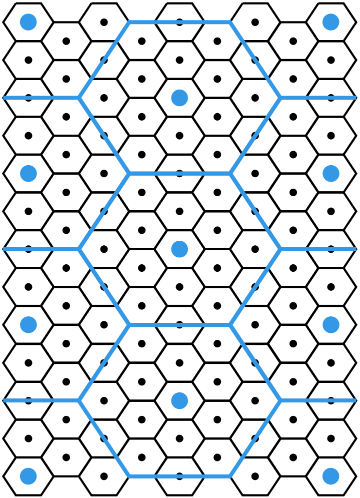

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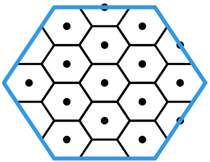

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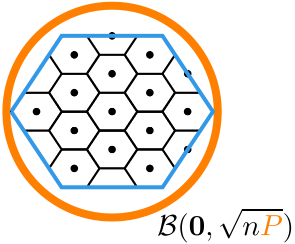

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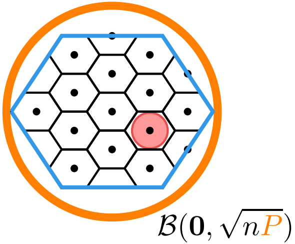
- Coarse lattice Λ_{C} enforces the power constraint.
- Fine lattice Λ_{F} protects against noise.



Modulo Operation

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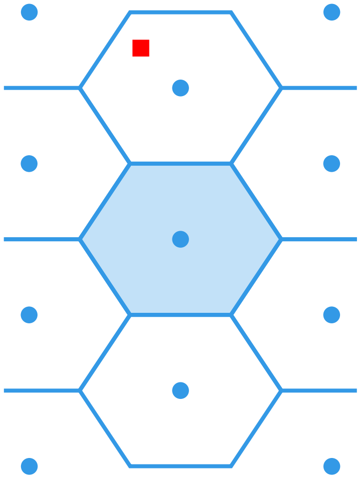

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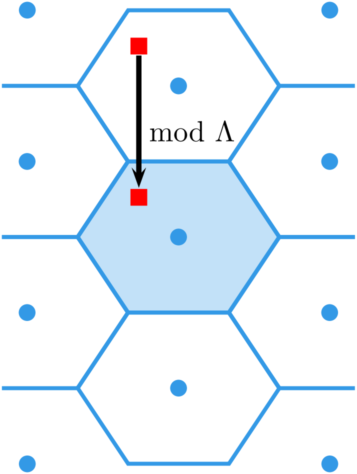

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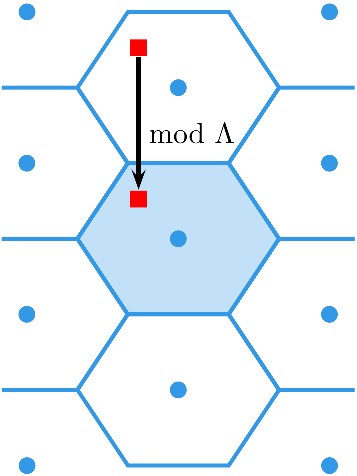
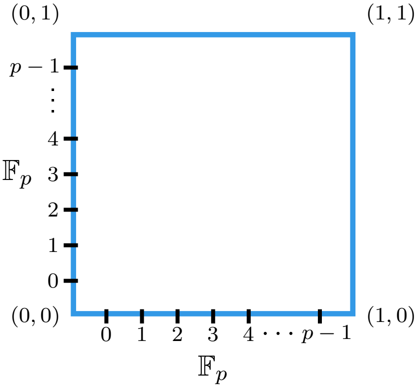


Construction A: Lattice Codes from Linear Codes

- Map elements $\{0, 1, 2, \ldots, p-1\}$ to equally spaced points on $[0, 1)$.



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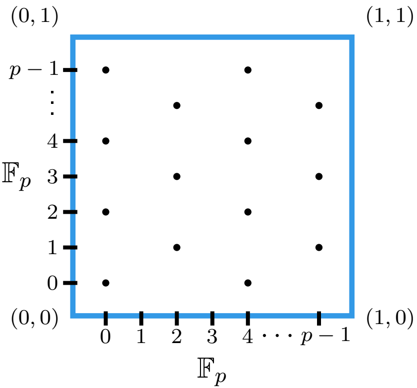

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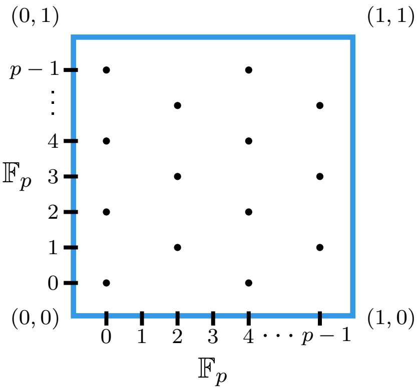

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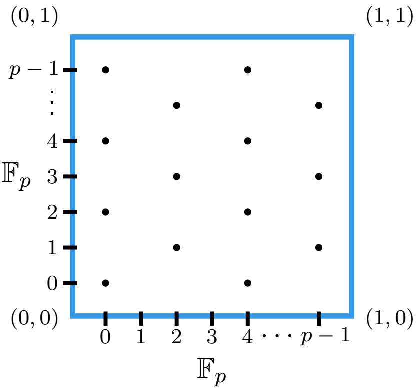

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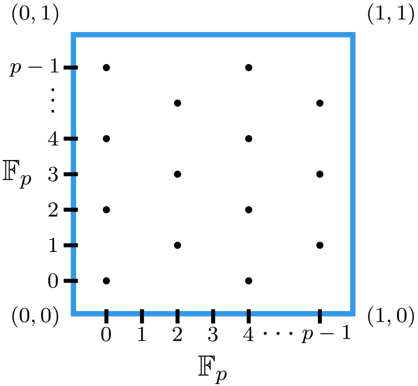

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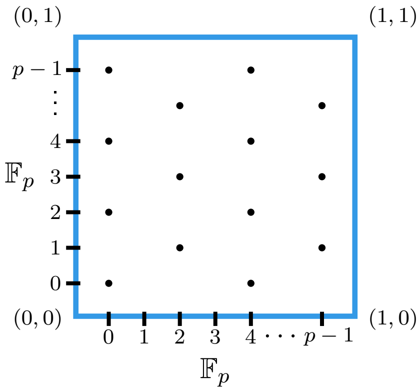

- Can design good coarse lattice via a *two-stage approach.*
- Existence of good nested lattice codes via Construction A:
  Loeliger '97, Forney-Trott-Chung '00, Erez-Zamir '04, Erez-Litsyn-Zamir '05.

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- Generate nested lattices $\Lambda_{\mathsf{C}} \subset \Lambda_{\mathsf{F}}$ by using $\mathbf{G}_{\mathsf{F}}$ in Construction A.
- Ideally, the resulting code meets the power constraint and tolerates effective noise, while maintaining a high rate. We would also like an isomorphism to $\mathbb{F}_p^k$.

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- Isomorphism exists: There is a function $\phi : \mathbb{F}_p^k \rightarrow \Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}$ such that if $\mathbf{t}_\ell = \phi(\mathbf{w}_\ell)$, then

$$\phi^{-1}\left(\left[\sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell\right] \bmod \Lambda_{\mathsf{C}}\right) = \bigoplus_{\ell=1}^{L} q_\ell \mathbf{w}_\ell$$

for any $a_\ell \in \mathbb{Z}$ and $q_\ell = [a_\ell] \bmod p$.

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- What happened to the "$1\,+$"?

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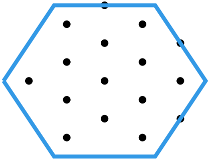
- Dithering can make the effective noise look independent from the desired lattice codeword.



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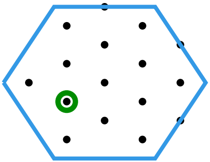

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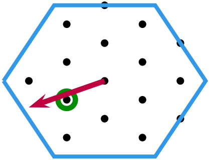

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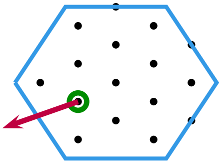

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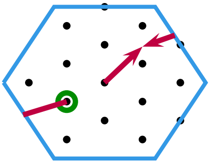

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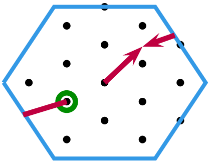
- $\mathbf{x}$ is now independent of the codeword $\mathbf{t}$.



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- **Decode:** $\hat{\mathbf{t}} = \big[Q_{\Lambda_{\mathsf{F}}}(\tilde{\mathbf{y}})\big] \bmod \Lambda_{\mathsf{C}}$.

Point-to-Point AWGN Channel: Lattice Encoding and Decoding

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- **Map message to lattice point:** $\mathbf{t} = \phi(\mathbf{w})$.
- **Dither + Transmit:** $\mathbf{x} = [\mathbf{t} + \mathbf{d}] \bmod \Lambda_{\mathsf{C}}$.
- **Receive:** $\mathbf{y} = \mathbf{x} + \mathbf{z}$.
- **Scale + Remove dithers:**

$$\begin{aligned}
& \tilde{\mathbf{y}} = [\alpha \mathbf{y} - \mathbf{d}] \bmod \Lambda_{\mathsf{C}} \\
(\mathbf{z}_{\mathsf{eff}} = (\alpha - 1)\mathbf{x} + \alpha \mathbf{z}) \quad & = [\mathbf{x} + \mathbf{z}_{\mathsf{eff}} - \mathbf{d}] \bmod \Lambda_{\mathsf{C}} \\
& = \big[[\mathbf{t} + \mathbf{d}] \bmod \Lambda_{\mathsf{C}} - \mathbf{d} + \mathbf{z}_{\mathsf{eff}}\big] \bmod \Lambda_{\mathsf{C}} \\
(\text{Distributive Law}) \quad & = [\mathbf{t} + \mathbf{z}_{\mathsf{eff}}] \bmod \Lambda_{\mathsf{C}}
\end{aligned}$$

- **Decode:** $\hat{\mathbf{t}} = \big[Q_{\Lambda_{\mathsf{F}}}(\tilde{\mathbf{y}})\big] \bmod \Lambda_{\mathsf{C}}$.
- **Map back to finite field:** $\hat{\mathbf{w}} = \phi^{-1}(\hat{\mathbf{t}})$.

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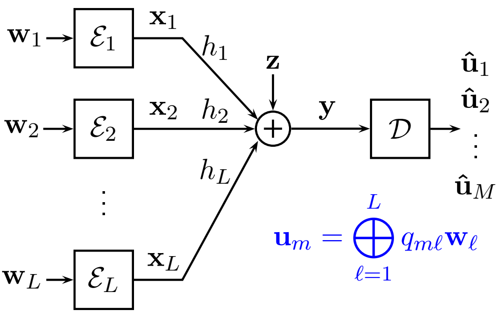
- **Map back to finite field:** $\hat{\mathbf{w}} = \phi^{-1}(\hat{\mathbf{t}})$.
- Decoding is successful with high probability if we set $\sigma^2_{\mathsf{eff}} > \frac{P}{1+P}$. This means that the rate $R = \frac{1}{2}\log(1+P)$ is achievable.

Refresher: Compute-and-Forward Problem Statement



$$\mathbf{u}_m = \bigoplus_{\ell=1}^{L} q_{m\ell} \mathbf{w}_\ell$$

- Messages are finite field vectors, $\mathbf{w}_\ell \in \mathbb{F}_p^k$.
- Real-valued inputs and outputs, $\mathbf{x}_\ell, \mathbf{y} \in \mathbb{R}^n$.
- Power constraint, $\frac{1}{n}\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq P$.
- Gaussian noise, $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.
- Equal rates: $R = \dfrac{k}{n} \log_2 p$

- Decoder wants M linear combinations of the messages with vanishing probability of error $\lim_{n \to \infty} \mathbb{P}\Big(\bigcup_m \{\hat{\mathbf{u}}_m \neq \mathbf{u}_m\} \Big) = 0$.
- The linear combination with integer coefficient vector $\mathbf{a}_m = [a_{m1} \ a_{m2} \ \cdots \ a_{mL}]^\mathsf{T} \in \mathbb{Z}^L$ corresponds to

$$\mathbf{u}_m = \bigoplus_{\ell=1}^{L} q_{m\ell} \mathbf{w}_\ell \qquad \text{where } q_{m\ell} = [a_{m\ell}] \bmod p \ .$$

Encoding operations at the ℓ^{th} transmitter:

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Compute-and-Forward: Lattice Encoding

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Encoding operations at the ℓ^{th} transmitter:

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- Notice that these operations do not depend on the channel gains.

Decoding operations at the receiver to recover the linear combination with integer coefficient vector $\mathbf{a}_m = [a_{m1} \ \cdots \ a_{mL}]^{\mathsf{T}}$:

Compute-and-Forward: Lattice Decoding

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Compute-and-Forward: Lattice Decoding

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- **Receive:** $\mathbf{y} = \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z}$.
- **Scale + Remove dithers:**

$$\begin{aligned} \tilde{\mathbf{y}} &= \left[\alpha \mathbf{y} - \sum_{\ell=1}^{L} a_{m\ell} \mathbf{d}_\ell \right] \bmod \Lambda_{\mathsf{C}} \\ &= \left[\sum_{\ell=1}^{L} a_{m\ell}(\mathbf{x}_\ell - \mathbf{d}_\ell) + \sum_{\ell=1}^{L} (\alpha h_\ell - a_{m\ell}) \mathbf{x}_\ell + \alpha \mathbf{z} \right] \bmod \Lambda_{\mathsf{C}} \\ &= [\mathbf{v} + \mathbf{z}_{\mathsf{eff}}] \bmod \Lambda_{\mathsf{C}} \quad (\text{Distributive Law}) \end{aligned}$$

$$\mathbf{v} = \left[\sum_{\ell=1}^{L} a_{m\ell} \mathbf{t}_\ell \right] \bmod \Lambda_{\mathsf{C}} \qquad \mathbf{z}_{\mathsf{eff}} = \sum_{\ell=1}^{L} (\alpha h_\ell - a_{m\ell}) \mathbf{x}_\ell + \alpha \mathbf{z}$$

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&= [\mathbf{v} + \mathbf{z}_{\mathsf{eff}}] \bmod \Lambda_{\mathsf{C}} \quad \text{(Distributive Law)}
\end{aligned}$$

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- **Decode:** $\hat{\mathbf{v}}_m = \left[Q_{\Lambda_{\mathsf{F}}}(\tilde{\mathbf{y}}) \right] \bmod \Lambda_{\mathsf{C}}$.

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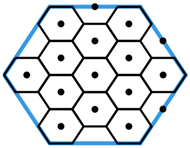
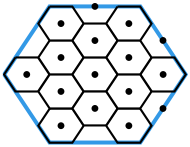
- **Decode:** $\hat{\mathbf{v}}_m = \left[Q_{\Lambda_{\mathsf{F}}}(\tilde{\mathbf{y}}) \right] \bmod \Lambda_{\mathsf{C}}$.
- **Map back to finite field:** $\hat{\mathbf{u}}_m = \phi^{-1}(\hat{\mathbf{v}})$.

Compute-and-Forward: Illustration

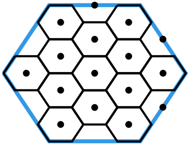
All users employ the same nested lattice code:

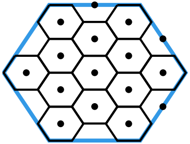




Compute-and-Forward: Illustration

Choose message vectors over finite field $\mathbf{w}_\ell \in \mathbb{F}_p^k$:

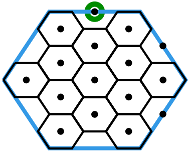
$\mathbf{w}_1 \longrightarrow$ 

$\mathbf{w}_2 \longrightarrow$ 

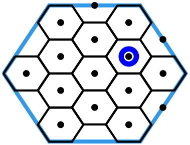
Compute-and-Forward: Illustration

Map $\mathbf{w}_\ell$ to lattice point $\mathbf{t}_\ell = \phi(\mathbf{w}_\ell)$:

$\mathbf{w}_1 \longrightarrow$

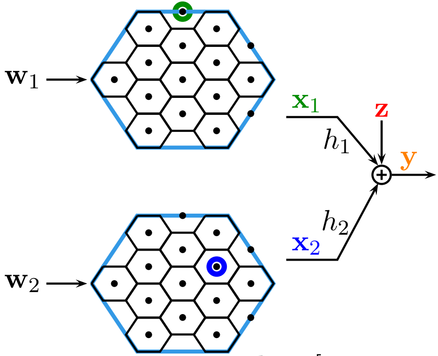


$\mathbf{w}_2 \longrightarrow$



Compute-and-Forward: Illustration

Transmit lattice points over the channel:

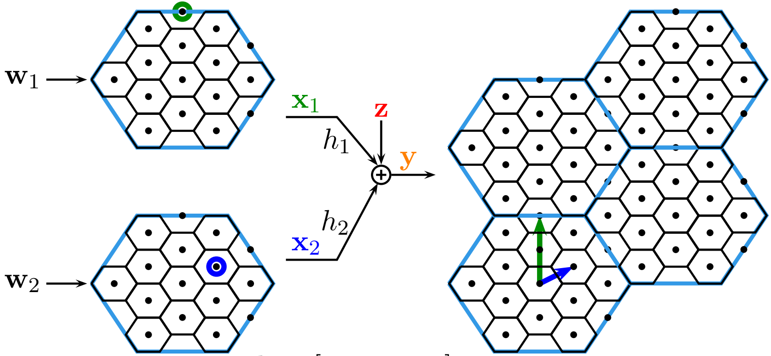


$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\mathbf{a}_m = [\ 2 \quad 3\ ]$$

Compute-and-Forward: Illustration

Transmit lattice points over the channel:



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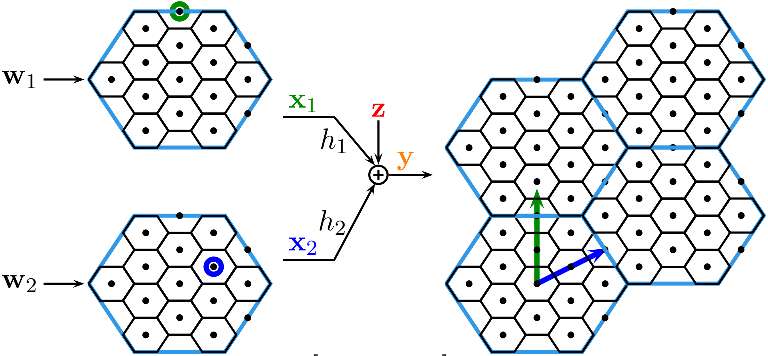
Lattice codewords are scaled by channel coefficients:



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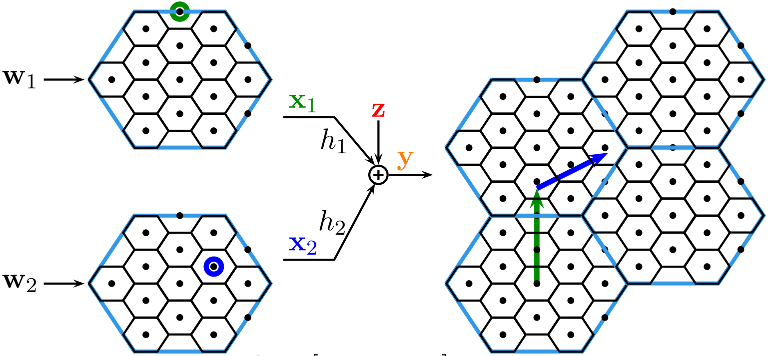
Compute-and-Forward: Illustration

Scaled codewords added together plus noise:



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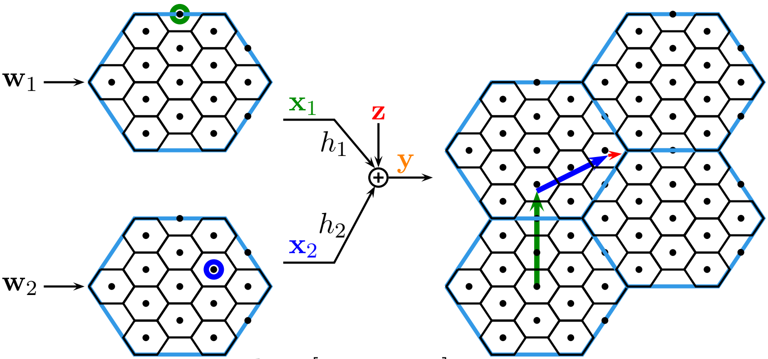

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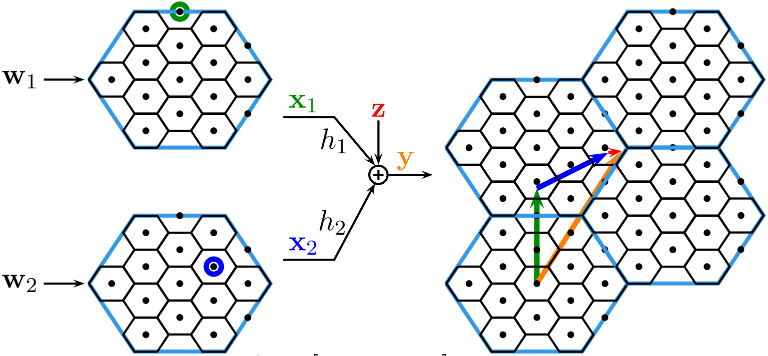
Extra noise penalty for non-integer channel coefficients:



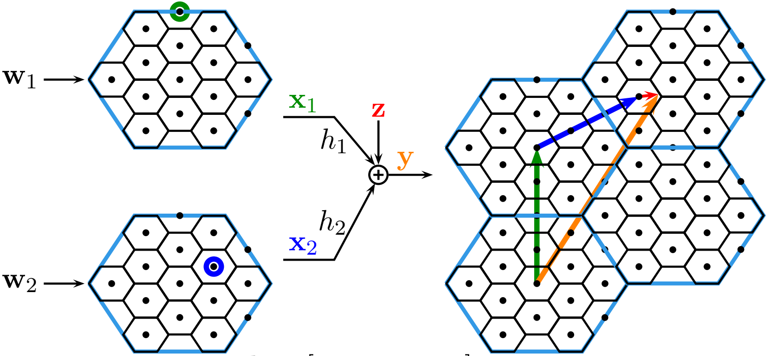
$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\mathbf{a}_m = [\ 2 \quad 3\ ]$$

Effective noise: $1 + P\|\mathbf{h} - \mathbf{a}_m\|^2$

Compute-and-Forward: Illustration

Scale output by α to reduce non-integer noise penalty:



$$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$$

$$\mathbf{a}_m = [\ \ 2 \qquad 3\ \ ]$$

Effective noise: $\alpha^2 + P\|\alpha\mathbf{h} - \mathbf{a}_m\|^2$

Compute-and-Forward: Illustration

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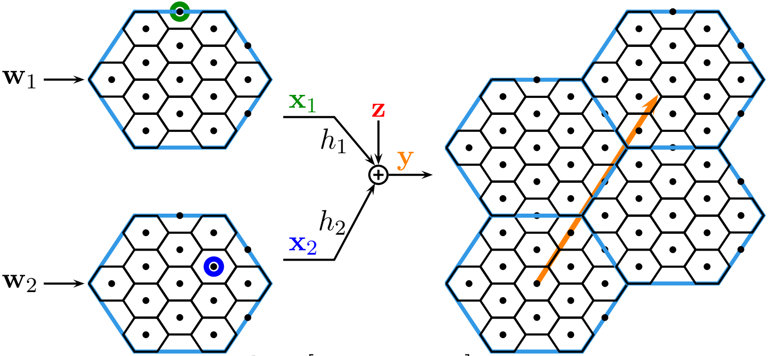

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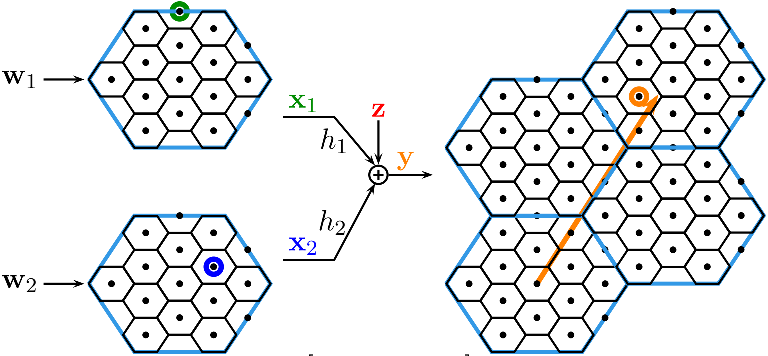
Compute-and-Forward: Illustration

Decode to the closest lattice point:



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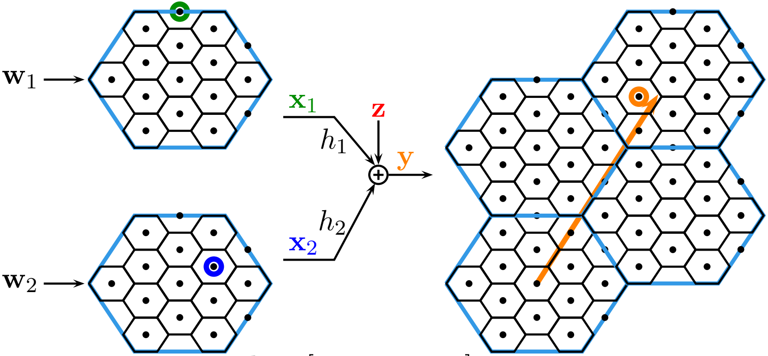
Recover integer linear combination $\mathrm{mod}\ \Lambda_{\mathsf{C}}$:



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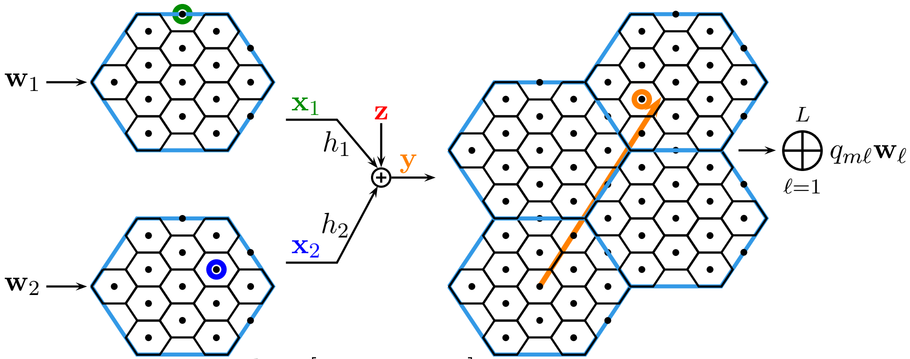
Compute-and-Forward: Illustration

Map back to linear combination of the messages:



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Compute-and-Forward: Effective Noise

- Overall, the linear combination with integer coefficient vector $\mathbf{a}_m$ can be successfully decoded if

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$$\alpha_{\text{MMSE}} = \frac{P\mathbf{a}_m^{\mathsf{T}}\mathbf{h}}{1 + P\|\mathbf{h}\|^2} \ .$$

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- Plugging this in and applying the Matrix Inversion Lemma, we get

$$\sigma^2_{\text{eff}} > \mathbf{a}_m^{\mathsf{T}}\big(P^{-1}\mathbf{I} + \ \mathbf{h}\mathbf{h}^{\mathsf{T}}\big)^{-1}\mathbf{a}_m \ .$$

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- Overall, we find that if the rate satisfies

$$R < \min_m \frac{1}{2}\log\left(\frac{P}{\mathbf{a}_m^{\mathsf{T}}\big(P^{-1}\mathbf{I} + \ \mathbf{h}\mathbf{h}^{\mathsf{T}}\big)^{-1}\mathbf{a}_m}\right)$$

we can successfully decode all M linear combinations.

Compute-and-Forward: Achievable Rates

Theorem (Nazer-Gastpar '11)

The computation rate region described by

$$R_{comp}(\mathbf{h}, \mathbf{a}) = \max_{\alpha \in \mathbb{R}} \frac{1}{2} \log^{+} \left(\frac{P}{\alpha^2 + P \|\alpha \mathbf{h} - \mathbf{a}\|^2} \right)$$

is achievable.

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Compute-and-Forward: Achievable Rates

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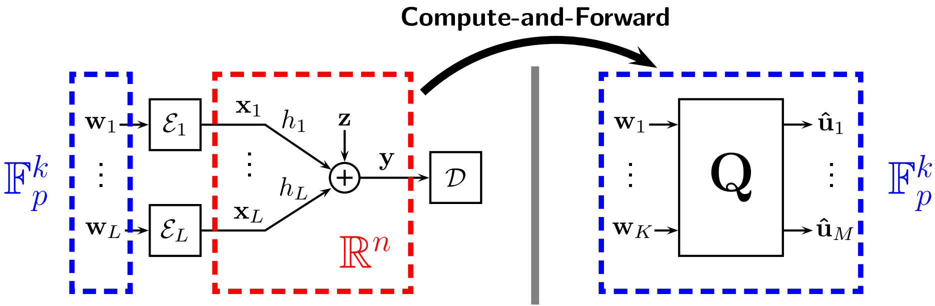

if $R < \min_m R_{\text{comp}}(\mathbf{h},\mathbf{a}_m)$ for some $\mathbf{a}_1,\ldots,\mathbf{a}_M \in \mathbb{Z}^L$ satisfying $[\mathbf{a}_m] \bmod p = \mathbf{q}_m$.

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is achievable.

Special Cases:

- Perfect Match: $R_{\mathsf{comp}}(\mathbf{a},\mathbf{a}) = \frac{1}{2}\log^{+}\left(\frac{1}{\|\mathbf{a}\|^2} + P\right)$

Compute-and-Forward: Achievable Rates

Theorem (Nazer-Gastpar '11)

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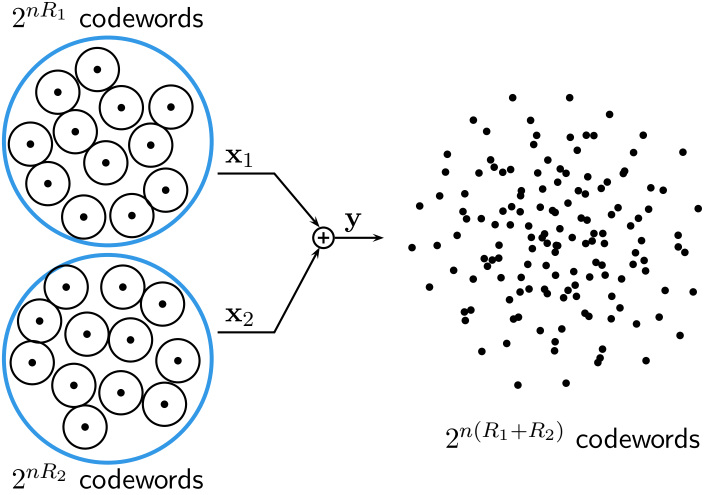
is achievable.

Special Cases:

- Perfect Match: $R_{\text{comp}}(\mathbf{a}, \mathbf{a}) = \frac{1}{2} \log^+ \left(\frac{1}{\|\mathbf{a}\|^2} + P \right)$
- Decode a Message:

$$R_{\text{comp}}\Big(\mathbf{h}, [\underbrace{0 \ \cdots \ 0}_{m-1 \text{ zeros}} \ 1 \ 0 \ \cdots \ 0]^\mathsf{T}\Big) = \frac{1}{2} \log \left(1 + \frac{h_m^2 P}{1 + P \sum_{\ell \neq m} h_\ell^2} \right)$$

Random i.i.d. codes are not good for computation



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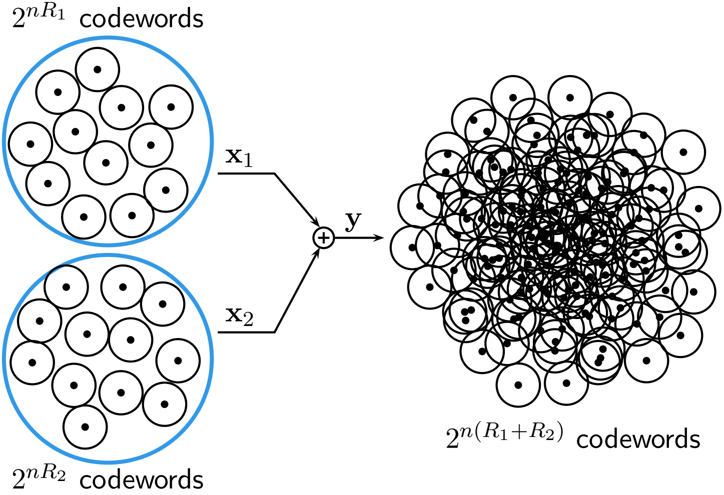
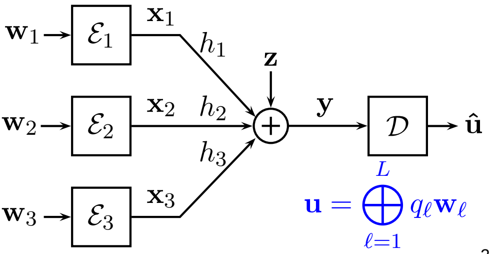
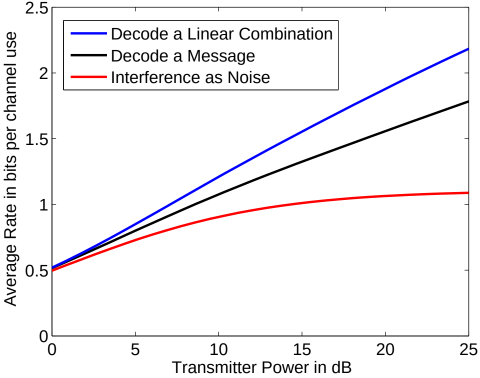


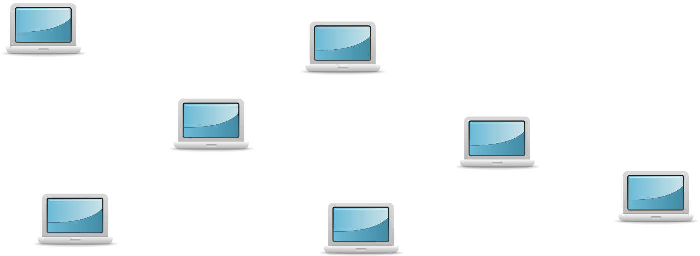


Relay either decodes some linear combination of messages or an individual message.

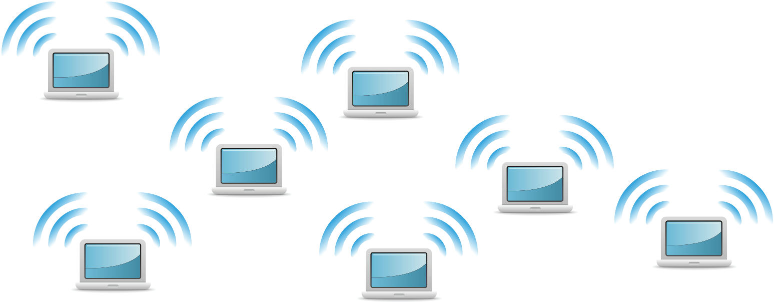
- Three transmitters that do not know the fading coefficients.
- Average rate plotted for i.i.d. Gaussian fading.



Physical-Layer Network Coding



- Usually fight interference and convert to network of bit pipes.



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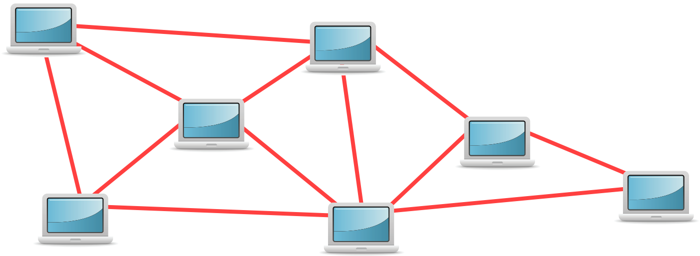

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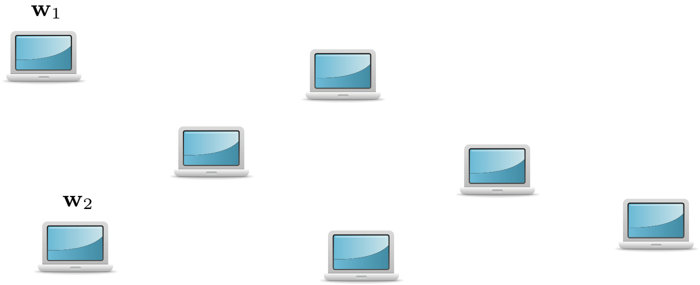
Physical-Layer Network Coding



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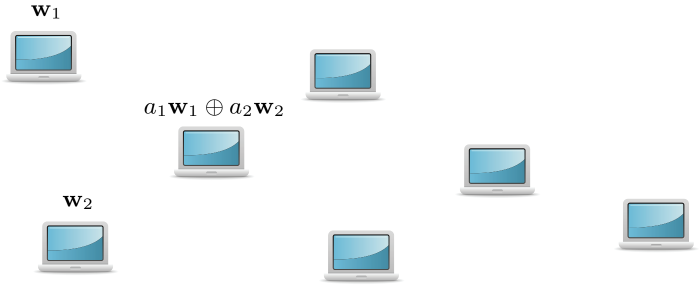

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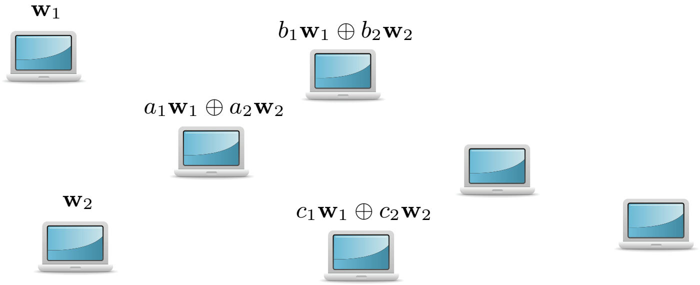

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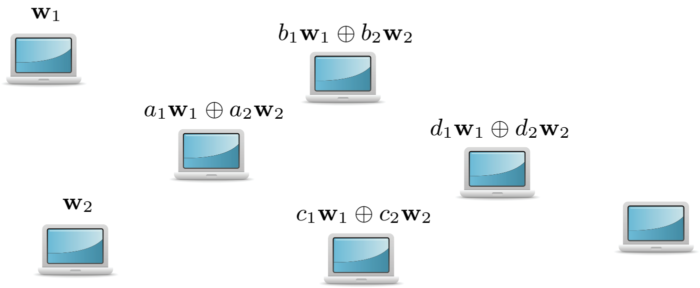

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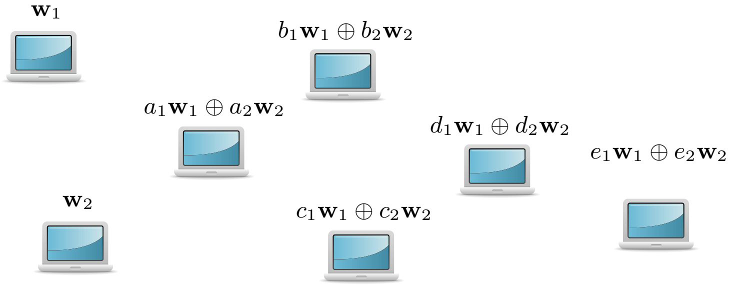

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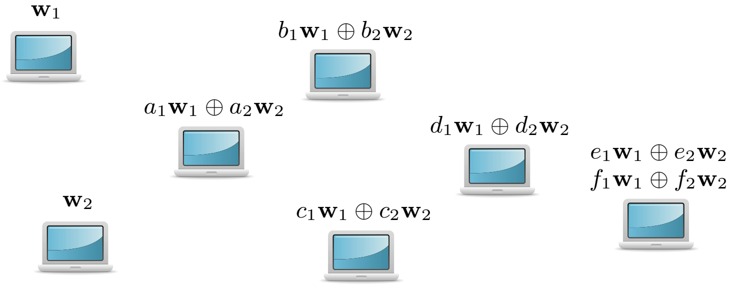

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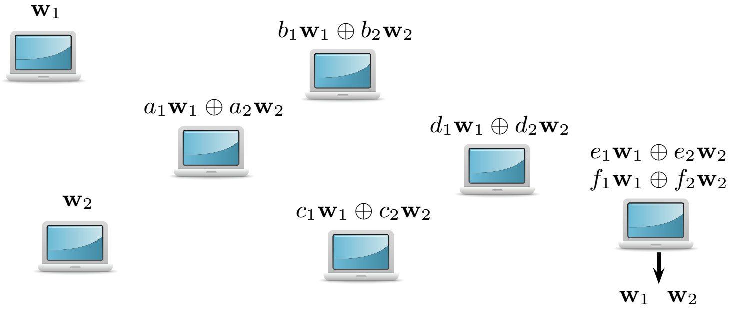

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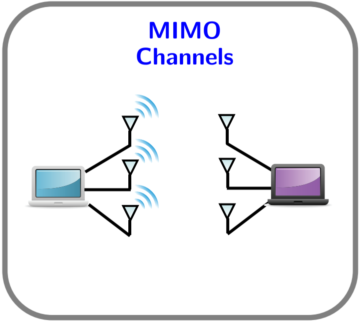
- Warm-up: Compute-and-forward over finite field channels.
- Compute-and-forward over Gaussian channels.
- Applications to communication across single-hop Gaussian networks.



Joint work with Jiening Zhan, Uri Erez, and Michael Gastpar.

- Increasing the number of antennas in a wireless system can significantly increase its capacity.
  Foschini '96, Foschini and Gans '98, Telatar '99.
- Enormous body of work has strived to develop receiver architectures that can approach these capacity gains with manageable complexity.

Vast majority of receiver architectures fall into these two categories (references at the end of the talk):

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We propose a new class of Integer-Forcing Linear Receivers.

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- Does this help beyond integer-valued channel matrices?

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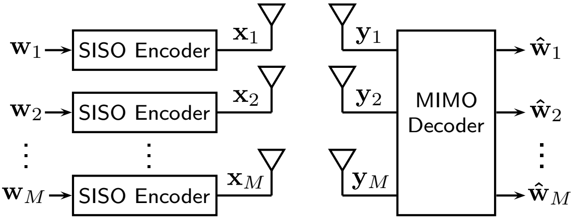

- Each antenna has an independent data stream $\mathbf{x}_\ell \in \mathbb{R}^n$ of rate R (e.g., V-BLAST setting, cellular uplink). $\mathbf{X} = [\mathbf{x}_1 \ \cdots \ \mathbf{x}_M]^\mathsf{T}$.
- Channel model: $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z}$ where $\mathbf{Z}$ is elementwise i.i.d. $\mathcal{N}(0,1)$.
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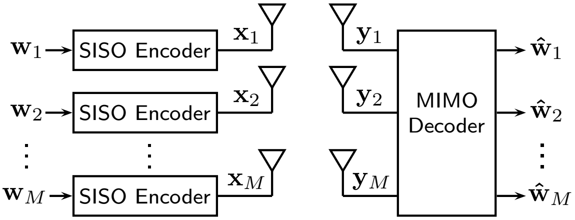

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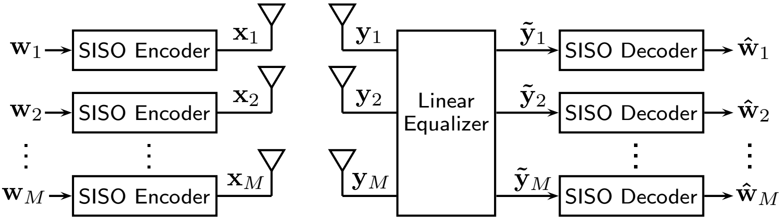
- Joint maximum likelihood decoding is optimal but has high implementation complexity.

Zero-Forcing Linear Receivers



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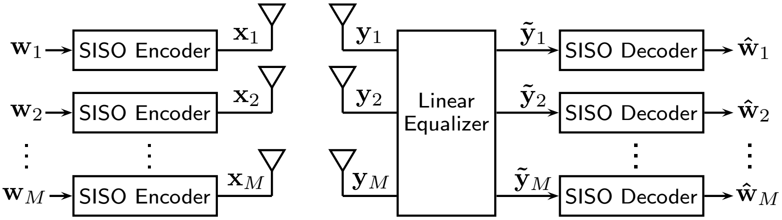

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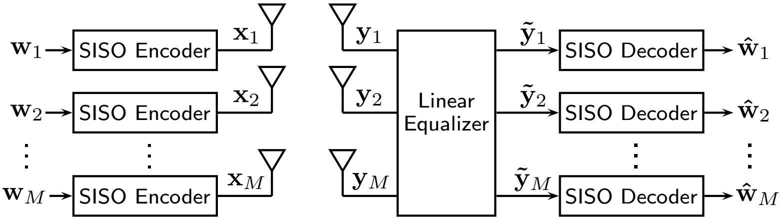

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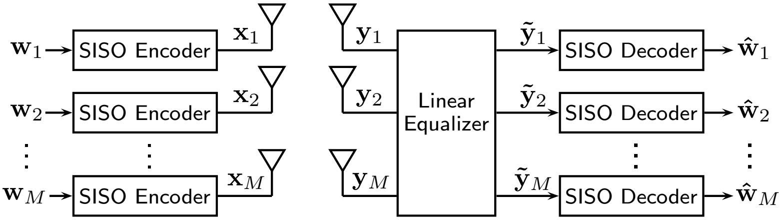

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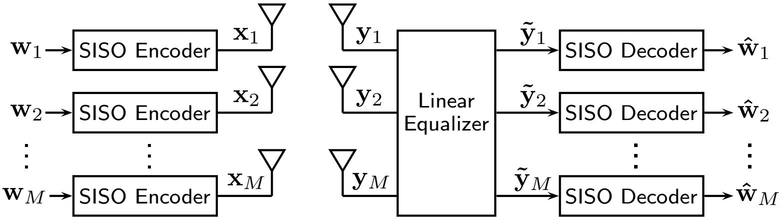

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  (Often called the linear MMSE receiver.)

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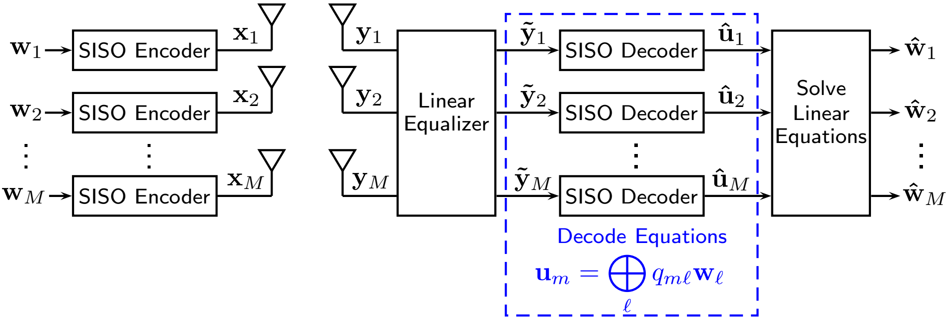

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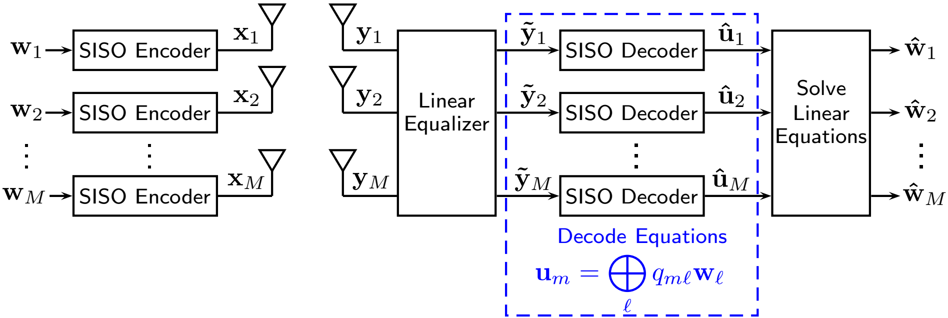

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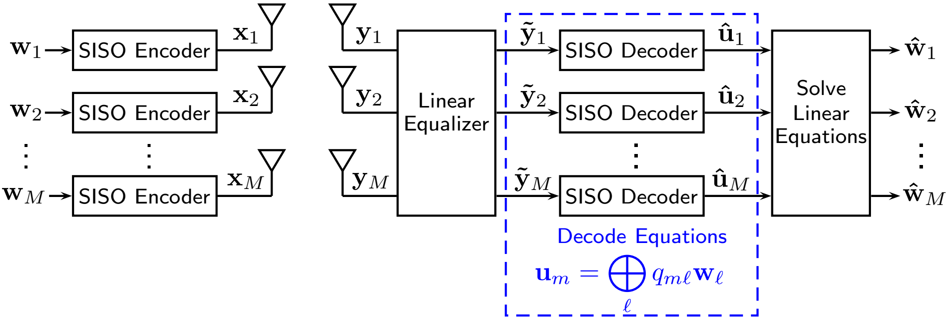

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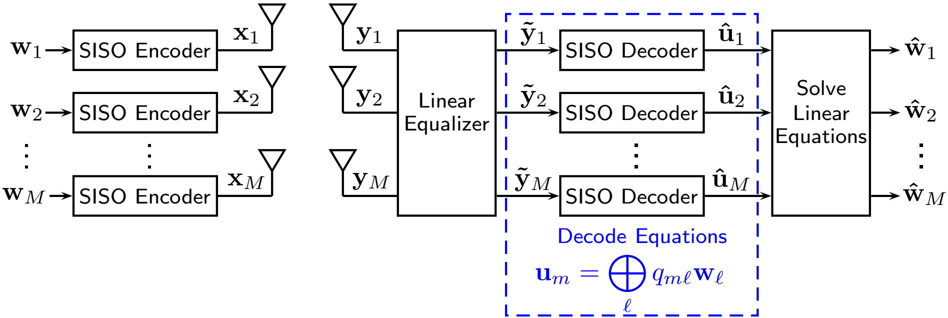

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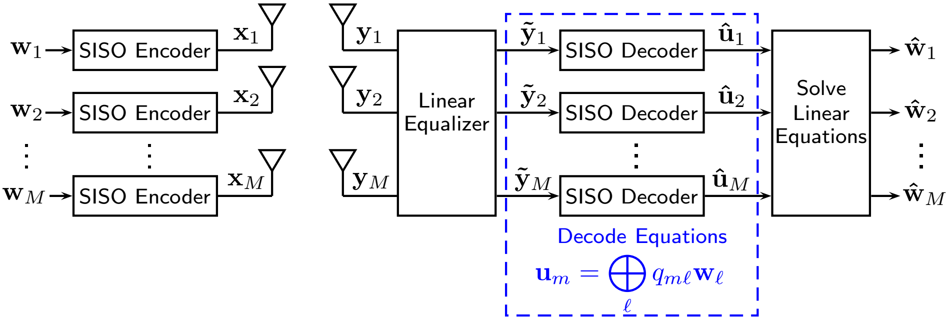

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Theorem (Zhan-Nazer-Erez-Gastpar '14)

The computation rate region described by

$$R_{\mathsf{comp}}(\mathbf{H}, \mathbf{a}) = \max_{\mathbf{b} \in \mathbb{R}^M} \frac{1}{2} \log^+ \left(\frac{P}{\|\mathbf{b}\|^2 + P\|\mathbf{H}^{\mathsf{T}}\mathbf{b} - \mathbf{a}\|^2} \right)$$

is achievable.

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$$R_{\mathsf{comp}}(\mathbf{H}, \mathbf{a}) = \frac{1}{2}\log^+\left(\frac{P}{\mathbf{a}^{\mathsf{T}}\left(P^{-1}\mathbf{I} + \ \mathbf{H}^{\mathsf{T}}\mathbf{H}\right)^{-1}\mathbf{a}}\right)$$

is achievable.

Integer-Forcing:

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z} \xrightarrow{\text{Project}} \mathbf{B}\mathbf{Y} = \mathbf{A}\mathbf{X} + \underbrace{(\mathbf{B}\mathbf{H} - \mathbf{A})\mathbf{X} + \mathbf{B}\mathbf{Z}}_{\mathbf{Z}_{\text{eff}}} \xrightarrow{\text{Decode}} \mathbf{A}\mathbf{X}$$

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- Achievable rate: $R_{\text{IF}}(\mathbf{H}) = M \max_{\substack{\mathbf{A} \in \mathbb{Z}^{M \times M} \\ \text{rank}(\mathbf{A}) = \text{M}}} \min_{m} R_{\text{comp}}(\mathbf{H}, \mathbf{a}_m)$

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Comparison

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Successive Interference Cancellation

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- V-BLAST III: Decodes and cancel the data streams in a predetermined order. The rate of each data stream is selected to maximize the sum rate. (Outside problem statement.)

Simulation: Outage Rates

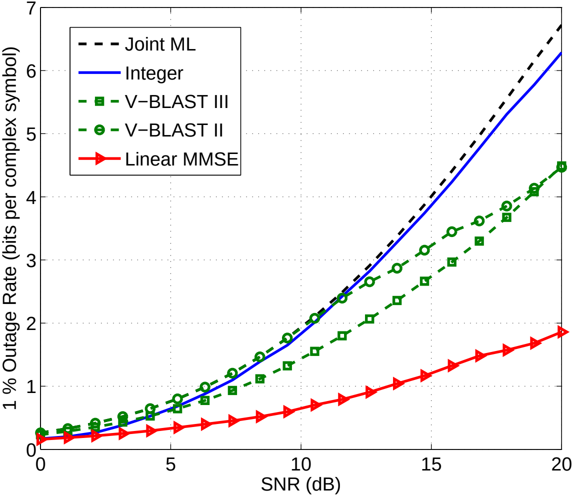
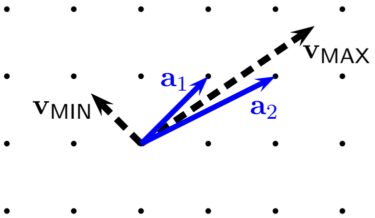


Figure: 1 percent outage rates for the 2×2 complex-valued MIMO channel with Rayleigh fading.

- Integer-forcing can adapt to the channel by choosing a basis (of integer vectors) close to the maximum singular vector.



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- Integer-forcing can adapt to the channel by choosing a basis (of integer vectors) close to the maximum singular vector.
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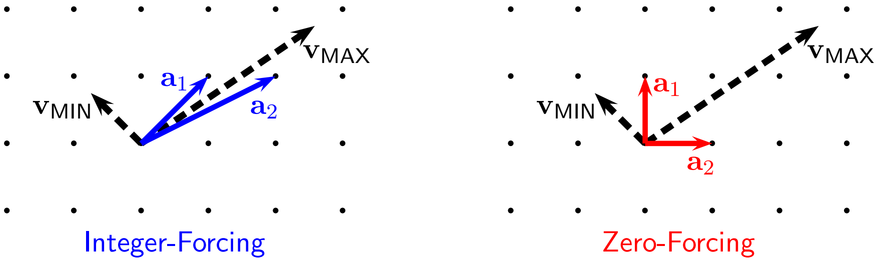

- **Zheng-Tse '03:** A family of codes is said to achieve spatial multiplexing gain r and diversity gain d if the total data rate and the average probability of error satisfy

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- **Zhan-Nazer-Erez-Gastpar '14:** Integer-forcing can attain the optimal DMT while conventional linear receivers cannot.

- **Zheng-Tse '03:** The DMTs achieved by the zero-forcing, linear MMSE, and successive interference cancellation architectures are

$$d_{\text{ZF}}(r) = d_{\text{LMMSE}}(r) = d_{\text{V-BLAST I}}(r) = \left(1 - \frac{r}{M}\right)$$

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- **Zhan-Nazer-Erez-Gastpar '14:** Integer-forcing recovers the optimal DMT for $N \geq M$ receive antennas:

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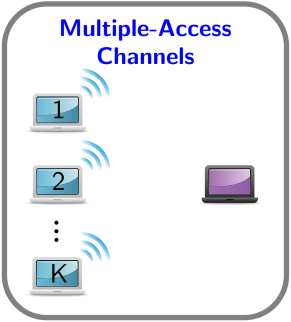
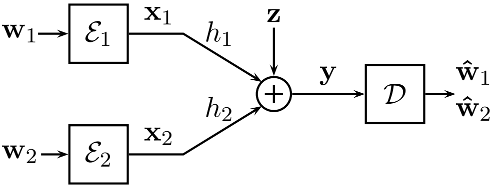
- **Ordentlich-Erez-Nazer '13:** Framework for IF-SIC and exact optimality proof.

- Warm-up: Compute-and-forward over finite field channels.
- Compute-and-forward over Gaussian channels.
- Applications to communication across single-hop Gaussian networks.



Joint work with Or Ordentlich and Uri Erez.

Gaussian Multiple-Access Channel



$$\|\mathbf{x}_\ell\|^2 \leq nP, \quad \mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}).$$

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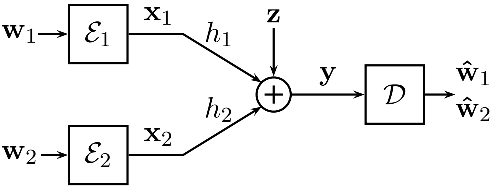

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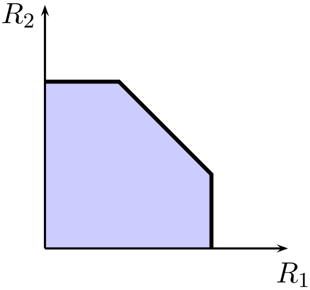

Theorem (Ahlswede '71, Liao '72, Wyner '74, Cover '75)

The capacity region is the set of all rate pairs (R_1, R_2) *satisfying:*

$$R_1 < \frac{1}{2}\log(1 + h_1^2 P) \qquad R_2 < \frac{1}{2}\log(1 + h_2^2 P)$$

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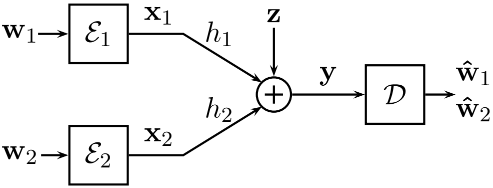

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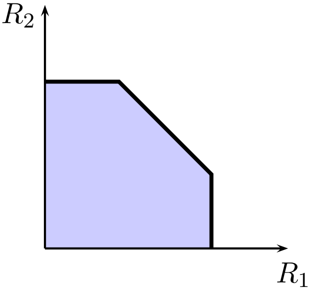

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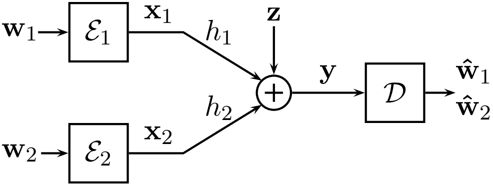
Achievable via joint decoding.

Successive Cancellation



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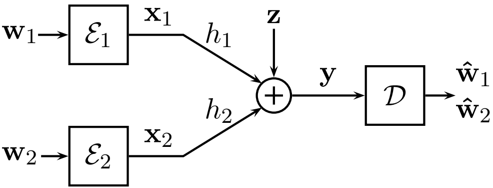

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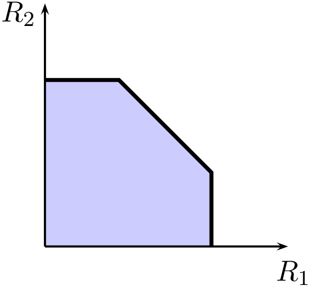

- Treat $\mathbf{x}_2$ as noise and decode $\mathbf{x}_1$, $R_1 < \frac{1}{2}\log\left(1 + \frac{h_1^2 P}{1 + h_2^2 P}\right)$.

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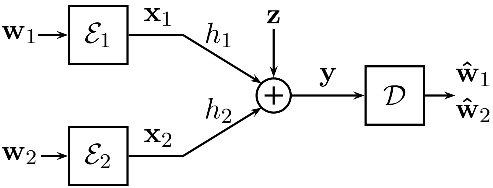

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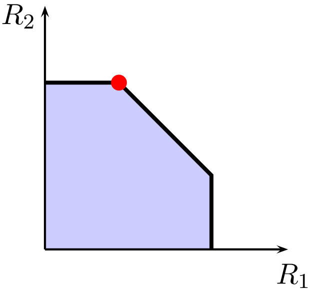

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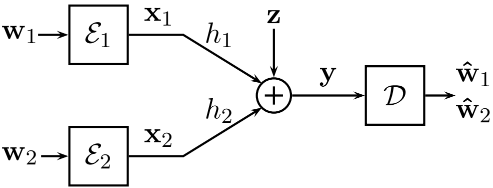

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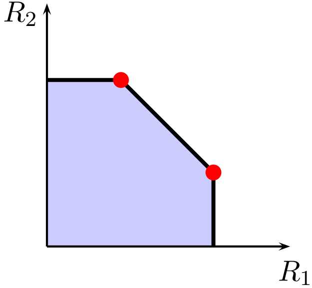

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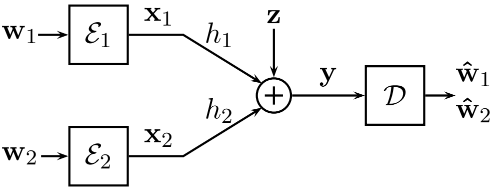

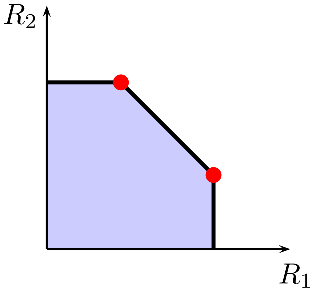

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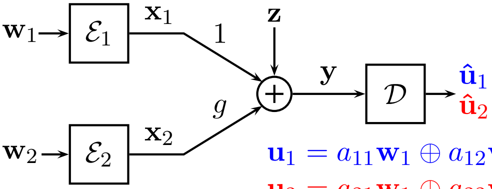
- Achieves capacity when combined with time-sharing or rate-splitting (**Rimoldi-Urbanke '96**).



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- Decode one linear combination.
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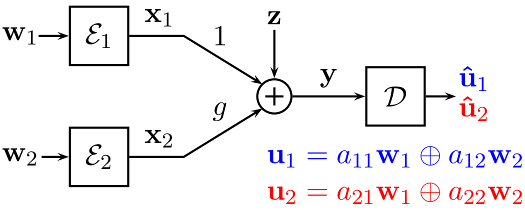
Two Linear Combinations



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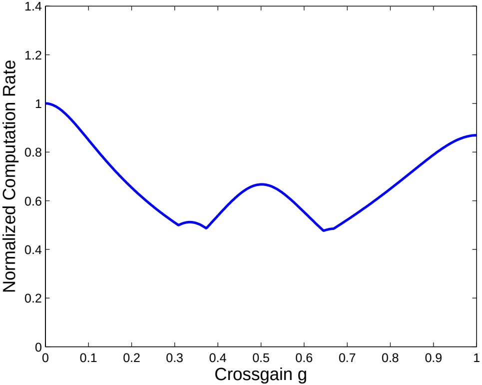

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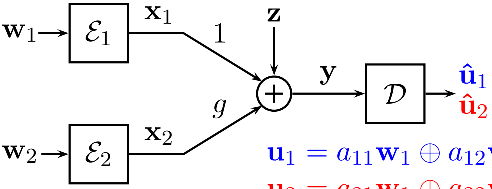

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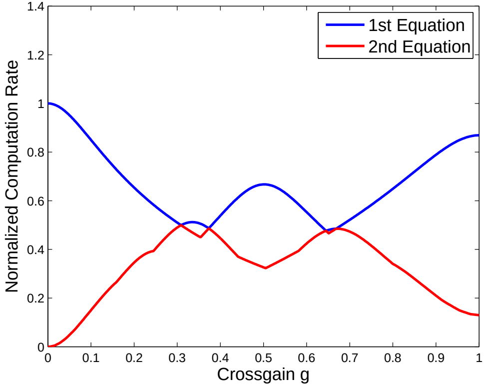

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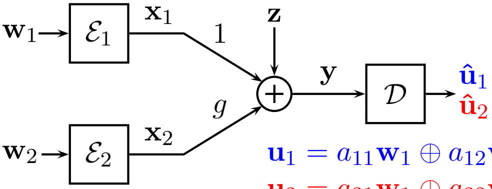

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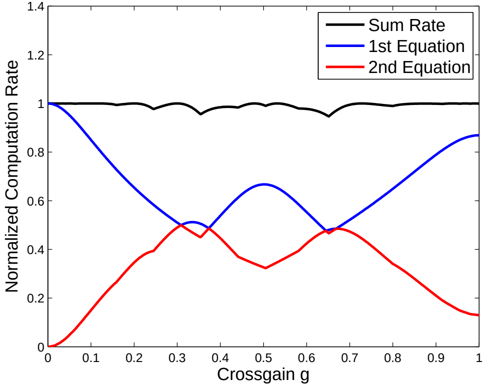

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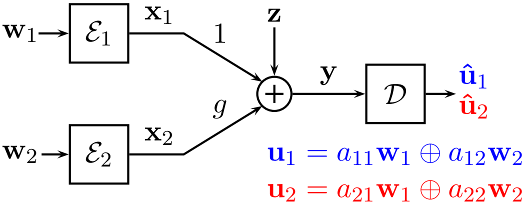

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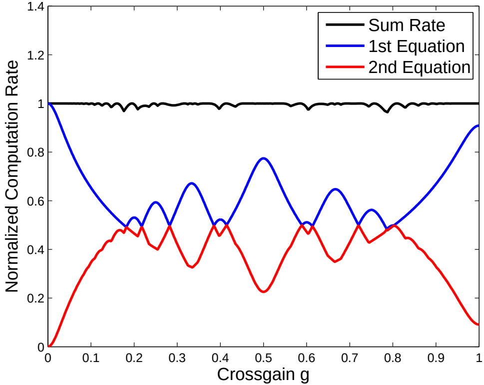
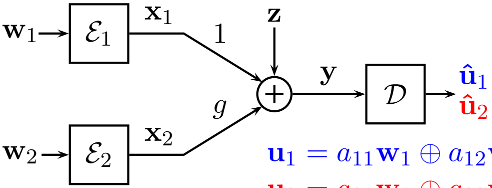
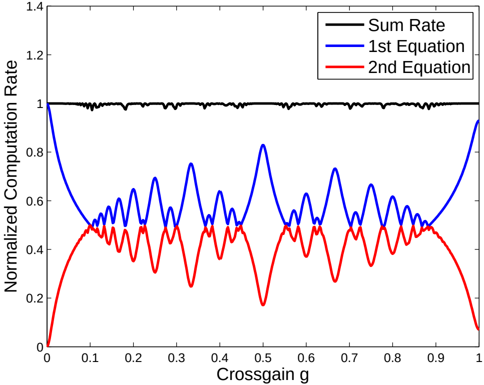


Two Linear Combinations



$$\mathbf{u}_1 = a_{11}\mathbf{w}_1 \oplus a_{12}\mathbf{w}_2$$
$$\mathbf{u}_2 = a_{21}\mathbf{w}_1 \oplus a_{22}\mathbf{w}_2$$

- Decode *two linearly independent* linear combinations.
- Plot rate normalized by the MAC sum rate $\frac{1}{2}\log(1 + (1 + g^2)P)$.



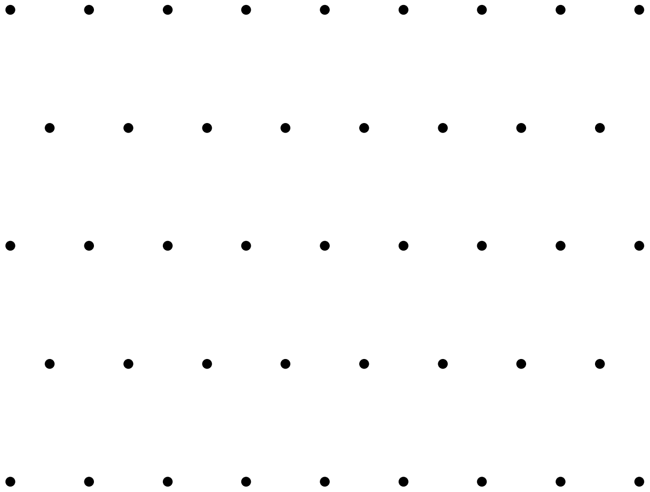
- Looks as if the sum of computation rates is nearly equal to the MAC sum capacity. Why is this happening?

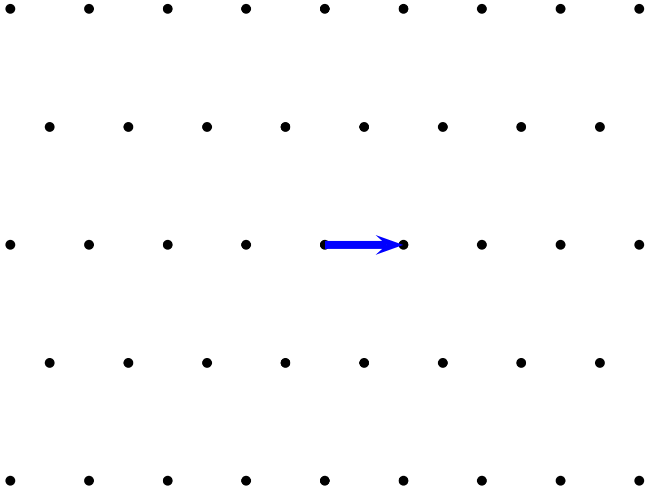
- Let $\mathbf{F} = \left(P^{-1/2}\mathbf{I} + \ \mathbf{h}\mathbf{h}^{\mathsf{T}}\right)^{-1/2}$. Then, each computation rate can be written as
$$R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a}_k) = \frac{1}{2}\log^{+}\left(\frac{P}{\|\mathbf{F}\ \mathbf{a}_k\|^2}\right) .$$

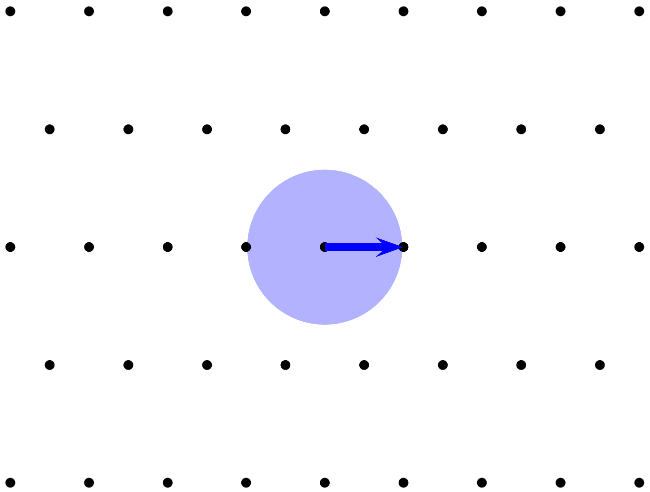
- Thus, decoding the best linear combinations is the same as finding the successive minima $\lambda_k(\mathbf{F})$ for the lattice $\Lambda(\mathbf{F}) = \mathbf{F}\mathbb{Z}^K$:

$$\lambda_k(\mathbf{F}) \triangleq \inf\left\{r : \dim\left(\operatorname{span}\left(\Lambda(\mathbf{F}) \ \cap \ \mathcal{B}(\mathbf{0}, r)\right)\right) \geq k\right\} .$$

Successive Minima







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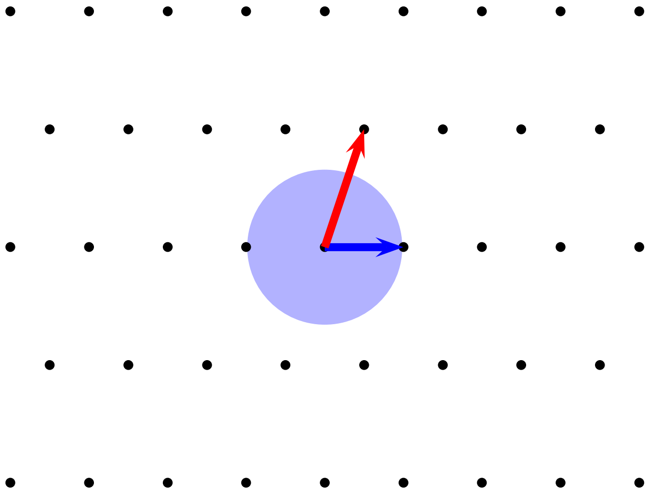

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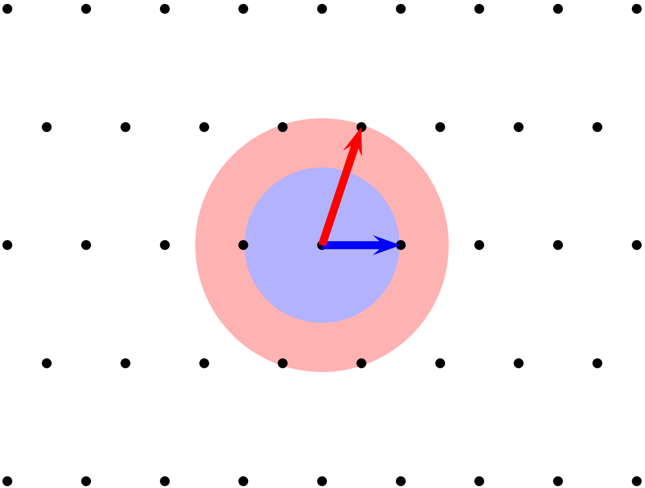

Minkowski's Theorem on Successive Minima

Theorem (Minkowski)

Let $\Lambda(\mathbf{F})$ be a lattice spanned by a full-rank $K \times K$ matrix $\mathbf{F}$. Its successive minima $\lambda_k(\mathbf{F})$ satisfy

$$\prod_{k=1}^{K} \lambda_k^2(\mathbf{F}) \leq K^K \big| \det(\mathbf{F}) \big|^2 \ .$$

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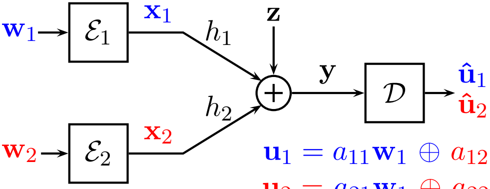
Theorem (Ordentlich-Erez-Nazer '14)

For any channel vector $\mathbf{h} \in \mathbb{R}^K$*, there exist linearly independent integer vectors* $\mathbf{a}_1, \ldots, \mathbf{a}_K \in \mathbb{Z}^K$ *satisfying*

$$\sum_{k=1}^{K} R_{\textit{comp}}(\mathbf{h}, \mathbf{a}_k) \geq \frac{1}{2} \log(1 + \|\mathbf{h}\|^2 \mathsf{SNR}) - \frac{K}{2} \log K \ .$$

Operational Interpretation: Multiple-Access



$$\mathbf{u}_1 = a_{11}\mathbf{w}_1 \oplus a_{12}\mathbf{w}_2$$
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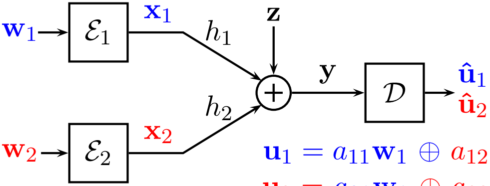
- Order linear combinations by descending computation rate.
- Associate each computation rate to a message.



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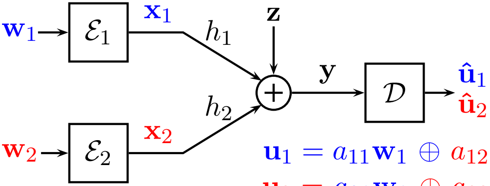

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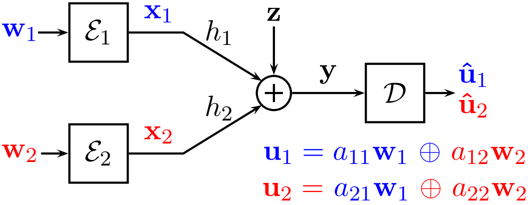

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Theorem (Ordentlich-Erez-Nazer '14)

For any linearly independent integer vectors $\mathbf{a}_1, \ldots, \mathbf{a}_K \in \mathbb{Z}^K$, there exists a permutation π such that the following rates are achievable:

$$R_\ell = R_{\textit{comp},\pi(\ell)} \ .$$

(Algebraic) Successive Cancellation

- After decoding the first linear combination, the receiver knows

$$\mathbf{v}_1 = \left[a_{11}\mathbf{t}_1 + a_{12}\mathbf{t}_2\right] \bmod \Lambda_{\mathsf{C}} \ .$$

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$$\tilde{\mathbf{y}}_2 = \left[a_{21}\mathbf{t}_1 + a_{22}\mathbf{t}_2 + \mathbf{z}_{\mathsf{effec}}(\mathbf{h}, \mathbf{a}_2)\right] \bmod \Lambda_{\mathsf{C}} \ .$$

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- Using $\mathbf{v}_1$, we can cancel out $\mathbf{t}_1$ from $\tilde{\mathbf{y}}_2$ without changing the effective noise.

$$\begin{aligned} \tilde{\mathbf{y}}_2^{\mathsf{SI}} &= \Big[\mathbf{s}_2 - b_1\mathbf{v}_1\Big] \bmod \Lambda_{\mathsf{C}} \\ &= \Big[\big(a_{22} - b_1 a_{12}\big)\mathbf{t}_2 + \mathbf{z}_{\text{effec}}(\mathbf{h}, \mathbf{a}_2)\Big] \bmod \Lambda_{\mathsf{C}} \ . \end{aligned}$$

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- Now, the receiver can decode since $R_2 < R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a}_2)$.

Using One Linear Combination to Get Another

- Basic Idea: After decoding the first linear combination with coefficients $\mathbf{a}$, we should create a new effective channel with coefficients $\mathbf{h} + \beta\mathbf{a}$ to make it easier to decode the second linear combination.

- We need the real sum of codewords $\sum_{\ell} a_\ell \mathbf{x}_\ell$.

- Issue: Our decoding scheme recovers the modulo sum of lattice points $\left[\sum_\ell a_\ell \mathbf{t}_\ell\right] \bmod \Lambda_{\mathsf{C}}$ on the way to the linear combination of messages, not the real sum.

- So far, we have only decoded a modulo sum of the lattice points:

$$\Big[\sum_{\ell} a_{\ell}\mathbf{t}_{\ell}\Big] \bmod \Lambda_{\mathsf{C}} \ .$$

Successive Computation

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- First, add back in the dithers to get the modulo sum of codewords:

$$\Big[\Big[\sum_{\ell} a_\ell \mathbf{t}_\ell\Big] \bmod \Lambda_{\mathsf{C}} + \Big[\sum_{\ell} a_\ell \mathbf{d}_\ell\Big] \bmod \Lambda_{\mathsf{C}}\Big] \bmod \Lambda_{\mathsf{C}} = \Big[\sum_{\ell} a_\ell \mathbf{x}_\ell\Big] \bmod \Lambda_{\mathsf{C}}$$

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- Subtract this from $\mathbf{y}$ to expose the coarse lattice point nearest to the real sum:

$$\mathbf{y} - \Big[\sum_{\ell} a_\ell \mathbf{x}_\ell\Big] \bmod \Lambda_{\mathsf{C}} = Q_{\Lambda_{\mathsf{C}}}\Big(\sum_{\ell} a_\ell \mathbf{x}_\ell\Big) + \sum_{\ell} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}$$

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- Coarse lattice point easier to decode than fine lattice point:

$$Q_{\Lambda_{\mathsf{C}}}\left(Q_{\Lambda_{\mathsf{C}}}\Big(\sum_{\ell} a_\ell \mathbf{x}_\ell\Big) + \sum_{\ell} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}\right) = Q_{\Lambda_{\mathsf{C}}}\Big(\sum_{\ell} a_\ell \mathbf{x}_\ell\Big) \quad \text{w.h.p.}$$

Successive Computation

- Modulo sum is just the quantization error of the real sum with respect to the coarse lattice.

- Combine the modulo sum with the quantized sum to get back the real sum:

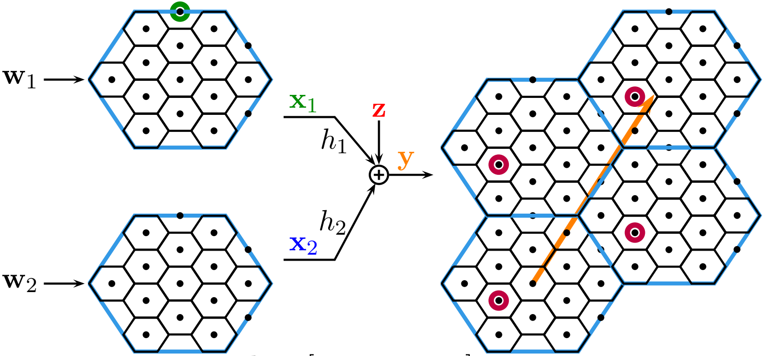
$$\Big[\sum_{\ell} a_\ell \mathbf{x}_\ell\Big] \bmod \Lambda_{\mathsf{C}} + Q_{\Lambda_{\mathsf{C}}}\Big(\sum_{\ell} a_\ell \mathbf{x}_\ell\Big) = \sum_{\ell} a_\ell \mathbf{x}_\ell$$

Lemma

In the compute-and-forward framework, if you can recover the modulo sum, you can also recover the real sum (with high probability).

Successive Computation Illustration

We have the modulo sum.



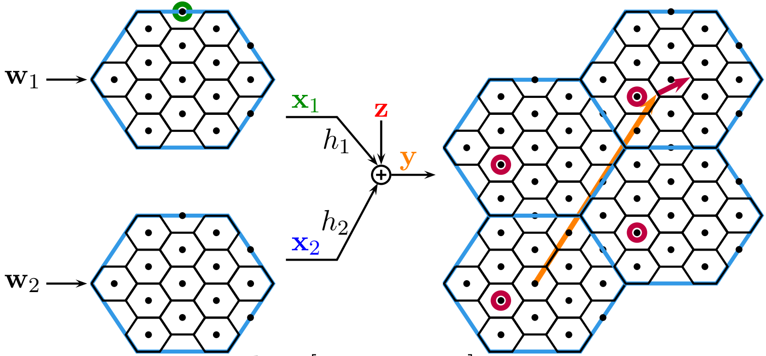
$$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

Effective noise: $\alpha^2 + \mathsf{SNR}\|\alpha\mathbf{h} - \mathbf{a}\|^2$

Successive Computation Illustration

Subtract modulo sum from the received signal.



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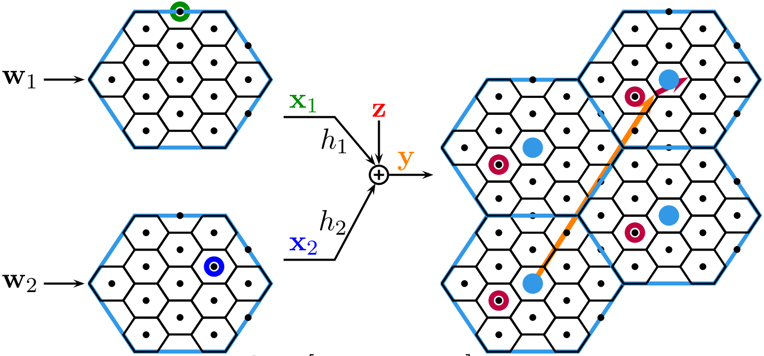
Successive Computation Illustration

Decode to the closest coarse lattice point.



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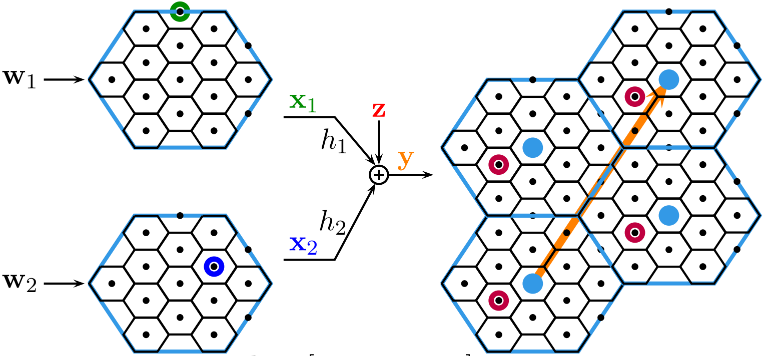

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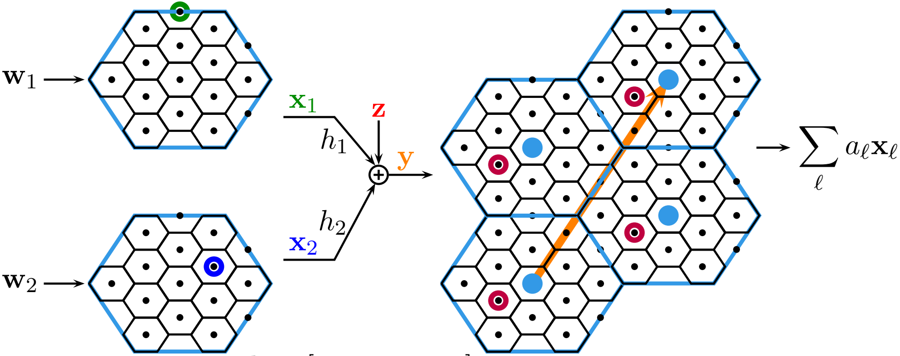
Successive Computation Illustration

Now we can infer the real sum.



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Successive Computation

- Receiver observes $\mathbf{y} = \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z}$

Successive cancellation:

- Decode $\mathbf{x}_i$.
- Calculate $\mathbf{y} - h_i \mathbf{x}_i$.
- Receiver now has

$$\sum_{\ell \neq i} h_\ell \mathbf{x}_\ell + \mathbf{z}$$

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Successive computation:

- Decode $\sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell$.
- Calculate $\mathbf{y} + \beta \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell$.
- Receiver now has

$$\sum_{\ell=1}^{L} (h_\ell + \beta a_\ell) \mathbf{x}_\ell + \mathbf{z}$$

Exact Sum-Rate Optimality

- **Key Idea:** Use recovered linear combinations to form a better effective channel for decoding subsequent linear combinations (as well as successive cancellation).

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$$\sigma^2_{\text{eff}}(\mathbf{h}, \mathbf{a}_m | \mathbf{a}_1, \ldots, \mathbf{a}_{m-1}) = \|\mathbf{C}_m^{\perp} \mathbf{F} \mathbf{a}_m\|^2$$

where $\mathbf{F} = \left(P^{-1}\mathbf{I} + \mathbf{h}\mathbf{h}^{\mathsf{T}}\right)^{-1/2}$ and $\mathbf{C}_m^{\perp}$ is the projection matrix for the nullspace of $\mathbf{F}[\mathbf{a}_1 \ \cdots \ \mathbf{a}_{m-1}]$.

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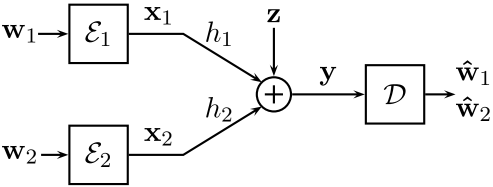
Theorem (Ordentlich-Erez-Nazer Allerton '13)

For any unimodular integer matrix $\mathbf{A} = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_K]^{\mathsf{T}} \in \mathbb{Z}^{K \times K}$ *with descending successive effective noise variances, we have that*

$$\sum_{m=1}^{K} \frac{1}{2} \log^+ \left(\frac{P}{\sigma^2_{\mathit{eff}}(\mathbf{h}, \mathbf{a}_m | \mathbf{a}_1, \ldots, \mathbf{a}_{m-1})} \right) = \frac{1}{2} \log \left(1 + \|\mathbf{h}\|^2 P\right) .$$

Moreover, there exists at least one permutation π *that associates each user's rate to a computation rate.*

Multiple-Access via Computation



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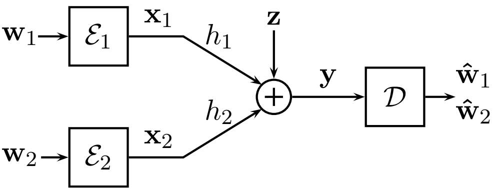
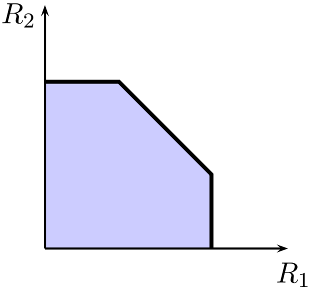
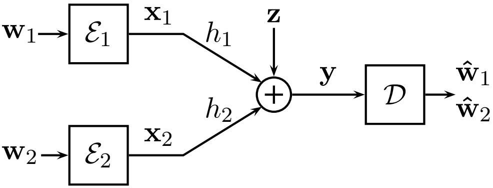
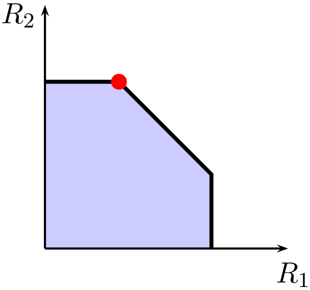




- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.





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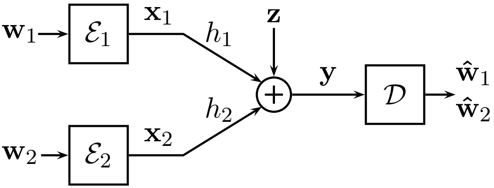

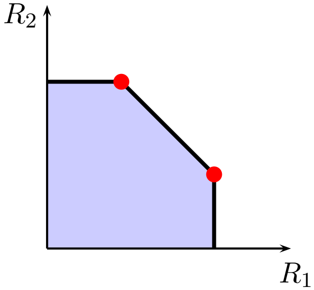

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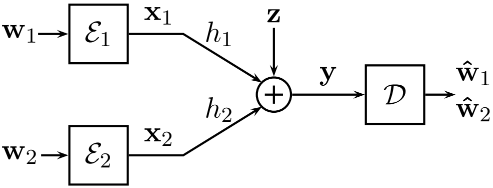

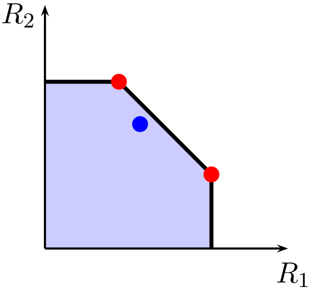

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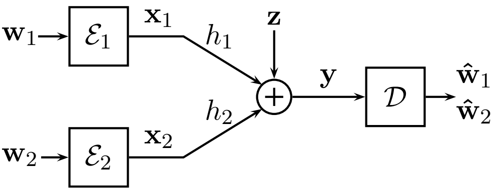

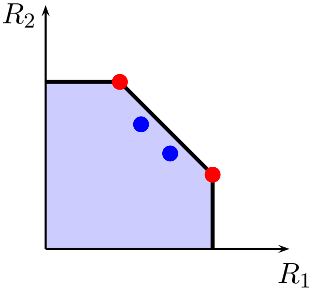

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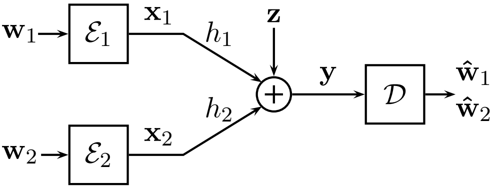

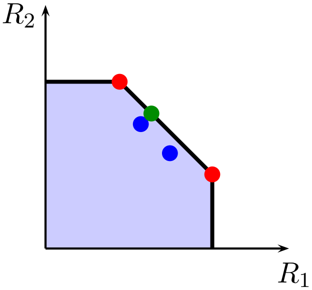

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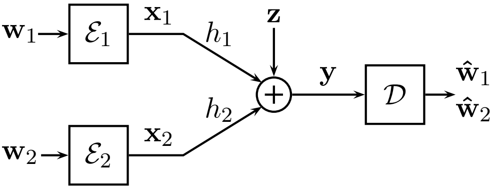

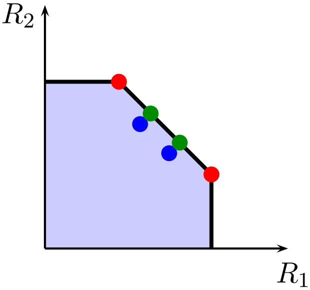

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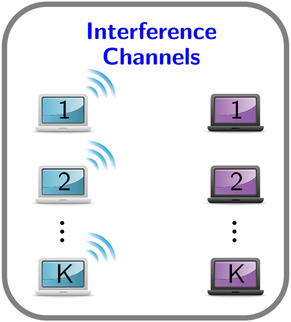
Road Map

- Warm-up: Compute-and-forward over finite field channels.
- Compute-and-forward over Gaussian channels.
- Applications to communication across single-hop Gaussian networks.



Joint work with:

Symmetric case: Or Ordentlich and Uri Erez.

Stream-by-stream case: Vasilis Ntranos, Viveck Cadambe, and Giuseppe Caire.

Interference-Free Capacity

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Time Division

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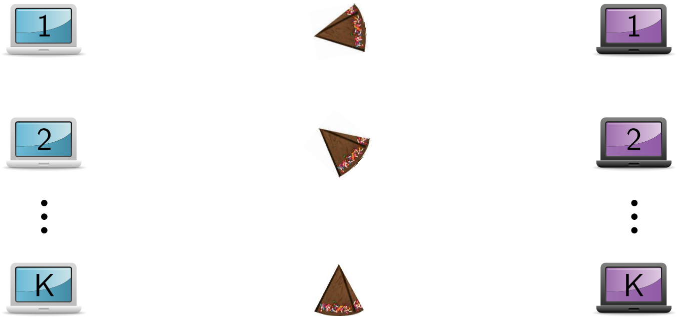

Interference Alignment

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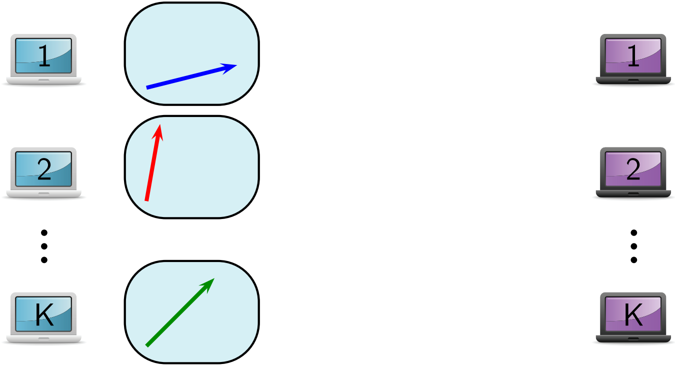
- **Birk-Kol '98:** Alignment for index coding. **Maddah-Ali - Motahari - Khandani '08:** Alignment for the MIMO X channel. See **Jafar '11** monograph (or recent e-book) for a richer history.



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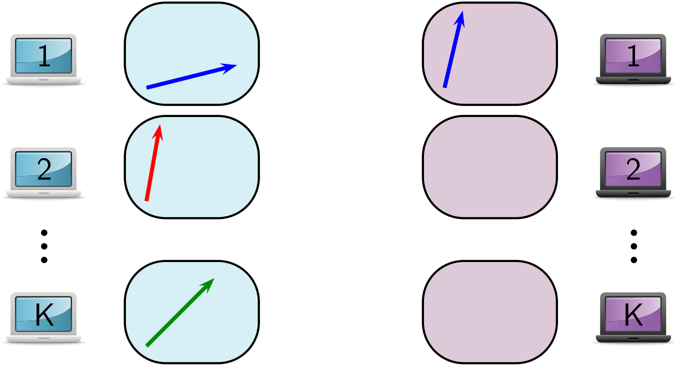

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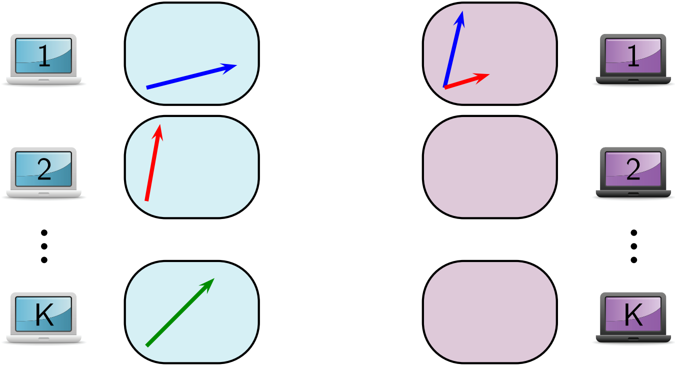

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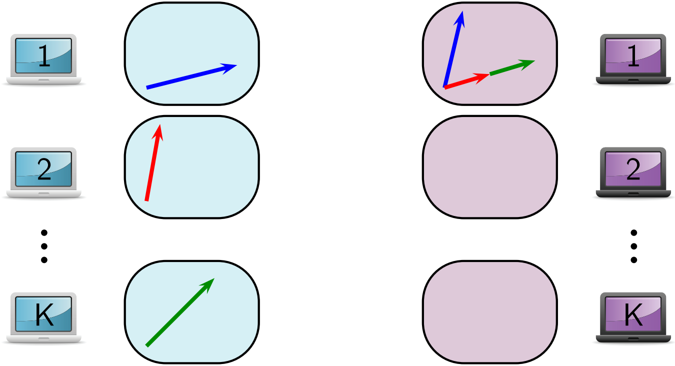

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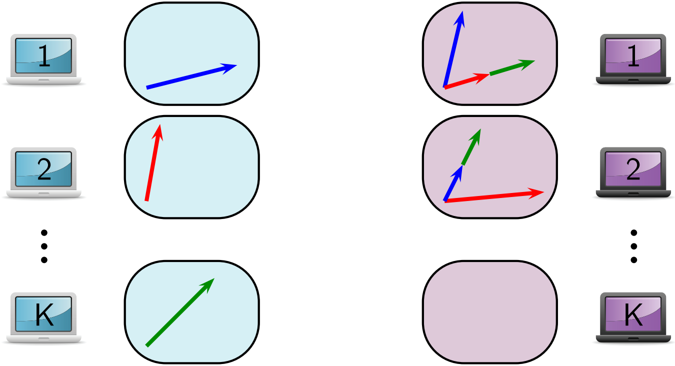

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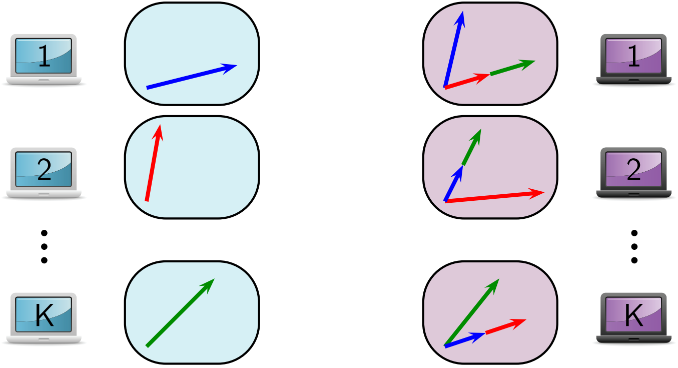

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- Very high SNR:
  - **Motahari, Gharan, Maddah-Ali, Khandani '09:** Real alignment. Achieves $\frac{K}{2}$ DoF over one channel realization using roughly 2^{K^2} codeword layers. Signal scale alignment.

Basic Coding Framework:

- Each transmitter has one or more data streams, each of which is drawn from an i.i.d. random codebook.
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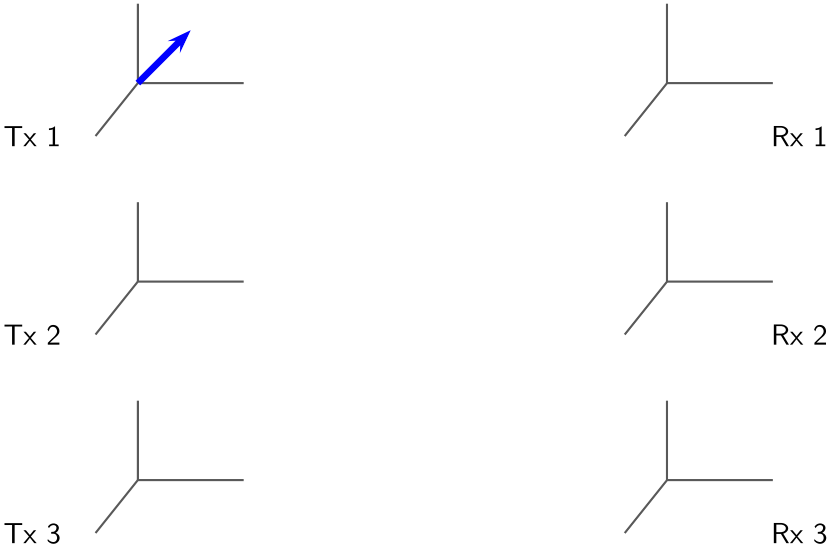

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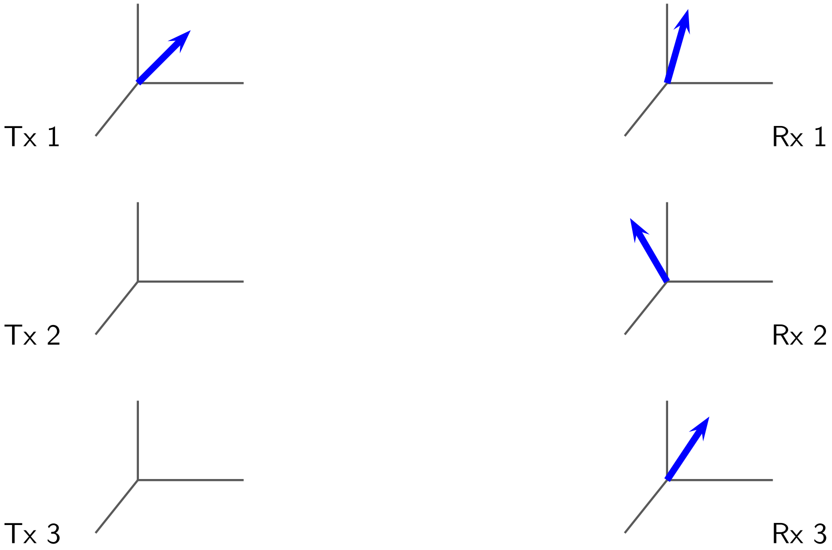

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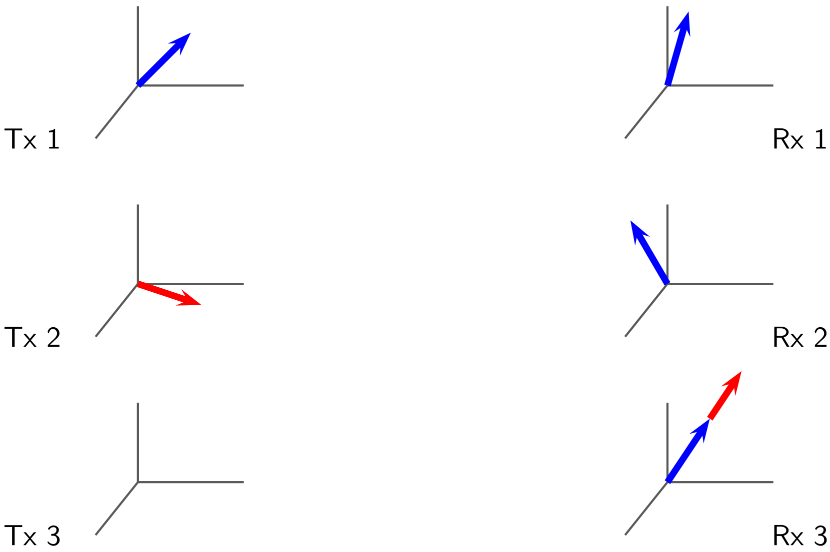

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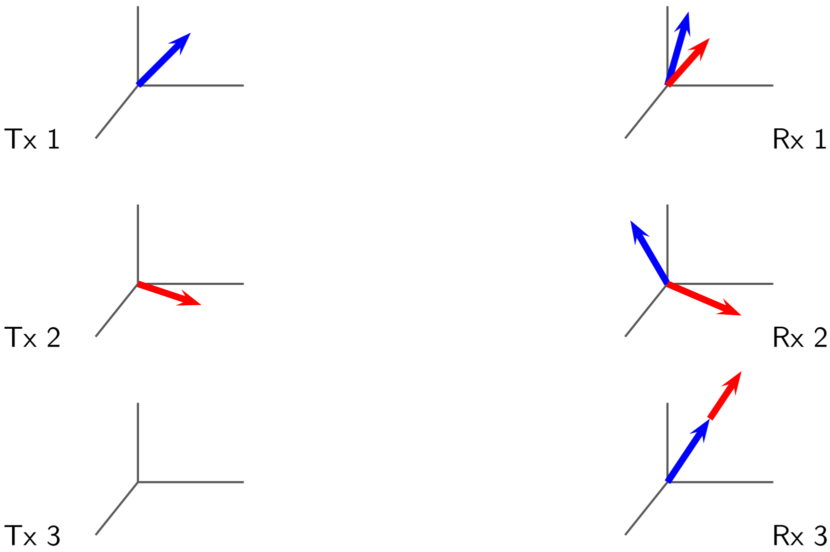

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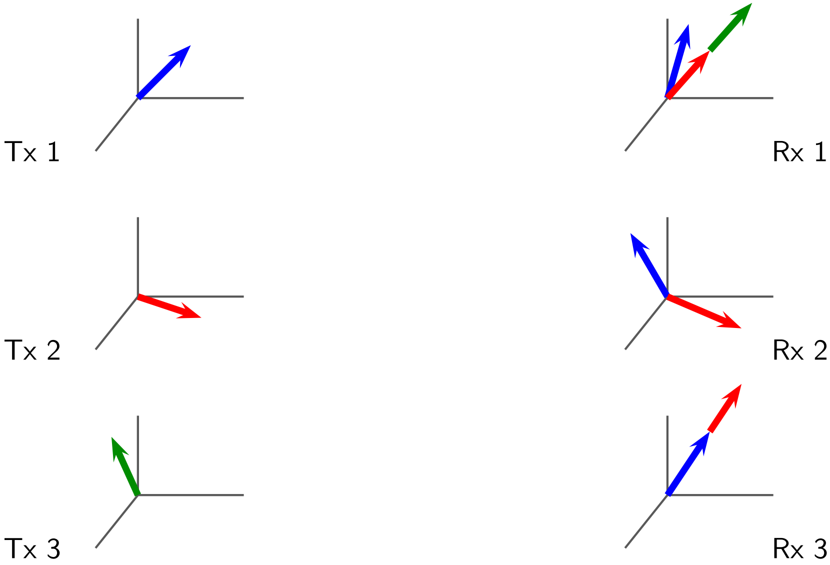

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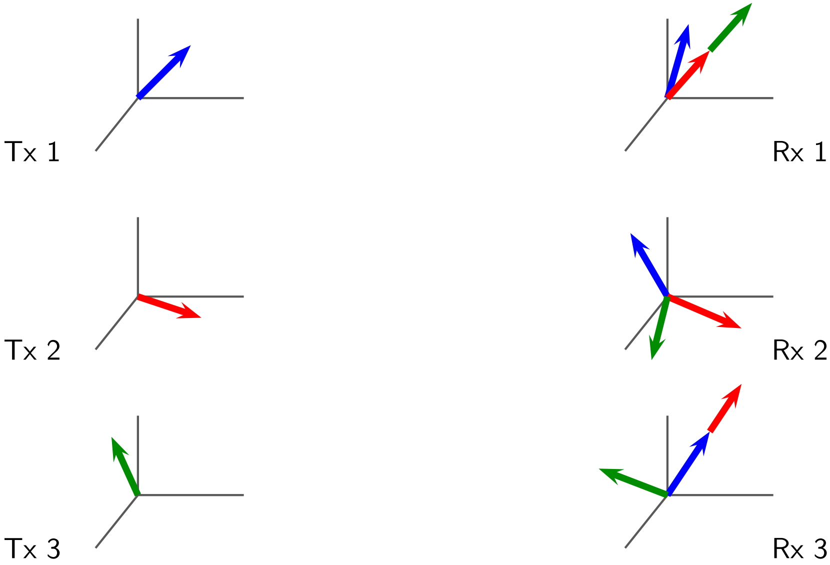

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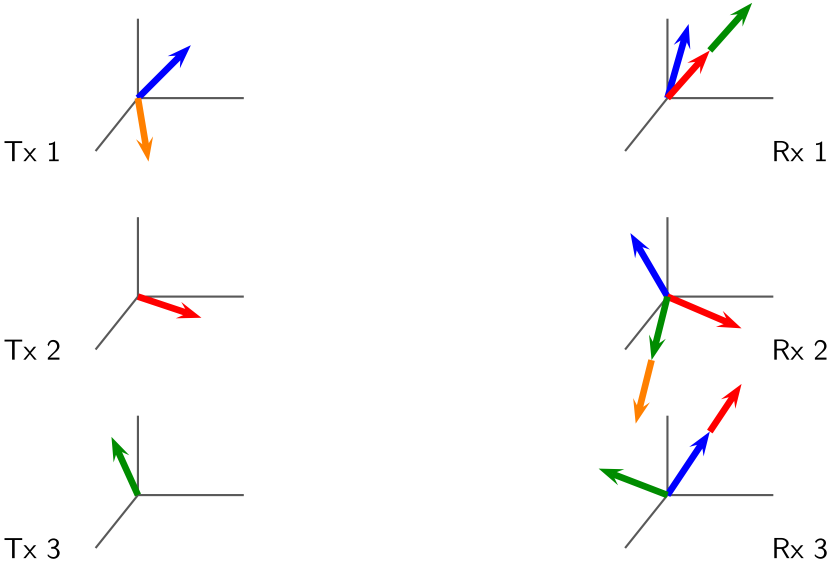

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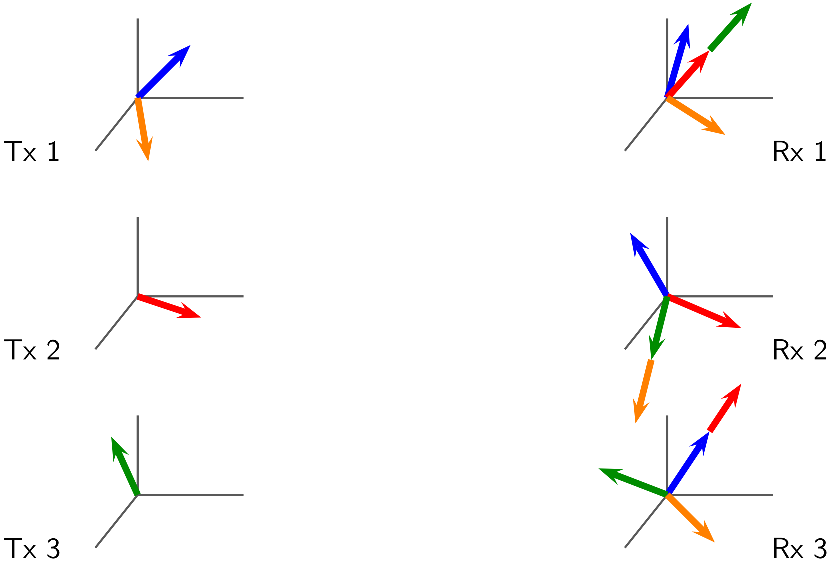

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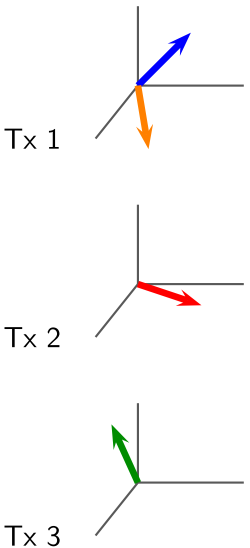
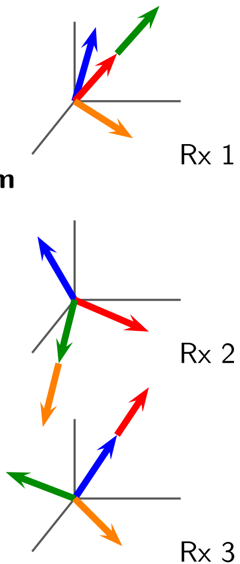


Total Degrees of Freedom

$$\text{DoF} = \frac{4 \text{ vectors}}{3 \text{ channel uses}} = \frac{4}{3}$$



Basic Coding Framework:

- Each transmitter has one or more codewords, each of which is drawn from a lattice codebook.
- Transmitter sends a linear combination of the codewords, with coefficients carefully chosen to align interference at the receivers.
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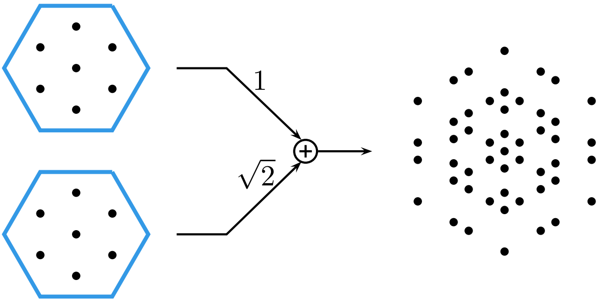
Disadvantages:

- Seems extremely sensitive to channel gains. DoF changes based on rationality/irrationality.
- Seems to require extremely high SNR.

Example: Two-User Lattice Alignment



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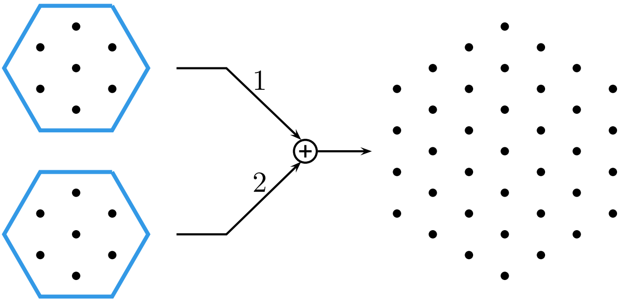

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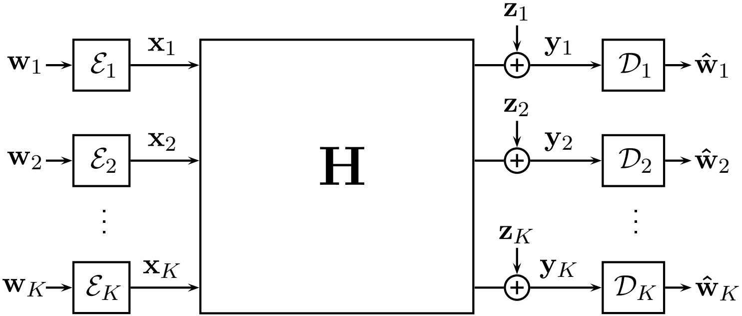
- If the ratio is rational, it is not always possible to uniquely identify the pair of codewords.

Symmetric K-User Gaussian Interference Channel



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- Signal scale alignment attains $K/2$ degrees-of-freedom for almost all channel gains, **Motahari et al. '09, Wu-Shamai-Verdu '11.**
- At finite SNR, the approximate capacity known in some special cases: two-user **Etkin-Tse-Wang '08**, many-to-one and one-to-many **Bresler-Parekh-Tse '10**, cyclic **Zhou-Yu '13**.

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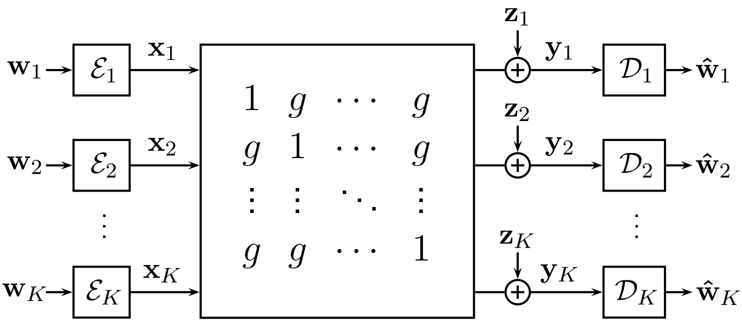

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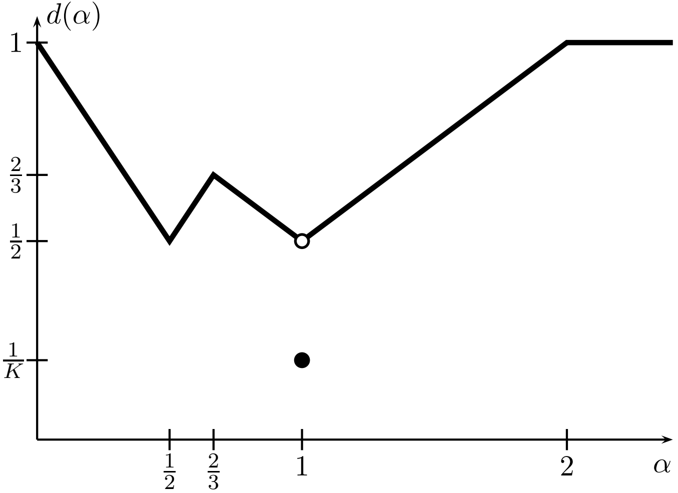
- Let's look at the symmetric case.



- Capacity understood in the high SNR regime. **Jafar-Vishwanath '10.**

$$\alpha = \frac{\log g^2 \mathsf{SNR}}{\log \mathsf{SNR}} \qquad d(\alpha) = \lim_{\mathsf{SNR}\to\infty} \frac{R(\mathsf{SNR})}{\frac{1}{2}\log \mathsf{SNR}}$$

Effective Multiple-Access Channel

- Each receiver sees an effective two-user multiple-access channel,

$$\mathbf{y}_k = \mathbf{x}_k + g \sum_{\ell \neq k} \mathbf{x}_\ell + \mathbf{z}_k \ .$$

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Joint Decoding:

- Direct analysis is hindered by dependencies between codeword pairs.
- Existing work only applies at very high SNR, **Ordentlich-Erez '13.**

Alignment via Two Equations

- **Ordentlich-Erez-Nazer '14:** Decode two linear combinations:

$$a_1 \mathbf{x}_k + a_2 \sum_{\ell \neq k} \mathbf{x}_\ell \qquad\qquad b_1 \mathbf{x}_k + b_2 \sum_{\ell \neq k} \mathbf{x}_\ell$$

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- Set of "bad rationals" depends on the SNR. Only rationals with denominator $\sqrt{\mathsf{SNR}}$ or smaller cause issues.

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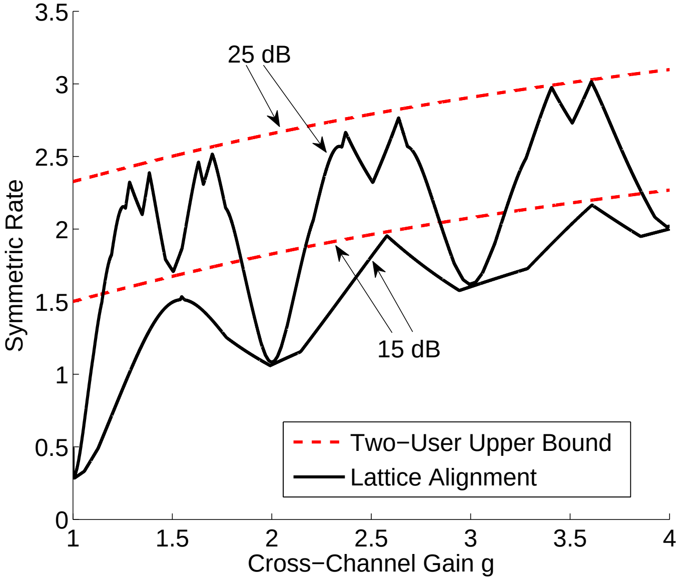

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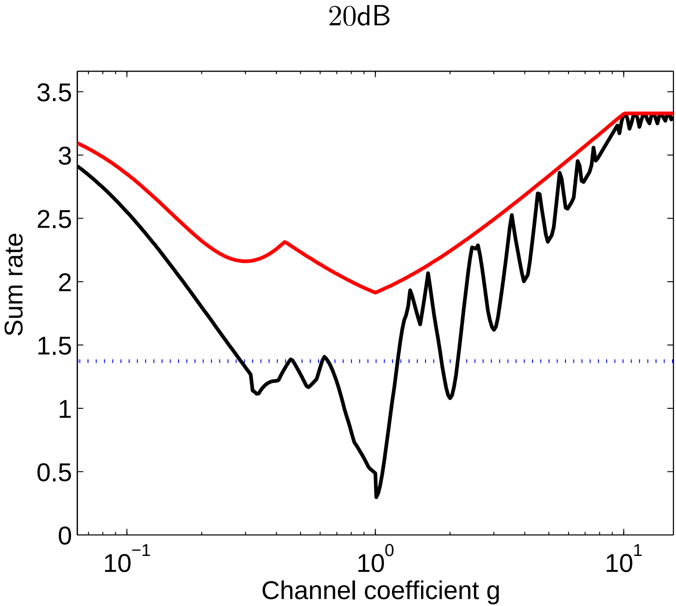
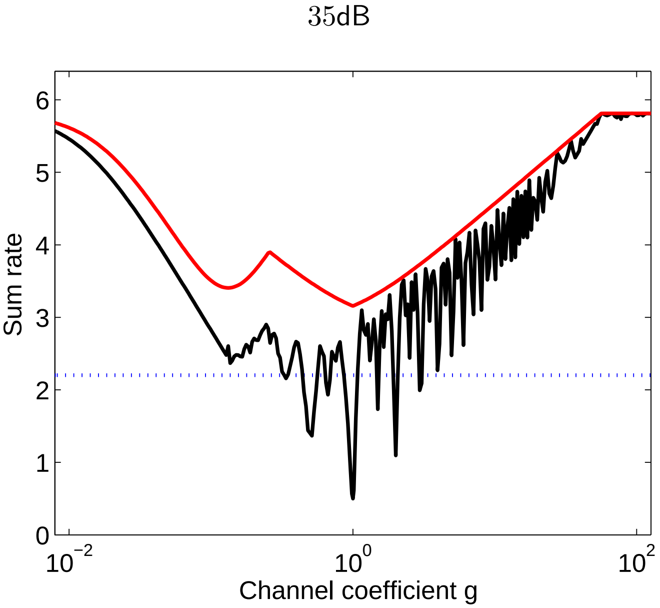
- **Ordentlich-Erez-Nazer '14:**
  - Noisy Regime: Decode one linear combination.
  - Moderately Weak and Weak Regimes: Send public and private lattice codewords. Decode three linear combinations.
  - Strong and Very Strong Regime: Decode two linear combinations.

Symmetric K-User Gaussian Interference Channel




35dB
Sum rate
Channel coefficient g
0
1
2
3
4
5
6
10^{-2}
10^{0}
10^{2}

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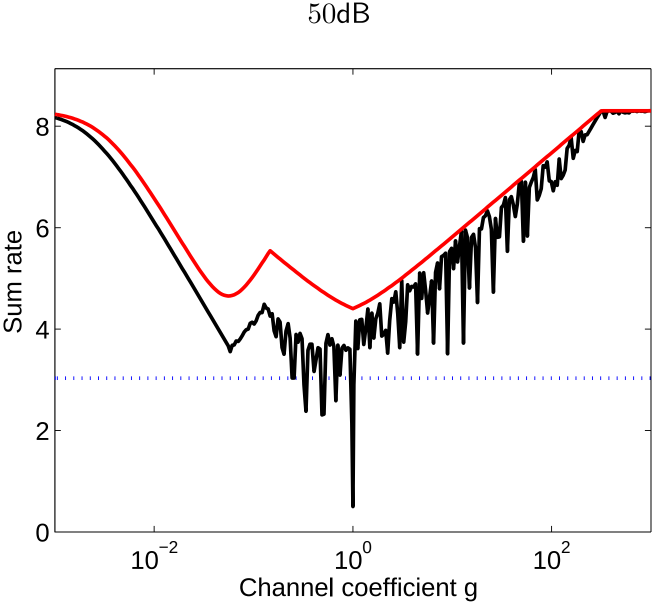

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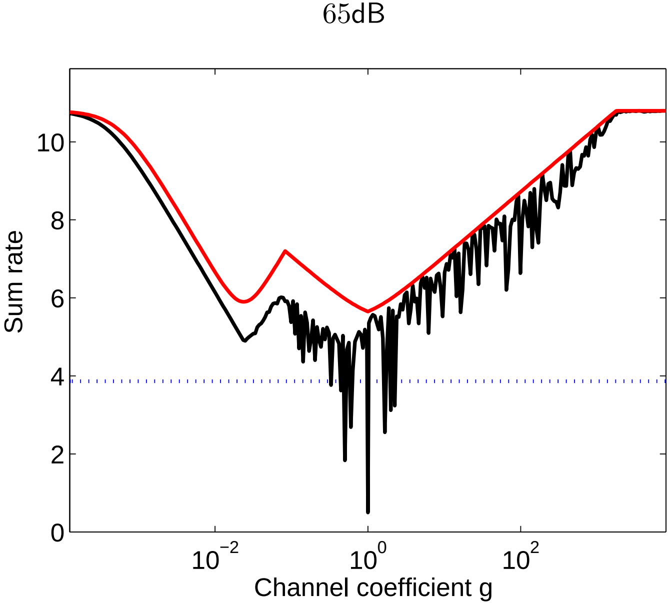

Approximate Capacity Results: Strong Regime

- Using the fact that the sum of the computation rates is nearly equal to the multiple-access sum capacity, we can approximate the sum capacity of the symmetric K-user Gaussian interference channel in all regimes.

$$R_{\mathsf{sym}} > \frac{1}{2}\log\left(1 + (1+2g^2)\mathsf{SNR}\right) - \max_{\mathbf{a}\in\mathbb{Z}^2} R_{\mathsf{comp}}\left([1 \ g]^{\mathsf{T}}, \mathbf{a}\right) - 1$$

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- **Sample Result:** In the strong interference regime,

$$\frac{1}{4}\log^{+}(g^2\mathsf{SNR}) - \frac{c}{2} - 3 \le C_{\mathsf{sym}} \le \frac{1}{4}\log^{+}(g^2\mathsf{SNR}) + 1$$

  for all channel gains except for an outage set whose measure is a fraction of 2^{-c} of the interval $1 < |g| < \sqrt{\mathsf{SNR}}$, for any $c > 0$.

Integer-Forcing Interference Alignment

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- Yields a new achievable rate region for any scenario which employs "stream-by-stream" alignment.

Problem Setting:

- Multiple data streams (i.e., codewords) $\mathbf{s}^{[\ell]} \in \mathbb{C}^T$, each assigned to its own beamforming vector $\mathbf{v}^{[\ell]} \in \mathbb{C}^M$.

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$$\mathbf{Y} = \sum_{\ell=1}^{L} \mathbf{H}_{\mathrm{D}}^{[\ell]} \mathbf{v}_{\mathrm{D}}^{[\ell]} \big(\mathbf{s}_{\mathrm{D}}^{[\ell]}\big)^{\mathsf{T}} + \sum_{j=1}^{J} \sum_{\ell=1}^{L} \mathbf{H}_{\mathrm{I}}^{[j,\ell]} \mathbf{v}_{\mathrm{I}}^{[j,\ell]} \big(\mathbf{s}_{\mathrm{I}}^{[j,\ell]}\big)^{\mathsf{T}} + \mathbf{Z} \ .$$

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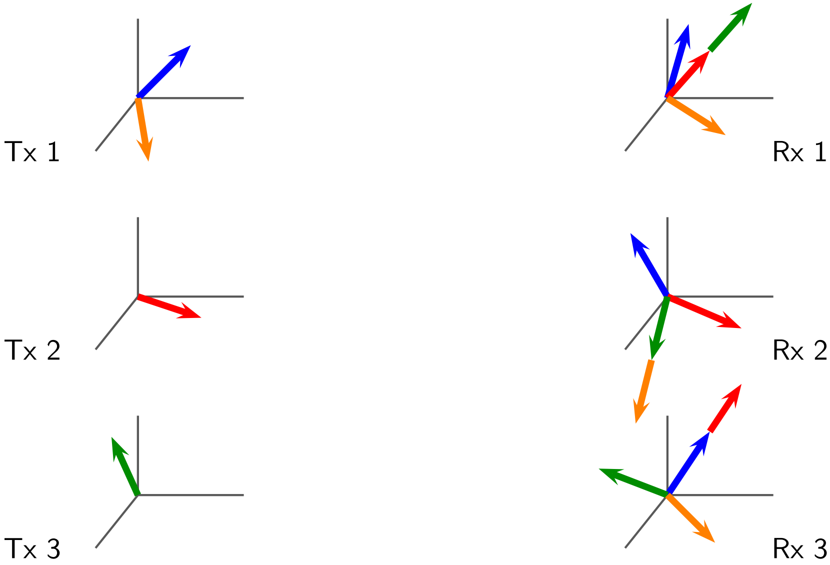
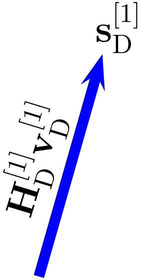
Example: Cadambe-Jafar '08 over 3 Channel Realizations



Received Signal



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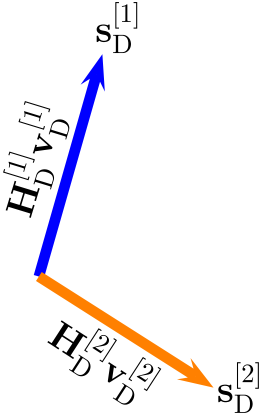

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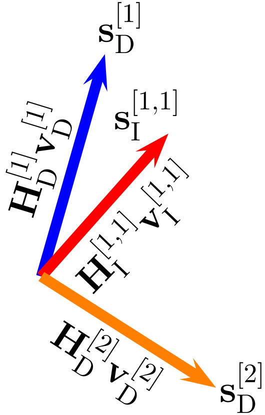

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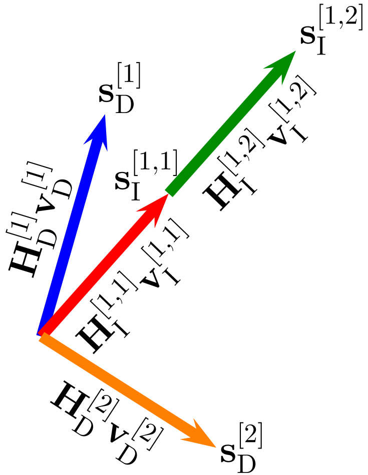

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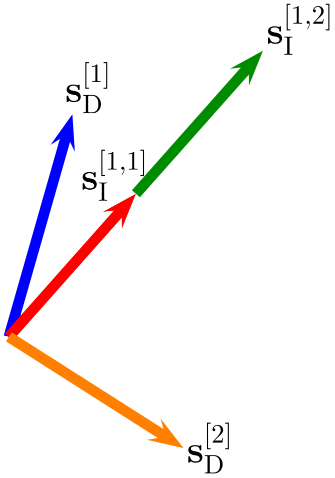

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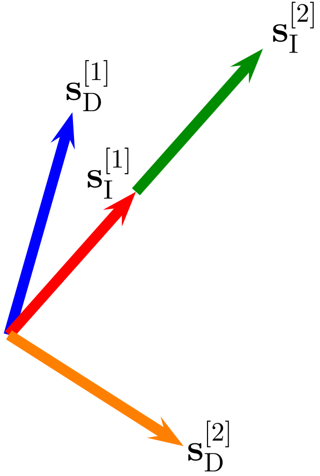

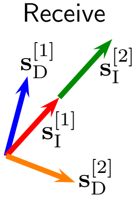

How should each receiver decoder its desired data streams?

Zero-Forcing Interference Alignment:

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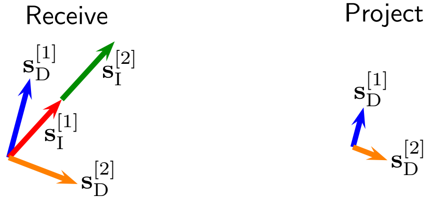

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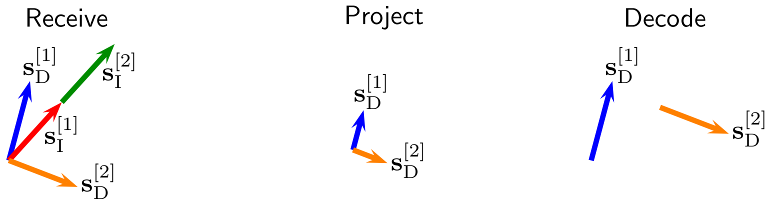

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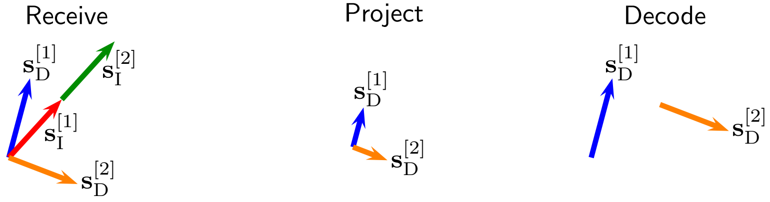

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- Suffices from a degrees-of-freedom perspective.

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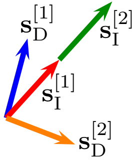

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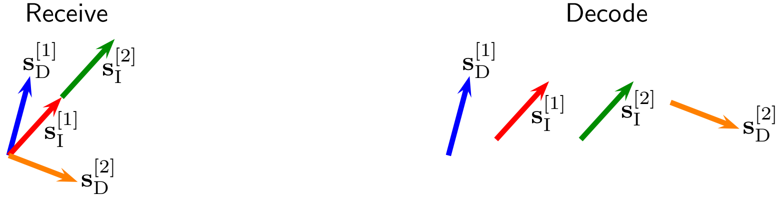
Joint Decoding (with i.i.d Random Codes)



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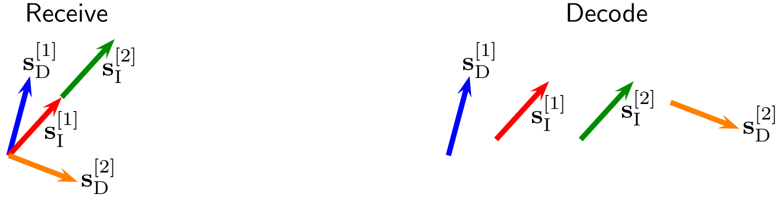

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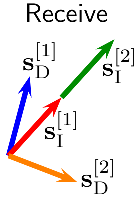
- Analyzing lattice-coded data streams is beyond the reach of current techniques owing to dependencies.



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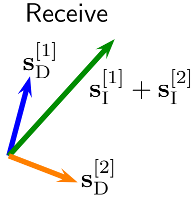

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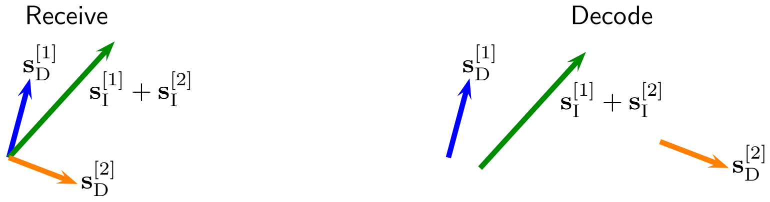

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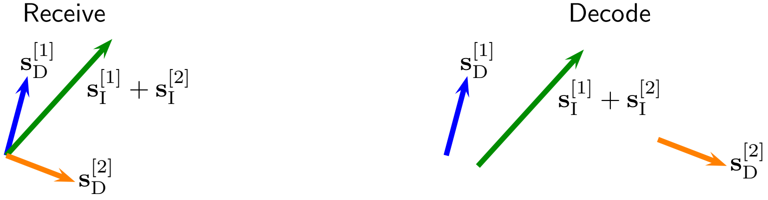

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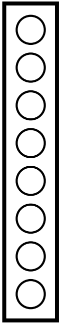
- Each codeword is assigned an effective noise tolerance $\sigma^2_{\mathsf{eff},\ell}$ and power level P_ℓ. Rate is $\frac{1}{2}\log(P_\ell/\sigma^2_{\mathsf{eff},\ell})$.



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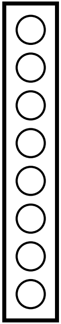

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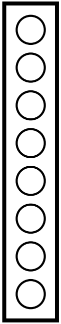

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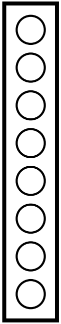

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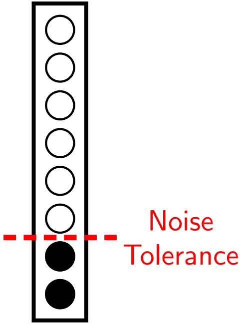

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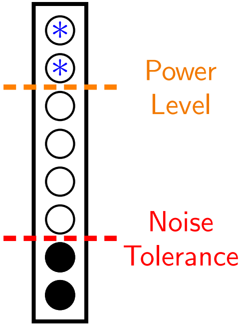

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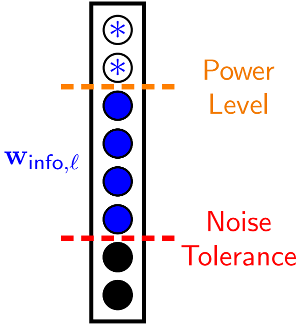

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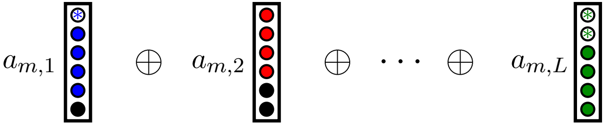

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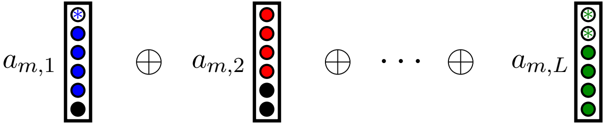


- In both cases, the linear combination with coefficient vector $\mathbf{a}_m^{\mathsf{T}} = [a_{m1} \ a_{m2} \ \cdots \ a_{mL}]$ can be decoded reliably if

$$\sigma^2_{\mathsf{eff},\ell} > \mathbf{a}_m^{\mathsf{T}} \big(\mathbf{P}^{-1} + \mathbf{H}\mathbf{H}^{\mathsf{T}} \big)^{-1} \mathbf{a}_m$$

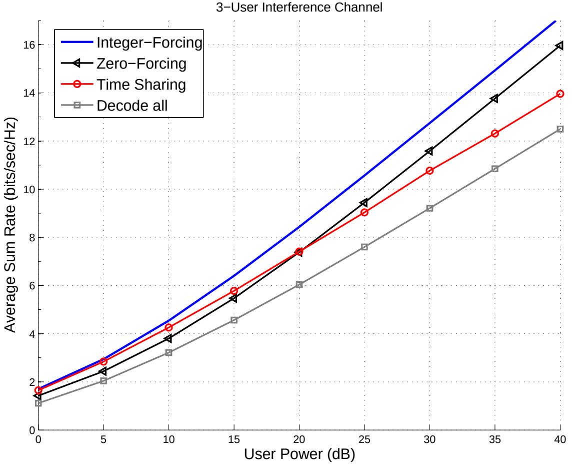
for all ℓ such that $a_{m\ell} \neq 0$.

Performance Comparison

- 3-user Gaussian interference channel.
- Can code over 3 independent fading realizations from an i.i.d. Rayleigh distribution.

Strategies:

- CJ '08 Beamforming $+$ Zero-Forcing Decoding.
- CJ '08 Beamforming $+$ Integer-Forcing Decoding.



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- and many more (such as joint decoding, non-unique decoding)...

Recent coding perspectives on compute-and-forward:

- **Feng-Silva-Kschischang '13:** General algebraic framework in terms of lattice partitions and R-modules.
- **Hern-Narayanan '13, Huang-Narayanan-Tunali '14:** Multilevel codes.
- **Ordentlich-Erez '12, Yang et al. '12:** Binary convolutional codes.
- **Hong and Caire '11, Ordentlich et al. '11:** Binary and p-ary LDPC codes.
- **Belfiore-Ling '12:** Code design criteria.
- **Tunali-Narayanan-Pfister '13:** Spatially-coupled LDPC codes.

Some topics we did not have a chance to cover:

- Distributed Source Coding: **Körner-Marton '79**, **Krithivasan-Pradhan '09,'11**, **Wagner '11**, **Tse-Maddah-Ali '10**
- Relaying: **Wilson-Narayanan-Pfister-Sprintson '10**, **Nam-Chung-Lee '10, '11**, **Goseling-Gastpar-Weber '11**, **Song-Devroye '13**, **Nokleby-Aazhang '12**
- Cellular Networks: **Sanderovich-Peleg-Shamai '11**, **Nazer-Sanderovich-Gastpar-Shamai '09**, **Hong-Caire '13**
- Distributed Dirty-Paper Coding: **Philosof-Zamir '09**, **Philosof-Zamir-Erez-Khisti '11**, **Wang '12**
- Joint Source-Channel Coding: **Kochman-Zamir '09**, **Nazer-Gastpar '07, '08**, **Soundararajan-Vishwanath '12**
- Physical-Layer Secrecy: **He-Yener '11, '14**, **Kashyap-Shashank-Thangaraj '12**

Concluding Remarks

- Even if you only want to recover messages, it can help to first decode linear combinations.
- Compute-and-forward creates a direct link between Gaussian interference networks and finite field ones.
- Enables more efficient encoding/decoding for networks where the capacity is already known.
- Yields new achievable rates for interference channels.
- Broader story: Algebraic Structure in Network Information Theory. ISIT '11 Tutorial. Survey on physical-layer network coding in Proceedings of the IEEE, March 2011.
- Upcoming textbook by Ram Zamir.

References - Intro

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