# Computing over Multiple-Access Channels with Connections to Wireless Network Coding

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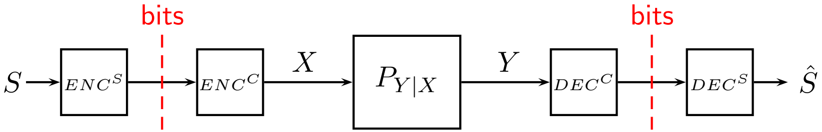
# Outline

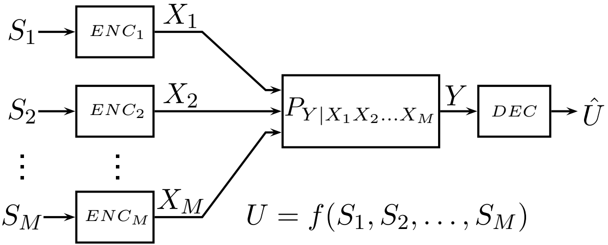
1. Source-Channel Separation Theorems
2. Computation Coding
3. Three Illustrative Examples
4. Multicast Capacity for Finite Field MAC Networks

# Source-Channel Separation Theorems (or lack thereof)



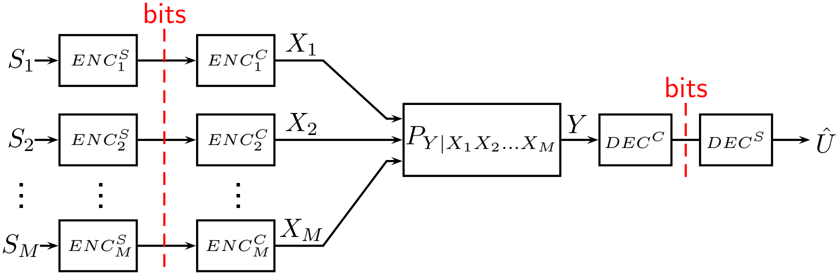

- Shannon (1948): separation optimal for point-to-point links
  - Reliable communication possible if $H(S) < \max_{p(x)} I(X;Y)$
- Cover-El Gamal-Salehi (1980): no separation theorem for multiple-access channels (MACs) with correlated sources.
- Is there a separation theorem for sending functions over MACs?

# Problem Statement: Reliable Computation over MACs




- $M$ users each observe a source
- Only want a function of the sources, $U = f(S_1, S_2, \ldots, S_M)$
- Reliable computation: $\lim_{k\to\infty} P(\hat{U}^k \neq U^k) = 0$
- Computation rate, $\kappa = \frac{k}{n}$, functions sent per channel use
- No separation theorem *even if the sources are independent*

# Separation-Based Scheme

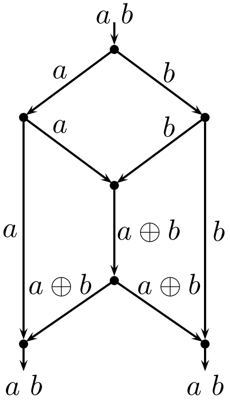



- $\mathcal{R}_U$: distributed compression rate region, only known in special cases (Slepian-Wolf, Körner-Marton)
- $\mathcal{R}_{\text{MAC}}$: MAC capacity region
  - 2-user MAC: convex closure of all $(R_1, R_2)$ satisfying:

    $$R_1 < I(X_1; Y|X_2) \quad R_2 < I(X_2; Y|X_1) \quad R_1 + R_2 < I(X_1, X_2; Y)$$

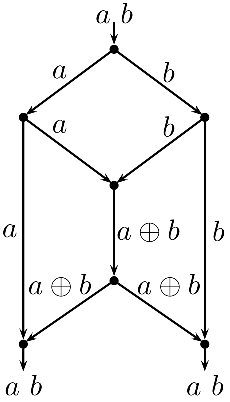
  - Separation-based computation possible if $\mathcal{R}_U \cap \mathcal{R}_{\text{MAC}} \neq \emptyset$

# Routing Is Suboptimal




- Butterfly example
- One source, two sinks, and noiseless rate $1$ links
- Max-flow min-cut bound $= 2$ bits per network use
- Routing suboptimal, only transmits $1.5$ bits
- Mixing optimal, send $a \oplus b$ down center path

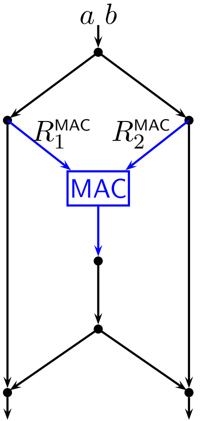
# Network Coding




- Ahlswede-Cai-Li-Yeung (2000): for any network of bit pipes, network coding achieves the max-flow min-cut bound
- Later works have generalized the setup and simplified the coding scheme:
  - Li-Yeung-Cai (2003): linear network coding
  - Ho et al., Jaggi et al., Sanders et al. (2003): distributed construction, bounds on field size
- Channel-network separation is optimal for multicasting over a network of point-to-point channels

# A Simple Network with a MAC

- Add a MAC into the usual butterfly network
- Separation can send $a$ and $b$ iff $R_1^{\text{MAC}} \geq 1$ and $R_2^{\text{MAC}} \geq 1$
- What if the MAC just takes the mod-2 sum: $Y = X_1 \oplus X_2$
- Uncoded transmission optimal
- Separation-based coding suboptimal
  - Must transmit $a$ and $b$ individually
  - $R_1^{\text{MAC}} + R_2^{\text{MAC}} \leq 1$
- What if there is noise?



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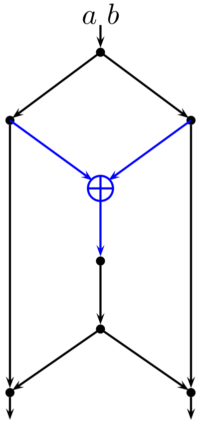

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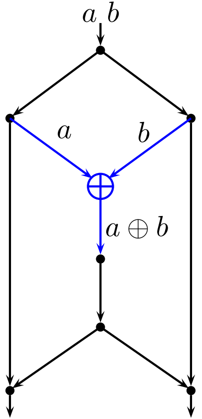
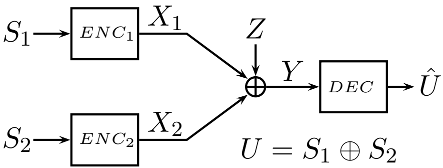


# A New Coding Technique: Computation Coding

- Many MACs compute a function of the sources then add noise (ex: Gaussian MAC)
- Uncoded transmission achieves rate gains but incurs a noise penalty
- Trick: Use structured codes
- Example: Mod-$2$ Noisy Adder MAC (M2MAC)



# Computation Coding for the M2MAC: Proof Sketch

- Only want $U = S_1 \oplus S_2$ at the decoder
- *Key idea:* Use the same linear codebook at each encoder
- Pick good source and channel coding matrices $\mathbf{H}$ and $\mathbf{G}$

$$\mathbf{x_1} = \mathbf{s_1 H G}$$
$$\mathbf{x_2} = \mathbf{s_2 H G}$$
$$\mathbf{y} = \mathbf{s_1 H G} \oplus \mathbf{s_2 H G} \oplus \mathbf{z} = \mathbf{u H G} \oplus \mathbf{z}$$

- After the channel, looks as if $U$ was jointly encoded
- Relies on low complexity codes

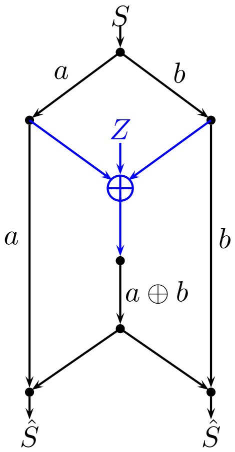
# Example 1: Butterfly Network with an M2MAC

- Add the M2MAC into a butterfly network
- Separation-based scheme requires:

$$H(S) < 1 + \left( \frac{1 - h_B(p)}{2} \right)$$

- Computation coding is optimal and meets the max-flow min-cut bound:

$$H(S) < 1 + (1 - h_B(p))$$



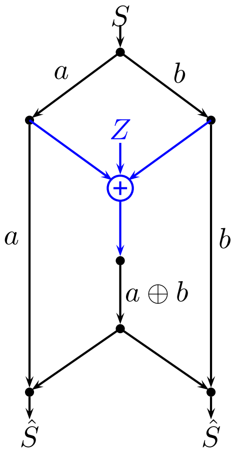
# Example 2: Not Just MACs over Fields

- Butterfly network with real-adder MAC
- Channel output is $X_1 + X_2$ passed through a symmetric DMC
- Example: Crossover probability $0.2$
- Separation-based scheme requires:

$$H(S) < 1.33$$

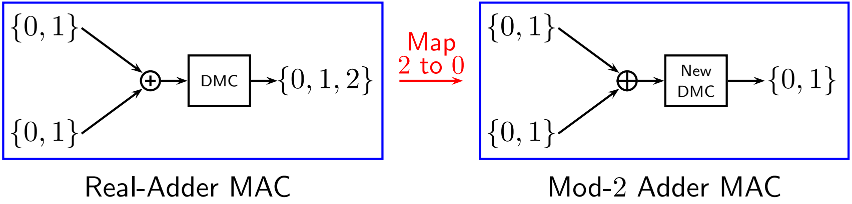
- Computation coding scheme:

$$H(S) < 1.40$$



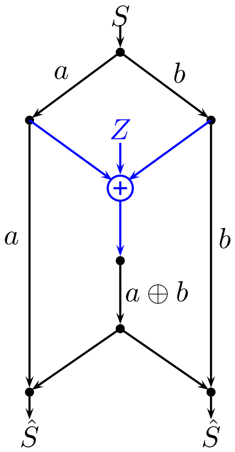
# Scheme: Transform Real-Adder to Mod-2 Adder

- Real-adder MAC is nearly a mod-2 adder
- Just map output symbol 2 to 0
- Uniform distribution is a good input distribution

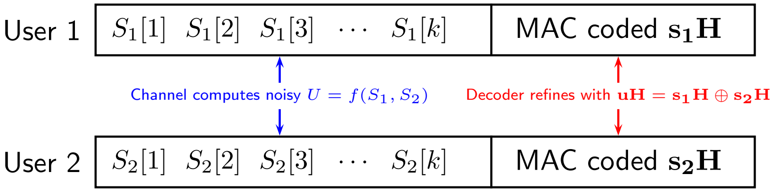


# What about Gaussian?

- Butterfly network with Gaussian MAC:
  $Y = X_1 + X_2 + Z, \qquad Z \sim \mathcal{N}(0, \frac{1}{3})$
- Usual power constraint on the inputs, $P = 1$
- Mapping to a mod-$2$ adder MAC does worse than separation
- Reason: bad input distribution
- Need a better strategy



# Systematic Computation




- Phase 1: Uncoded transmission. Receiver gets a noisy version of function.
- Phase 2: Separation-based scheme sends linear update bins
- Binning only needs a mapping to a linear function over *some* field
- Trades off between using channel function and optimal MAC input distribution

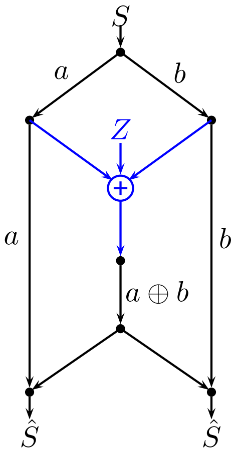
# Systematic Computation for the Gaussian MAC

- Butterfly network with Gaussian MAC:
  $Y = X_1 + X_2 + Z, \qquad Z \sim \mathcal{N}(0, \frac{1}{3})$
- Usual power constraint on the inputs, $P = 1$
- Separation-based scheme requires:

$$H(S) < 1.70$$

- Systematic computation coding scheme:

$$H(S) < 1.76$$



# Multicasting over Finite Field MAC Networks

- Usual network coding setup: Single source, $L$ receivers, encoder/decoder nodes, and directed point-to-point channels
- Add any number of MACs of the form:
  - $Y = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_M X_M + Z$
  - $\alpha_i \in \mathbb{F} \setminus \{0\}$
  - $X_i, Z \in \mathbb{F}$
  - Addition over $\mathbb{F}$
- All MACs operate over the same field
- No broadcast constraint

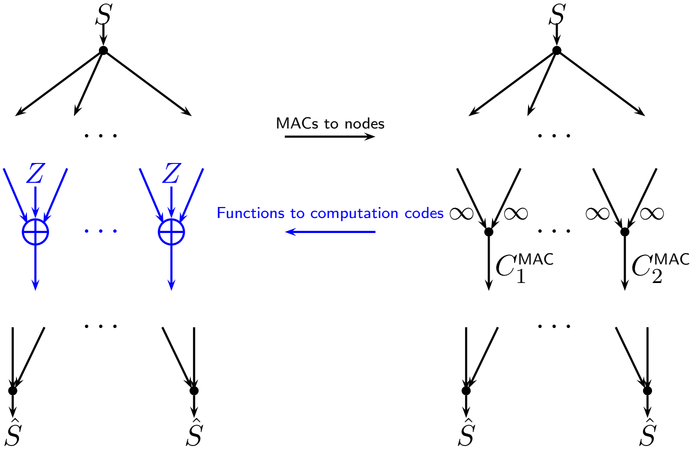
**Theorem**

*If the field size of the MACs, $|\mathbb{F}|$, is larger than the number of receivers, $L$, then the multicast capacity is given by the max-flow min-cut bound.*

# Proof Outline

- Need result of Ho et al. (2003): Linear solution for a network of directed point-to-point links exists if the alphabet size is larger than the number of receivers
- Apply network transformation:
  - Replace each MAC with a node whose output has a capacity equal to the original sum-rate capacity
  - Each incoming link to the original MAC is replaced with an infinite capacity link to the new node
- Find a linear solution for this network
- Find a computation code for each MAC to duplicate the function of its replacement

# Proof Outline



# Conclusions

- New technique: computation coding
  - Sometimes optimal, often helps
  - Relies on low complexity, structured codes
- Interference useful for wireless network coding
- Structural considerations may be necessary in large networks