Algebraic Structure in Network Information Theory

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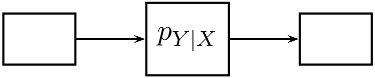
Motivation



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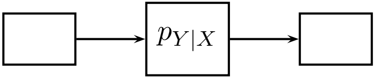

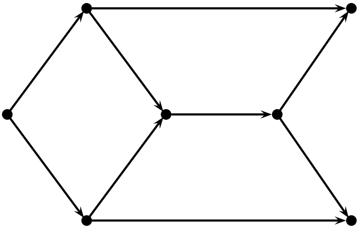

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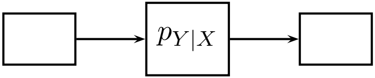

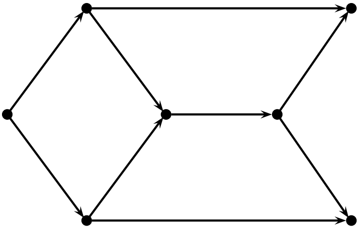

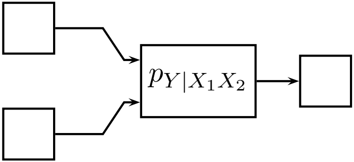

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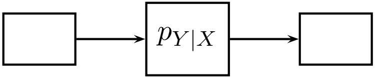
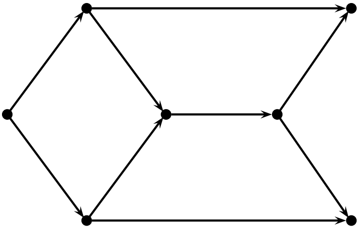
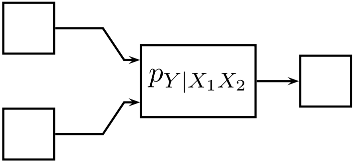
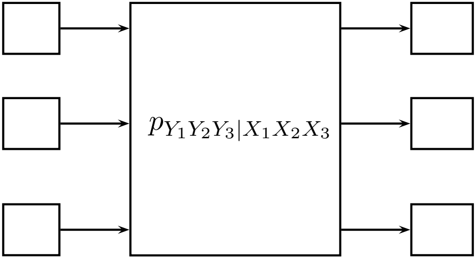








Disclaimer

In the interest of telling a certain story,

- this tutorial does not attempt to provide an authoritative chronological account of the results;
- this tutorial does not claim to be complete (although a certain effort in this direction was made);

What This Tutorial Is Not *About*

We will *not* address the following very interesting questions (and apologize for a potentially misleading title):

- Complexity of coding schemes
- New families of algebraic codes
- Algebraic coding theory
-

What This Tutorial Is *About*

- Achievable rates that seem out of reach for "classical" arguments.
- Novel communication strategies where algebraic arguments appear to be of key importance.
- Recipes for how to apply these strategies to networks.
- Elements missing from Information Theory books.

I. Discrete Alphabets

II. AWGN Channels

III. Network Applications

Point-to-Point Channels

$$\mathbf{w} \longrightarrow \boxed{\mathcal{E}} \xrightarrow{\mathbf{x}} \boxed{p_{Y|X}} \xrightarrow{\mathbf{y}} \boxed{\mathcal{D}} \longrightarrow \hat{\mathbf{w}}$$

The Usual Suspects:

- Message $\mathbf{w} \in \{0,1\}^k$
- Encoder $\mathcal{E} : \{0,1\}^k \to \mathcal{X}^n$
- Input $\mathbf{x} \in \mathcal{X}^n$
- Estimate $\hat{\mathbf{w}} \in \{0,1\}^k$
- Decoder $\mathcal{D} : \mathcal{Y}^n \to \{0,1\}^k$
- Output $\mathbf{y} \in \mathcal{Y}^n$

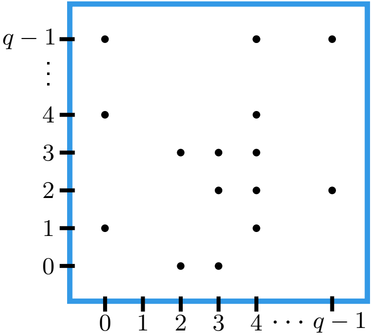
- Memoryless Channel $p(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^{n} p(y_i|x_i)$
- Rate $R = \dfrac{k}{n}$.
- (Average) Probability of Error: $\mathbb{P}\{\hat{\mathbf{w}} \neq \mathbf{w}\} \to 0$ as $n \to \infty$. Assume $\mathbf{w}$ is uniform over $\{0,1\}^k$.

i.i.d. Random Codes

- Generate 2^{nR} codewords $\mathbf{x} = [X_1 \ X_2 \ \cdots \ X_n]$ independently and elementwise i.i.d. according to some distribution p_X

$$p(\mathbf{x}) = \prod_{i=1}^{n} p_X(x_i)$$

- Bound the average error probability for a random codebook.
- If the average performance over codebooks is good, there must exist at least one good fixed codebook.



(Weak) Joint Typicality

- Two sequences $\mathbf{x}$ and $\mathbf{y}$ are (weakly) jointly typical if

$$
\begin{aligned}
\left| -\frac{1}{n}\log p(\mathbf{x}) - H(X) \right| &< \epsilon \\
\left| -\frac{1}{n}\log p(\mathbf{y}) - H(Y) \right| &< \epsilon \\
\left| -\frac{1}{n}\log p(\mathbf{x},\mathbf{y}) - H(X,Y) \right| &< \epsilon
\end{aligned}
$$

- For our considerations, weak typicality is convenient as it can also be stated in terms of differential entropies.

- If $\mathbf{x}$ and $\mathbf{y}$ are i.i.d. sequences, the probability that they are jointly typical goes to 1 as n goes to infinity.

Decoder looks for a codeword that is jointly typical with the received sequence $\mathbf{y}$

Error Events

1. Transmitted codeword $\mathbf{x}$ is not jointly typical with $\mathbf{y}$.
   $\Longrightarrow$ Low probability by the Weak Law of Large Numbers.
2. Another codeword $\tilde{\mathbf{x}}$ is jointly typical with $\mathbf{y}$.

Cuckoo's Egg Lemma

Let $\tilde{\mathbf{x}}$ be an i.i.d. sequence that is independent from the received sequence $\mathbf{y}$.

$$\mathbb{P}\Big\{(\tilde{\mathbf{x}}, \mathbf{y}) \text{ is jointly typical}\Big\} \leq 2^{-n(I(X;Y)-3\epsilon)}$$

See **Cover and Thomas**.

- We can upper bound the probability of error via the union bound:

$$\begin{aligned}\mathbb{P}\{\hat{\mathbf{w}} \neq \mathbf{w}\} &\leq \sum_{\tilde{\mathbf{w}} \neq \mathbf{w}} \mathbb{P}\Big\{(\mathbf{x}(\tilde{\mathbf{w}}), \mathbf{y}) \ \text{ is jointly typical.}\Big\} \\ &\leq 2^{-n(I(X;Y)-R-3\epsilon)} \qquad \leftarrow \text{Cuckoo's Egg Lemma}\end{aligned}$$

- If $R < I(X;Y)$, then the probability of error can be driven to zero as the blocklength increases.

Theorem (Shannon '48)

The capacity of a point-to-point channel is $C = \max_{p_X} I(X;Y)$.

- Linear Codebook: A linear map between messages and codewords (instead of a lookup table).

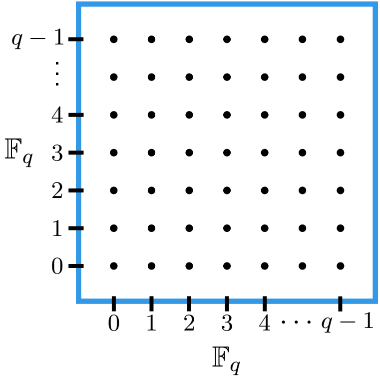
q-ary Linear Codes

- Represent message $\mathbf{w}$ as a length-k vector over $\mathbb{F}_q$.
- Codewords $\mathbf{x}$ are length-n vectors over $\mathbb{F}_q$.
- Encoding process is just a matrix multiplication, $\mathbf{x} = \mathbf{G}\mathbf{w}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1k} \\ g_{21} & g_{22} & \cdots & g_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nk} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}$$

- Recall that, for prime q, operations over $\mathbb{F}_q$ are just $\mod q$ operations over the reals.
- Rate $R = \dfrac{k}{n} \log q$

- Linear code looks like a regular subsampling of the elements of $\mathbb{F}_q^n$.
- **Random linear code:** Generate each element g_{ij} of the generator matrix $\mathbf{G}$ elementwise i.i.d. according to a uniform distribution over $\{0, 1, 2, \ldots, q-1\}$.
- How are the codewords distributed?



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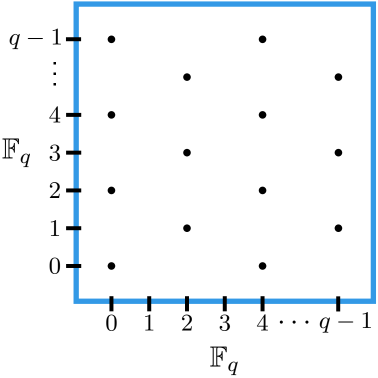


It is convenient to instead analyze the shifted ensemble $\bar{\mathbf{x}} = \mathbf{G}\mathbf{w} \oplus \mathbf{v}$ where $\mathbf{v}$ is an i.i.d. uniform sequence. (See **Gallager.**)

Shifted Codeword Properties

1. Marginally uniform over $\mathbb{F}_q^n$. For a given message $\mathbf{w}$, the codeword $\bar{\mathbf{x}}$ looks like an i.i.d. uniform sequence.

$$\mathbb{P}\{\bar{\mathbf{x}} = \mathsf{x}\} = \frac{1}{q^n} \quad \text{for all } \mathsf{x} \in \mathbb{F}_q^n$$

2. Pairwise independent. For $\mathbf{w}_1 \neq \mathbf{w}_2$, codewords $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2$ are independent.

$$\mathbb{P}\{\bar{\mathbf{x}}_{\mathbf{1}} = \mathsf{x}_1, \bar{\mathbf{x}}_{\mathbf{2}} = \mathsf{x}_2\} = \frac{1}{q^{2n}} = \mathbb{P}\{\bar{\mathbf{x}}_1 = \mathsf{x}_1\}\mathbb{P}\{\bar{\mathbf{x}}_2 = \mathsf{x}_2\}$$

Achievable Rates

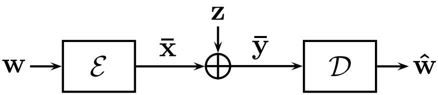
- Cuckoo's Egg Lemma only requires independence between the true codeword $\mathbf{x}(\mathbf{w})$ and the other codeword $\mathbf{x}(\tilde{\mathbf{w}})$. From the union bound:

$$\begin{aligned}\mathbb{P}\{\hat{\mathbf{w}} \neq \mathbf{w}\} &\leq \sum_{\tilde{\mathbf{w}} \neq \mathbf{w}} \mathbb{P}\Big\{(\mathbf{x}(\tilde{\mathbf{w}}), \mathbf{y}) \ \text{ is jointly typical.}\Big\} \\ &\leq 2^{-n(I(X;Y) - R - 3\epsilon)}\end{aligned}$$

- This is exactly what we get from pairwise independence.

- Thus, there exists a good fixed generator matrix $\mathbf{G}$ and shift $\mathbf{v}$ for any rate $R < I(X;Y)$ where X is uniform.

Removing the Shift



- For a binary symmetric channel (BSC), the output can be written as the modulo sum of the input plus i.i.d. Bernoulli(p) noise,

$$\bar{\mathbf{y}} = \bar{\mathbf{x}} \oplus \mathbf{z}$$
$$\bar{\mathbf{y}} = \mathbf{G}\mathbf{w} \oplus \mathbf{v} \oplus \mathbf{z}$$

- Due to this symmetry, the probability of error depends *only* on the realization of the noise vector $\mathbf{z}$.
  $\implies$ For a BSC, $\mathbf{x} = \mathbf{G}\mathbf{w}$ is a good code as well.

- We can now assume the existence of good generator matrices for channel coding.

Random I.I.D. vs. Random Linear

- What have we gotten for linearity (so far)?
  Simplified encoding. (Decoder is still quite complex.)

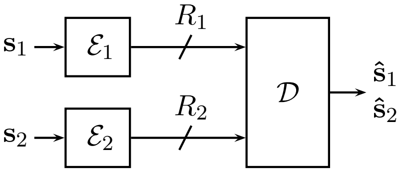
- What have we lost?
  Can only achieve $R = I(X;Y)$ for uniform X instead of $\max\limits_{p_X} I(X;Y)$.

- In fact, this is a fundamental limitation of group codes, **Ahlswede '71**.

- Workarounds: symbol remapping **Gallager '68**, nested linear codes

- Are random linear codes strictly worse than random i.i.d. codes?

Slepian-Wolf Problem



- Joint i.i.d. sources $p(\mathbf{s}_1, \mathbf{s}_2) = \prod_{i=1}^{m} p_{S_1 S_2}(s_{1i}, s_{2i})$
- **Rate Region:** Set of rates (R_1, R_2) such that the encoders can send $\mathbf{s}_1$ and $\mathbf{s}_2$ to the decoder with vanishing probability of error

$$\mathbb{P}\{(\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2) \neq (\mathbf{s}_1, \mathbf{s}_2)\} \to 0 \text{ as } m \to \infty$$

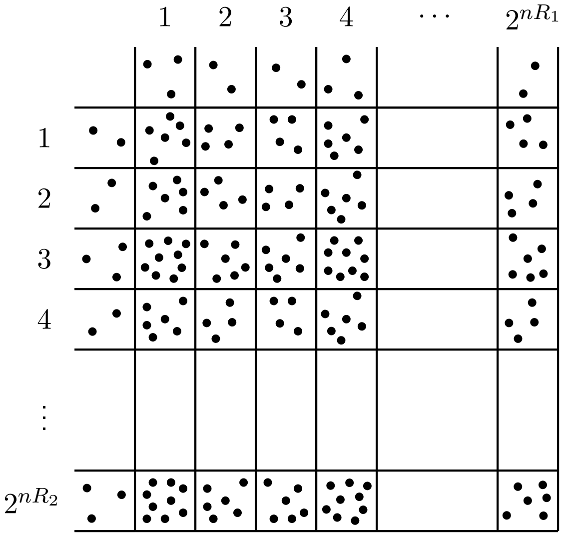
Random Binning

- Codebook 1: Independently and uniformly assign each source sequence $\mathbf{s}_1$ to a label $\{1, 2, \ldots, 2^{mR_1}\}$
- Codebook 2: Independently and uniformly assign each source sequence $\mathbf{s}_2$ to a label $\{1, 2, \ldots, 2^{mR_2}\}$
- Decoder: Look for jointly typical pair $(\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2)$ within the received bin. Union bound:

$$
\begin{aligned}
&\mathbb{P}\Big\{\text{jointly typical } (\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2) \neq (\mathbf{s}_1, \mathbf{s}_2) \text{ in bin } (\ell_1, \ell_2)\Big\} \\
&\leq \sum_{\text{jointly typical } (\tilde{\mathbf{s}}_1, \tilde{\mathbf{s}}_2)} 2^{-m(R_1+R_2)} \\
&\leq 2^{m(H(S_1,S_2)+\epsilon)} 2^{-m(R_1+R_2)}
\end{aligned}
$$

- Need $R_1 + R_2 > H(S_1, S_2)$.
- Similarly, $R_1 > H(S_1|S_2)$ and $R_2 > H(S_2|S_1)$

Slepian-Wolf Problem: Binning Illustration



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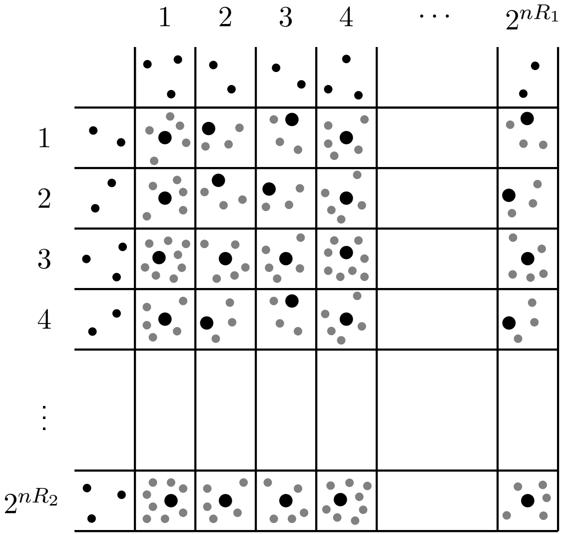


Random Linear Binning

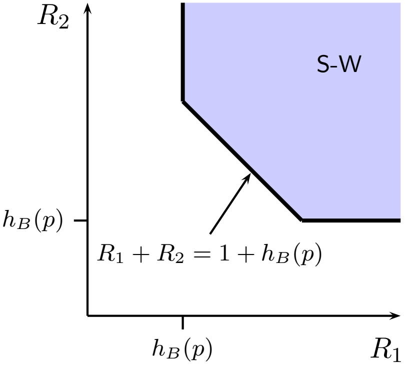
- Assume source symbols take values in $\mathbb{F}_q$.
- Codebook 1: Generate matrix $\mathbf{G}_1$ with i.i.d. uniform entries drawn from $\mathbb{F}_q$. Each sequence $\mathbf{s}_1$ is binned via matrix multiplication, $\mathbf{w}_1 = \mathbf{G}_1 \mathbf{s}_1$.
- Codebook 2: Generate matrix $\mathbf{G}_2$ with i.i.d. uniform entries drawn from $\mathbb{F}_q$. Each sequence $\mathbf{s}_2$ is binned via matrix multiplication, $\mathbf{w}_2 = \mathbf{G}_2 \mathbf{s}_2$.
- Bin assignments are uniform and pairwise independent (except for $\mathbf{s}_\ell = \mathbf{0}$)
- Can apply the same union bound analysis as random binning.

Slepian-Wolf Rate Region

Slepian-Wolf Theorem

Reliable compression possible if and only if:

$$R_1 \geq H(S_1|S_2) = h_B(p)$$
$$R_2 \geq H(S_2|S_1) = h_B(p)$$
$$R_1 + R_2 \geq H(S_1, S_2) = 1 + h_B(p)$$



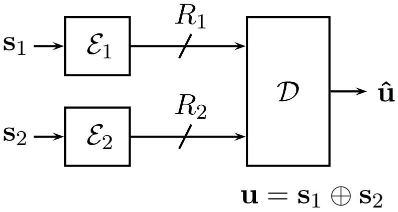
Random linear binning is as good as random i.i.d. binning!

Example: Doubly Symmetric Binary Source

$S_1 \sim \text{Bern}(1/2)$ $\quad$ $U \sim \text{Bern}(p)$ $\quad$ $S_2 = S_1 \oplus U$

Körner-Marton Problem

- Binary sources
- $\mathbf{s}_1$ is i.i.d. Bernoulli$(1/2)$
- $\mathbf{s}_2$ is $\mathbf{s}_1$ corrupted by Bernoulli(p) noise
- Decoder wants the modulo-2 sum .



Rate Region: Set of rates (R_1, R_2) such that there exist encoders and decoders with vanishing probability of error

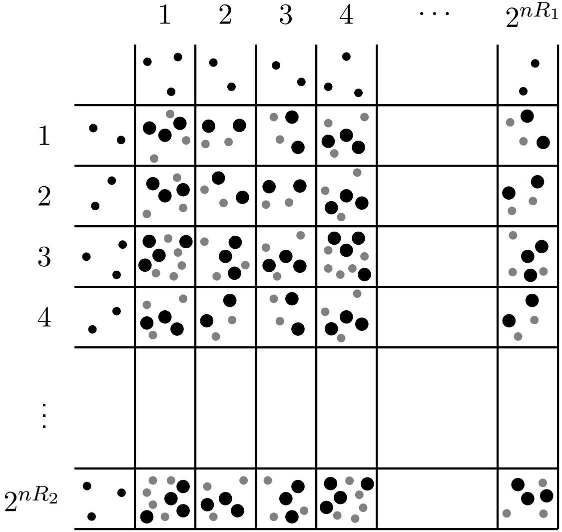
$$\mathbb{P}\{\hat{\mathbf{u}} \neq \mathbf{u}\} \to 0 \text{ as } m \to \infty$$

Are any rate savings possible over sending $\mathbf{s}_1$ and $\mathbf{s}_2$ in their entirety?

Random Binning

- Sending $\mathbf{s}_1$ and $\mathbf{s}_2$ with random binning requires $R_1 + R_2 > 1 + h_B(p)$?
- What happens if we use rates such that $R_1 + R_2 < 1 + h_B(p)$?
- There will be exponentially many pairs $(\mathbf{s}_1, \mathbf{s}_2)$ in each bin!
- This would be fine if all pairs in a bin have the same sum, $\mathbf{s}_1 + \mathbf{s}_2$. But this probability goes to zero exponentially fast!

Körner-Marton Problem: Random Binning Illustration



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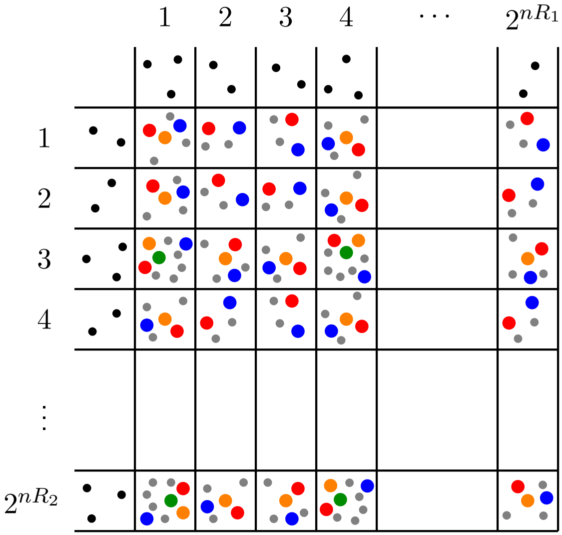


Linear Binning

- Use the same random matrix $\mathbf{G}$ for linear binning at each encoder:

$$\mathbf{w}_1 = \mathbf{G}\mathbf{s}_1 \qquad \mathbf{w}_2 = \mathbf{G}\mathbf{s}_2$$

- Idea from **Körner-Marton '79:** Decoder adds up the bins.

$$\begin{aligned} \mathbf{w}_1 \oplus \mathbf{w}_2 &= \mathbf{G}\mathbf{s}_1 \oplus \mathbf{G}\mathbf{s}_2 \\ &= \mathbf{G}(\mathbf{s}_1 \oplus \mathbf{s}_2) \\ &= \mathbf{G}\mathbf{u} \end{aligned}$$

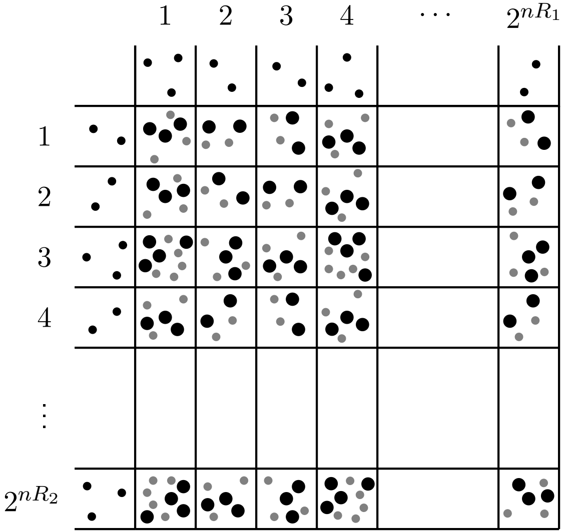
- $\mathbf{G}$ is good for compressing $\mathbf{u}$ if $R > H(U) = h_B(p)$.

Körner-Marton Theorem

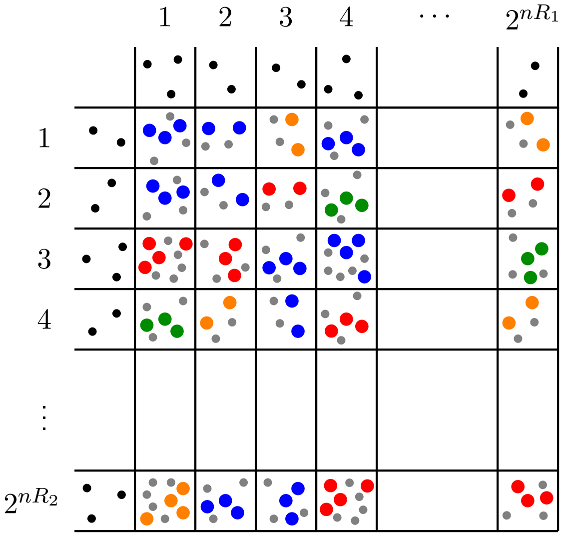
Reliable compression of the sum is possible if and only if:

$$R_1 \geq h_B(p) \qquad R_2 \geq h_B(p) \ .$$

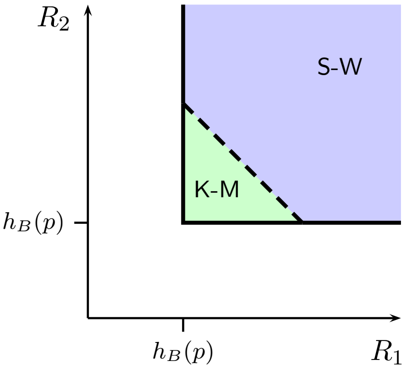
Körner-Marton Problem: Linear Binning Illustration



Körner-Marton Problem: Linear Illustration

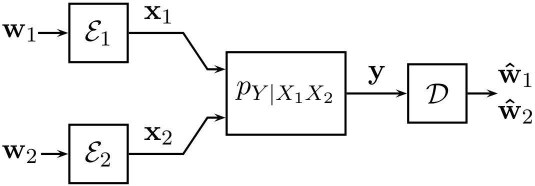


Körner-Marton Rate Region



Linear codes can improve performance!

(for distributed computation of dependent sources)



- **Rate Region:** Set of rates (R_1, R_2) such that the encoders can send $\mathbf{w}_1$ and $\mathbf{w}_2$ to the decoder with vanishing probability of error

$$\mathbb{P}\{(\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2) \neq (\mathbf{w}_1, \mathbf{w}_2)\} \to 0 \text{ as } m \to \infty$$

- Cuckoo's egg lemma applies to all three error events.
- For example, event that only $\hat{\mathbf{w}}_1$ is wrong:

$$\begin{aligned}\mathbb{P}\{\hat{\mathbf{w}}_1 \neq \mathbf{w}_1, \hat{\mathbf{w}}_2 = \mathbf{w}_2\} &\leq \sum_{\tilde{\mathbf{w}}_1 \neq \mathbf{w}_1} \mathbb{P}\Big\{(\mathbf{x}_1(\tilde{\mathbf{w}}_1), \mathbf{x}_2(\mathbf{w}_2), \mathbf{y}) \text{ jointly typical}\Big\} \\ &\leq 2^{-n(I(X_1;Y|X_2)-R_1-3\epsilon)}\end{aligned}$$

Rate Region (Ahlswede, Liao)

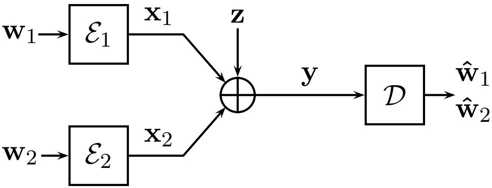
Convex closure of all (R_1, R_2) satisfying

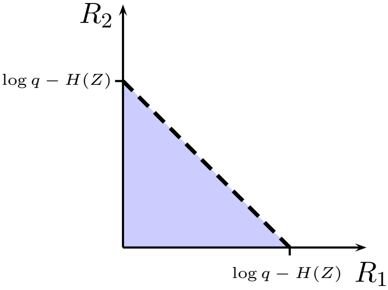
$$\begin{aligned} R_1 &< I(X_1;Y|X_2) \\ R_2 &< I(X_2;Y|X_1) \\ R_1 + R_2 &< I(X_1, X_2; Y) \end{aligned}$$

for some $p(x_1)p(x_2)$.

Finite-Field Multiple-Access Channels

- Linear codes can achieve any rate available for uniform $p(x_1), p(x_2)$.
- For finite field MACs, can achieve the whole capacity region.





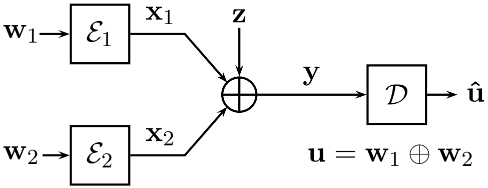
- Receiver observes noisy modulo sum of codewords $\mathbf{y} = \mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{z}$

Finite Field MAC Rate Region

All rates (R_1, R_2) satisfying

$$R_1 + R_2 \leq \log q - H(Z)$$

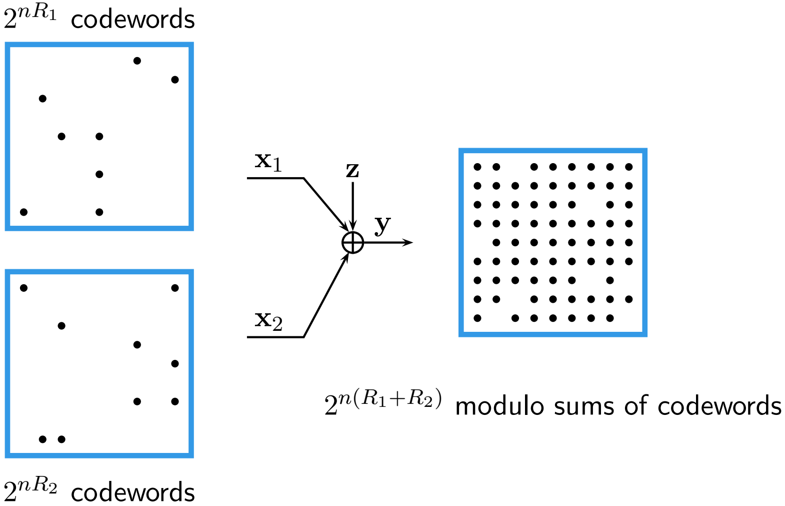
- Independent msgs $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{F}_q^k$.
- Want the sum $\mathbf{u} = \mathbf{w}_1 \oplus \mathbf{w}_2$ with vanishing prob. of error $\mathbb{P}\{\hat{\mathbf{u}} \neq \mathbf{u}\} \to 0$



I.I.D. Random Coding

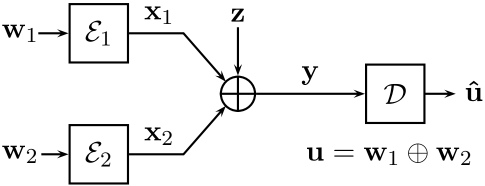
- Generate 2^{nR_1} i.i.d. uniform codewords for user 1.
- Generate 2^{nR_2} i.i.d. uniform codewords for user 2.
- With high probability, (nearly) all sums of codewords are distinct.
- This is ideal for multiple-access but not for computation.
- Need $R_1 + R_2 \leq \log q - H(Z)$

Random i.i.d. codes are not good for computation



Independent msgs $\mathbf{w}_1, \mathbf{w}_2$.

Want the sum $\mathbf{u} = \mathbf{w}_1 \oplus \mathbf{w}_2$ with vanishing prob. of error $\mathbb{P}\{\hat{\mathbf{u}} \neq \mathbf{u}\} \to 0$



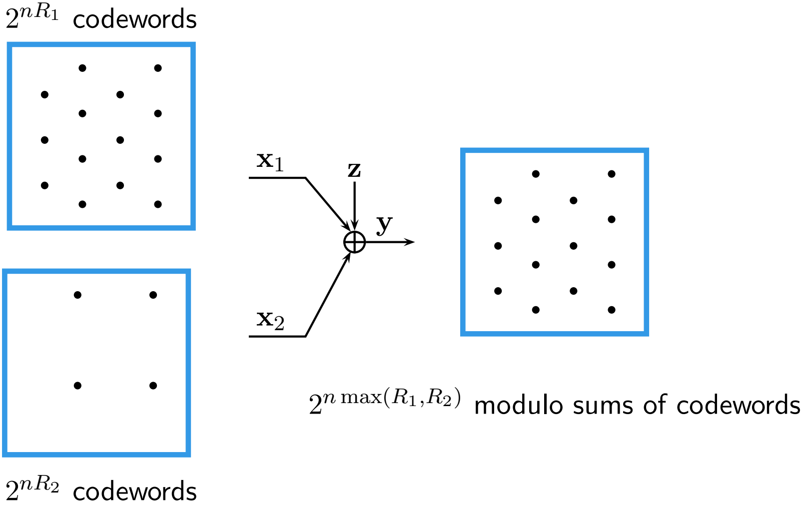
Random Linear Coding

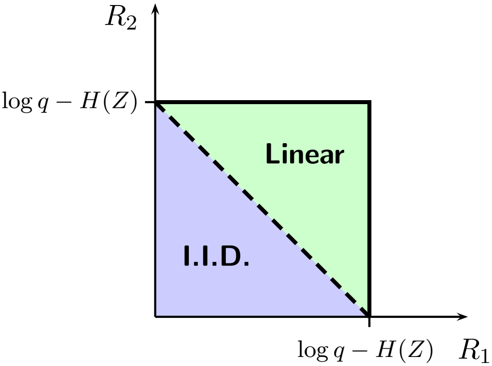
- Same linear code at both transmitters $\mathbf{x}_1 = \mathbf{G}\mathbf{w}_1$, $\mathbf{x}_2 = \mathbf{G}\mathbf{w}_2$.
- Sums of codewords are themselves codewords:

$$\begin{aligned} \mathbf{y} &= \mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{z} \\ &= \mathbf{G}\mathbf{w}_1 \oplus \mathbf{G}\mathbf{w}_2 \oplus \mathbf{z} \\ &= \mathbf{G}(\mathbf{w}_1 \oplus \mathbf{w}_2) \oplus \mathbf{z} \\ &= \mathbf{G}\mathbf{u} \oplus \mathbf{z} \end{aligned}$$

- Need $\max(R_1, R_2) \leq \log q - H(Z)$

Random linear codes are good for computation

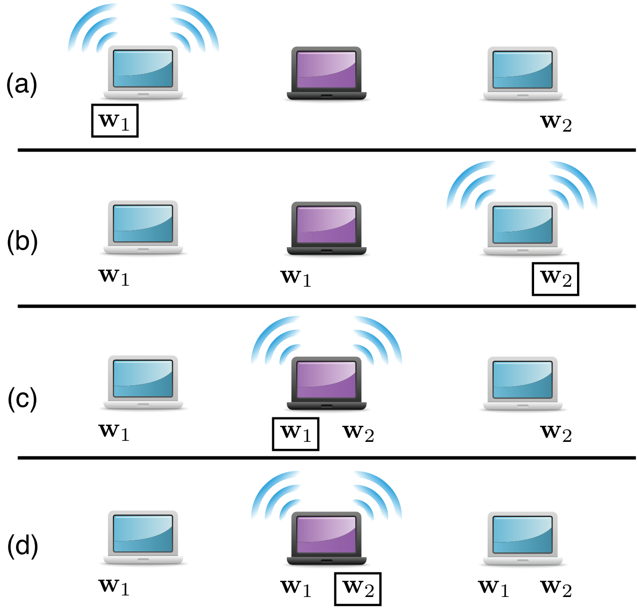




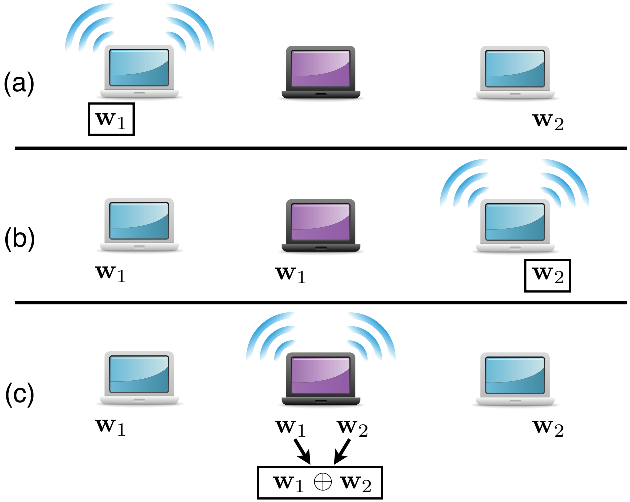
- **I.I.D. Random Coding**: $R_1 + R_2 \leq \log q - H(Z)$
- **Random Linear Coding**: $\max(R_1, R_2) \leq \log q - H(Z)$
- Linear codes double the sum rate *without any dependency*.
- Is this useful for *sending messages* (no computation)?

- Elegant example proposed by **Wu-Chou-Kung '04.**
- Closely related to butterfly network from **Ahlswede-Cai-Li-Yeung '00.**

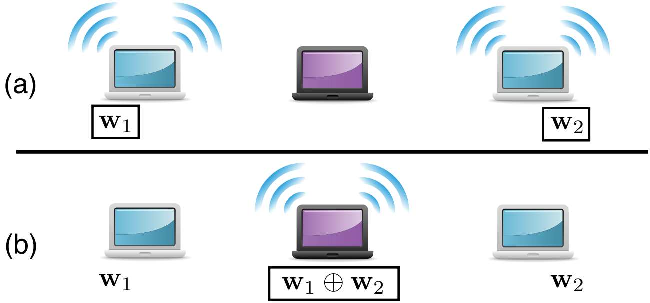
Two-Way Relay Channel – Time-Division

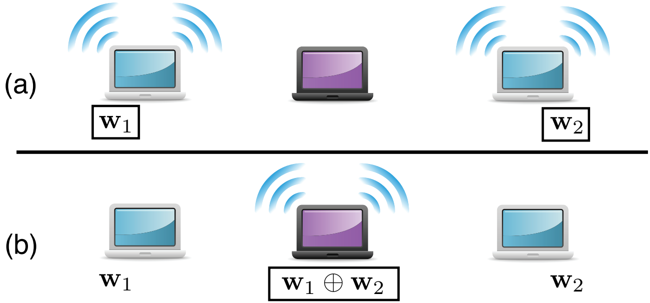


Two-Way Relay Channel – Network Coding



Two-Way Relay Channel – Physical-Layer Network Coding

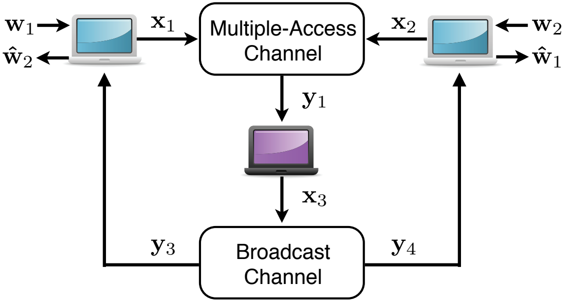




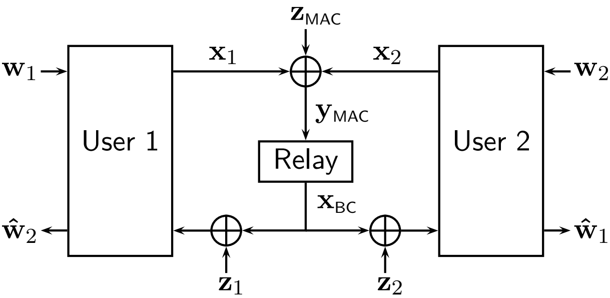
- Physical-layer network coding: exploiting the wireless medium for network coding. Independently and concurrently proposed by **Zhang-Liew-Lam '06, Popovski-Yomo '06, Nazer-Gastpar '06.**
- Sometimes referred to as Analog Network Coding **Katti-Gollakota-Katabi '08.**
- Some recent surveys **Liew-Zhang-Lu '11, Nazer-Gastpar '11.**

q-ary Two-Way Relay Channel

q-ary Two-Way Relay Channel

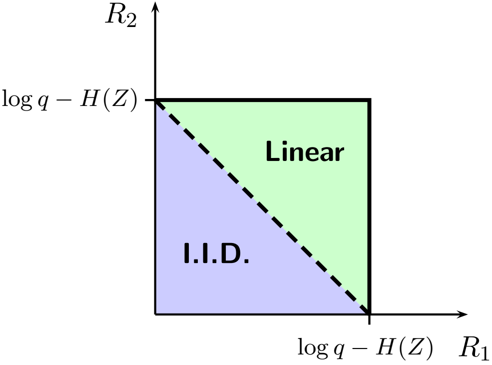


q-ary Two-Way Relay Channel



- i.i.d. noise sequences with entropy $H(Z)$.
- Rates R_1 and R_2.

- Upper Bound:
  $\max(R_1, R_2) \leq \log q - H(Z)$
- Random i.i.d.: Relay decodes $\mathbf{w}_1, \mathbf{w}_2$ and transmits $\mathbf{w}_1 \oplus \mathbf{w}_2$.
  $R_1 + R_2 \leq \log q - H(Z)$
- Random linear: Relay decodes and retransmits $\mathbf{w}_1 \oplus \mathbf{w}_2$
  $\max(R_1, R_2) \leq \log q - H(Z)$



- **I.I.D. Random Coding**: $R_1 + R_2 \leq \log q - H(Z)$
- **Random Linear Coding**: $\max(R_1, R_2) \leq \log q - H(Z)$
- Linear codes can double the sum rate *for exchanging messages.*

- *Observation:* For linear codes, the codeword statistics are uniform. This follows straightforwardly from the fact that the sum of any two codewords is again a codeword.
- *Question:* Can we retain some algebraic structure *and* have non-uniform codeword statistics?
- Idea: Nested Linear Codes (see, for instance, **Conway and Sloane '92, Forney '89, Zamir-Shamai-Erez '02** ...):

- Consider a linear code $\mathcal{C}_c$ of rate $1 - k/n$:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1,n-k} \\ g_{21} & g_{22} & \cdots & g_{2,n-k} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{n,n-k} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_{n-k} \end{bmatrix}$$

  with parity check matrix $\mathbf{H}_c$.

- For every binary sequence $\mathbf{u}$ of length k, define its coset as

$$\mathcal{C}_c(\mathbf{u}) = \{\mathbf{x} : \mathbf{H}_c\mathbf{x} = \mathbf{u}\}$$

- The *coset leader* is the one sequence in $\mathcal{C}_c(\mathbf{u})$ that has the *smallest* Hamming weight.

- For any sequence $\mathbf{x}$ we write $\mathbf{x} \bmod \mathcal{C}_c$ to denote the coset leader corresponding to $\mathbf{H}_c\mathbf{x}$.
- **Observation:** This satisfies all the usual properties of the modulo operation, such as

$$(\mathbf{x} \oplus \mathbf{y}) \bmod \mathcal{C}_c = (\mathbf{x} \bmod \mathcal{C}_c \oplus \mathbf{y} \bmod \mathcal{C}_c) \bmod \mathcal{C}_c$$

Theorem

There exists a binary linear code of rate $1 - k/n$ *such that all* 2^k *coset leaders satisfy* $w_{Hamming} \leq m$, *where*

$$k/n \geq H_b(m/n) - \epsilon$$

Note: Such a code is thus a good *covering* code.

Next step: *Decimate* coset leaders: retain only those belonging to a ("fine") code.

That way, we end up with a code of $2^{k-k'}$ codewords satisfying two properties:

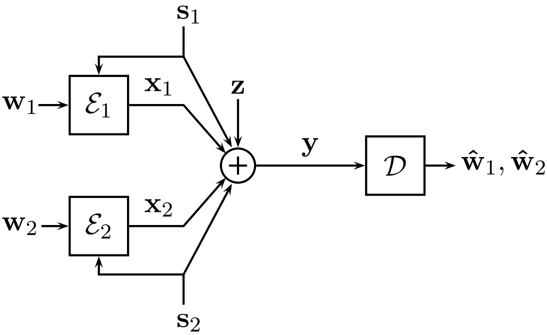
1. Noise protection just like the fine code
2. The sum of any two codewords, modulo "the coarse code," is again a codeword

On the BSC with crossover probability p, this code achieves a rate

$$R \;=\; H_b(m/n) - H_b(p).$$

Note that this is *not* the capacity of this channel.

Philosof-Zamir '09, Philosof-Zamir-Erez '09:



Without *input constraints*, the problem is trivial.

But now, consider

$$w_H(\mathbf{x}_1) \leq m \quad \text{and} \quad w_H(\mathbf{x}_2) \leq m.$$

- Choose codewords $\mathbf{t}_1$ and $\mathbf{t}_2$. Transmit

$$\mathbf{x}_1 = (\mathbf{t}_1 \oplus \mathbf{s}_1) \text{ mod } \mathcal{C}_c \quad \text{and} \quad \mathbf{x}_2 = (\mathbf{t}_2 \oplus \mathbf{s}_2) \text{ mod } \mathcal{C}_c$$

- Choose coarse code to satisfy Hamming input constraints. Receive:

$$\mathbf{y} = [(\mathbf{x}_1 \oplus \mathbf{s}_1) \text{ mod } \mathcal{C}_c] \oplus [(\mathbf{x}_2 \oplus \mathbf{s}_2) \text{ mod } \mathcal{C}_c] \oplus \mathbf{s}_1 \oplus \mathbf{s}_2 \oplus \mathbf{z}$$

- The key step is the following pre-processing step at the decoder:

$$\begin{aligned} \mathbf{y} \text{ mod } \mathcal{C}_c &= (\mathbf{x}_1 \oplus \mathbf{s}_1 \oplus \mathbf{x}_2 \oplus \mathbf{s}_2 \oplus \mathbf{s}_1 \oplus \mathbf{s}_2 \oplus \mathbf{z}) \text{ mod } \mathcal{C}_c \\ &= (\mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{z}) \text{ mod } \mathcal{C}_c \end{aligned}$$

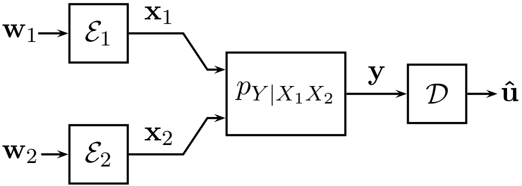
- Last step: show that the noise is essentially unchanged by the modulo operation.

- Can show that this achieves the capacity (see **Philosof-Zamir-Erez '09**.)

Beyond Linear

Independent msgs $\mathbf{w}_1, \mathbf{w}_2$.

Want the sum $\mathbf{u} = \mathbf{w}_1 \oplus \mathbf{w}_2$ with vanishing prob. of error $\mathbb{P}\{\hat{\mathbf{u}} \neq \mathbf{u}\} \to 0$



Achievable Strategy (Nazer-Gastpar '08)

Use the same linear code, $\max(R_1, R_2) \leq I(X_1 \oplus X_2; Y)$ (for binary, uniform inputs)

- General Functions: $U_i = f(W_{1i}, W_{2i})$
- Some achievable strategies, very hard in general (functional compression is a special case)
- For network communication, don't really care what functions in the middle, only care about msgs

I. **Discrete Alphabets**

II. **AWGN Channels**

III. **Network Applications**

Nested lattice results in this section are almost entirely drawn from:

- U. Erez and R. Zamir, *Achieving* $\frac{1}{2}\log(1+\mathsf{SNR})$ *on the AWGN channel with lattice encoding and decoding*, IEEE Transactions on Information Theory, vol. 50, pp. 2293-2314, October 2004.
- U. Erez, S. Litsyn, and R. Zamir, *Lattices which are good for (almost) everything*, IEEE Transactions on Information Theory, vol. 51, pp. 3401-3416, October 2005.
- R. Zamir, *Lattices are everywhere*, in Proceedings of the 4th Annual Workshop on Information Theory and its Applications, La Jolla, CA, February 2009.

Gaussian MMSE Estimation

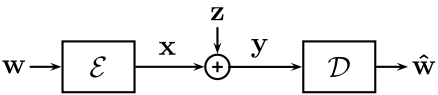
- Signal X is a scalar Gaussian r.v. with mean 0 and variance P.
- Noise Z is an independent scalar Gaussian r.v. with mean 0 and variance N.
- Estimate X from noisy observation $Y = X + Z$.
- Mean-squared error: $\mathbb{E}[(Y - X)^2] = \mathbb{E}[Z^2] = N$.
- Minimum mean-squared error (MMSE):

$$\begin{aligned} \mathbb{E}[(\alpha Y - X)^2] &= \mathbb{E}[(\alpha X + \alpha Z - X)^2] \\ &= \mathbb{E}[\alpha^2 Z^2 + (1-\alpha)^2 X^2] \qquad \text{Part of error due to } X \\ &= \alpha^2 N + (1-\alpha)^2 P \end{aligned}$$

- Optimal $\alpha = \dfrac{P}{N+P}$ yields $\mathbb{E}[(\alpha Y - X)^2] = \dfrac{PN}{N+P}$.

- Codewords must satisfy power constraint:

$$\|\mathbf{x}\|^2 \leq nP \ .$$



- i.i.d. Gaussian noise with variance N:

$$\mathbf{z} \sim \mathcal{N}(\mathbf{0}, N\mathbf{I}) \ .$$

- **Shannon '48**: Channel capacity:

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N}\right)$$

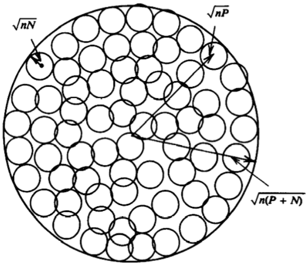


Figure 10.2. Sphere packing for the Gaussian channel.

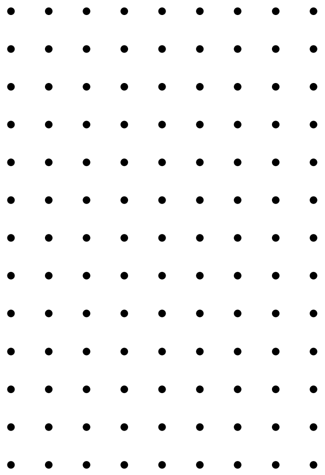
(Cover and Thomas, *Elements of Information Theory*)

- In high dimensions, noise starts to look spherical.

- A lattice Λ is a discrete subgroup of $\mathbb{R}^n$.
- Can write a lattice as a linear transformation of the integer vectors,
$$\Lambda = \mathbf{B}\mathbb{Z}^n \ ,$$
for some $\mathbf{B} \in \mathbb{R}^{n\times n}$.

Lattice Properties

- Closed under addition: $\lambda_1, \lambda_2 \in \Lambda \implies \lambda_1 + \lambda_2 \in \Lambda$.
- Symmetric: $\lambda \in \Lambda \implies -\lambda \in \Lambda$



$\mathbb{Z}^n$ is a simple lattice.

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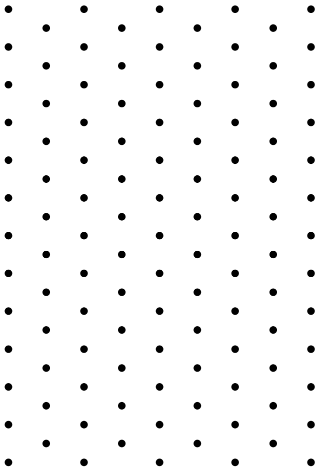


$\mathbf{B}\mathbb{Z}^n$

Voronoi Regions

- Nearest neighbor quantizer:

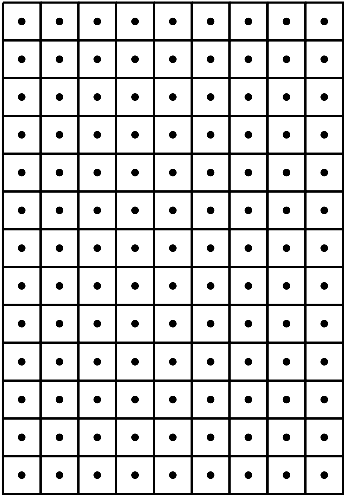
$$Q_\Lambda(\mathbf{x}) = \arg\min_{\lambda \in \Lambda} \|\mathbf{x} - \lambda\|_2$$

- The Voronoi region of a lattice point is the set of all points that quantize to that lattice point.

- Fundamental Voronoi region $\mathcal{V}$: points that quantize to the origin,

$$\mathcal{V} = \{\mathbf{x} : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$$

- Each Voronoi region is just a shift of the fundamental Voronoi region $\mathcal{V}$



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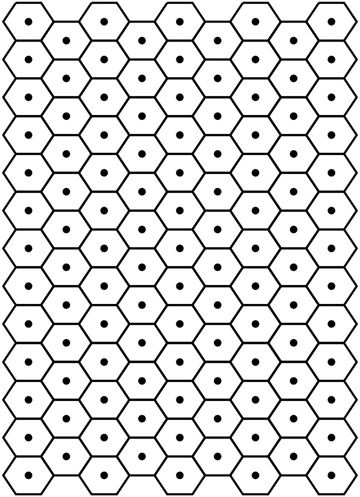


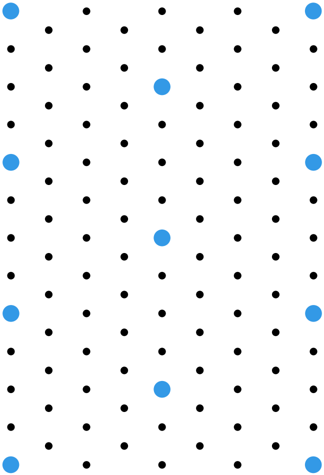
Nested Lattices

- Two lattices Λ and Λ_{FINE} are nested if $\Lambda \subset \Lambda_{\text{FINE}}$

- **Nested Lattice Code:** All lattice points from Λ_{FINE} that fall in the fundamental Voronoi region $\mathcal{V}$ of Λ.

- $\mathcal{V}$ acts like a power constraint

$$\text{Rate} = \frac{1}{n} \log \left(\frac{\text{Vol}(\mathcal{V})}{\text{Vol}(\mathcal{V}_{\text{FINE}})} \right)$$



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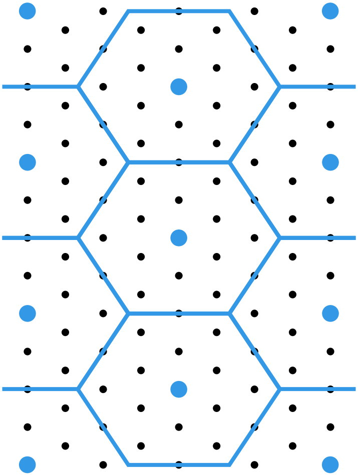

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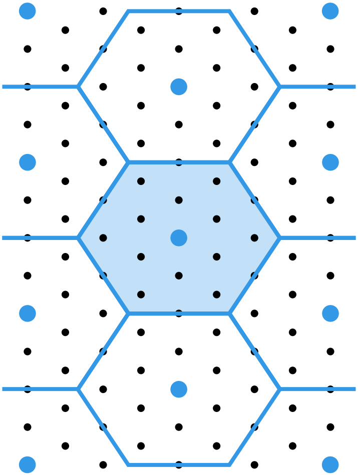

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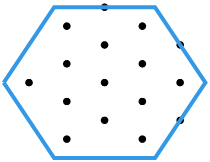

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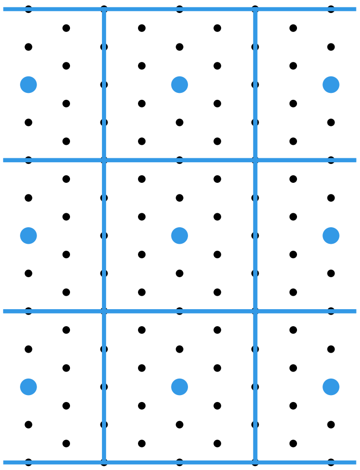
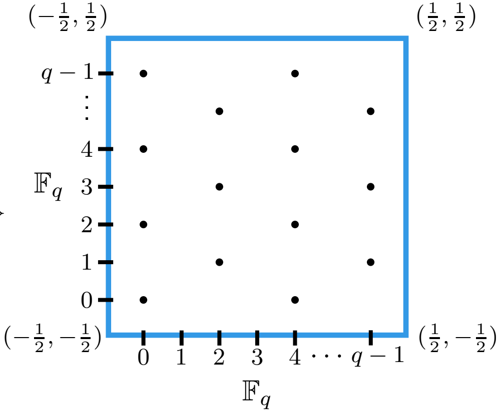


Nested Lattice Codes from q-ary Linear Codes

- Choose an $n \times k$ generator matrix $\mathbf{G} \in \mathbb{F}_q^{n \times k}$ for q-ary code.
- Integers serve as coarse lattice, $\Lambda = \mathbb{Z}^n$.
- Map elements $\{0, 1, 2, \ldots, q-1\}$ to equally spaced points between $-1/2$ and $1/2$.
- Place codewords $\mathbf{x} = \mathbf{G}\mathbf{w}$ into the fundamental Voronoi region $\mathcal{V} = [-1/2, 1/2)^n$



Modulo Operation

- Modulo operation with respect to lattice Λ is just the residual quantization error,

$$[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}) \ .$$

- Mimics the role of $\bmod q$ in q-ary alphabet.

- Distributive Law:

$$\Big[\mathbf{x}_1 + [\mathbf{x}_2] \bmod \Lambda\Big] \bmod \Lambda$$
$$= [\mathbf{x}_1 + \mathbf{x}_2] \bmod \Lambda$$

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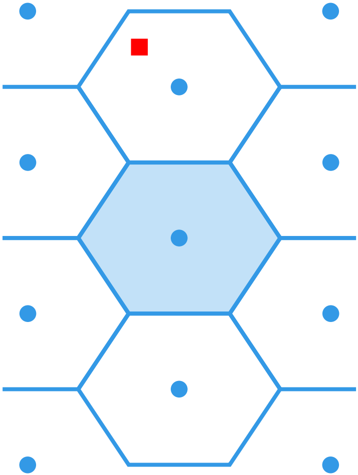

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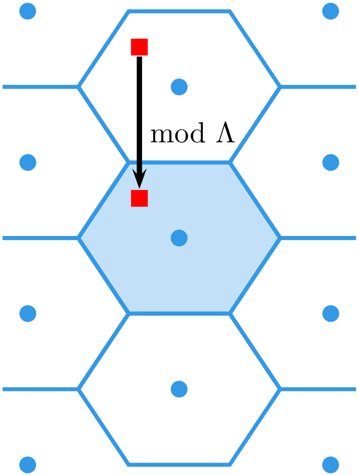
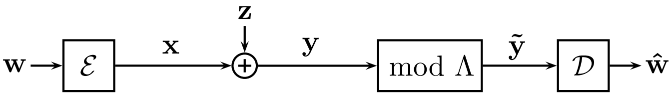
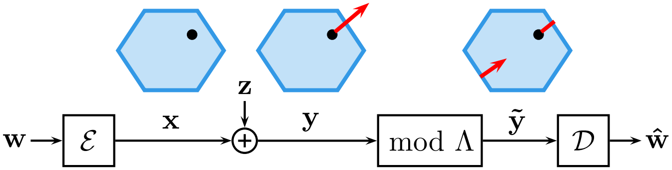


$\bmod \Lambda$ *AWGN Channel*



- Codebook lives on Voronoi region $\mathcal{V}$ of coarse lattice Λ.
- Take $\bmod \Lambda$ of received signal prior to decoding.
- What is the capacity of the $\bmod \Lambda$ channel?



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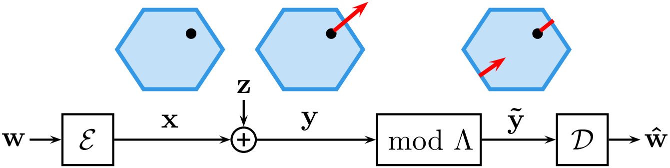

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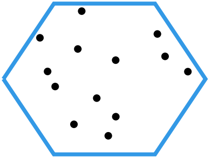
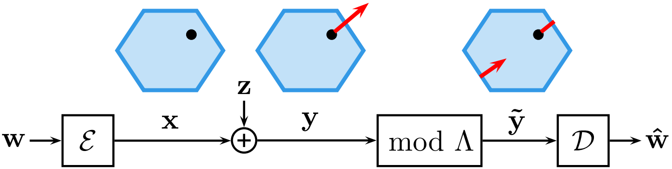
- Take $\bmod \Lambda$ of received signal prior to decoding.
- What is the capacity of the $\bmod \Lambda$ channel?

Using random i.i.d. code drawn over $\mathcal{V}$:

$$C = \frac{1}{n} \max_{p(\mathbf{x})} I(\mathbf{x}; \tilde{\mathbf{y}})$$

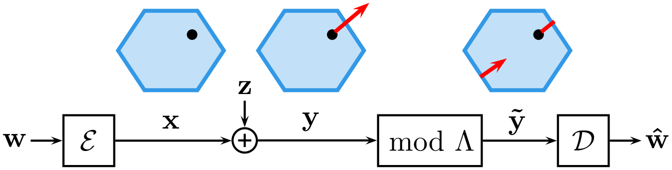


$\bmod \Lambda$ *AWGN Channel Capacity*



$$
\begin{aligned}
nC &= \max_{p(\mathbf{x})} I(\mathbf{x}; \tilde{\mathbf{y}}) \\
&= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\tilde{\mathbf{y}}|\mathbf{x}) \Big)
\end{aligned}
$$

$\mathrm{mod}\ \Lambda$ *AWGN Channel Capacity*

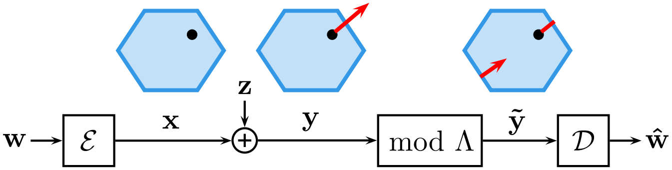


$$nC = \max_{p(\mathbf{x})} I(\mathbf{x}; \tilde{\mathbf{y}})$$

$$= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\tilde{\mathbf{y}}|\mathbf{x}) \Big)$$

$$= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h\Big([\mathbf{z}] \bmod \Lambda \Big) \Big)$$ Distributive Law

mod Λ *AWGN Channel Capacity*



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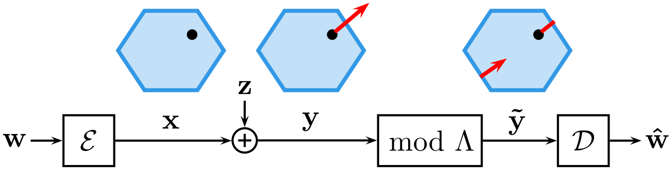
&= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h\Big([\mathbf{z}] \bmod \Lambda\Big)\Big) && \text{Distributive Law} \\
&\geq \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\mathbf{z}) \Big) && \text{Point Symmetry of Voronoi Region}
\end{aligned}
$$



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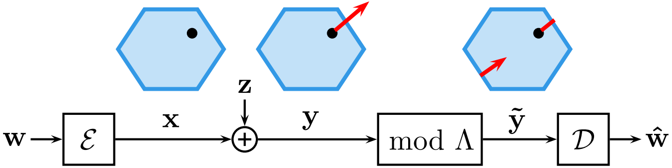
&\geq \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - h(\mathbf{z}) \Big) && \text{Point Symmetry of Voronoi Region} \\
&= \max_{p(\mathbf{x})} \Big(h(\tilde{\mathbf{y}}) - \frac{n}{2} \log(2\pi e N) \Big) && \text{Entropy of Gaussian Noise}
\end{aligned}
$$

$\bmod \Lambda$ *AWGN Channel Capacity*



- Channel output entropy is equal to the logarithm of the Voronoi region volume if it is uniform over $\mathcal{V}$:

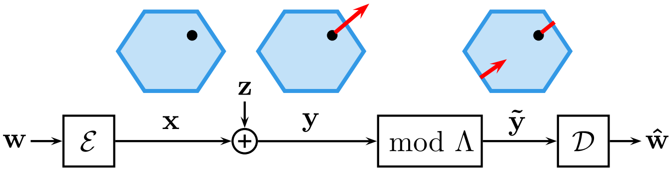
$$h(\tilde{\mathbf{y}}) = \log(\mathsf{Vol}(\mathcal{V})) \qquad \text{if } \tilde{\mathbf{y}} \sim \mathsf{Unif}(\mathcal{V})$$

- $\tilde{\mathbf{y}} = [\mathbf{x} + \mathbf{z}] \bmod \Lambda$ is uniform over $\mathcal{V}$ if $\mathbf{x}$ is uniform over $\mathcal{V}$.

- Random i.i.d. coding over the Voronoi region $\mathcal{V}$ can achieve:

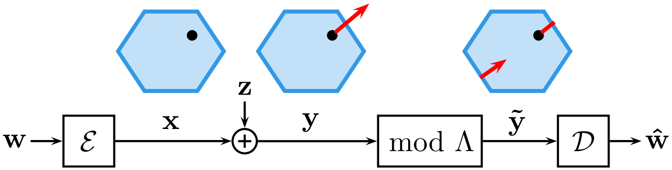
$$R = \frac{1}{n}\log(\mathsf{Vol}(\mathcal{V})) - \frac{1}{2}\log(2\pi e N)$$

Power Constraints and Second Moments



- Must scale lattice Λ so that the uniform distribution over the Voronoi region $\mathcal{V}$ meets the power constraint P.
- Set second moment $\sigma_\Lambda^2 = \frac{1}{n\mathsf{Vol}(\mathcal{V})}\int_{\mathcal{V}} \|\mathbf{x}\|^2 d\mathbf{x}$ equal to P.

Power Constraints and Second Moments



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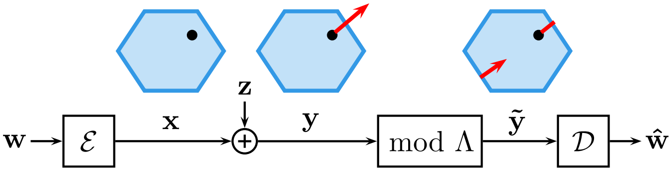
- Set second moment $\sigma_\Lambda^2 = \dfrac{1}{n\mathsf{Vol}(\mathcal{V})}\displaystyle\int_{\mathcal{V}} \|\mathbf{x}\|^2 d\mathbf{x}$ equal to P.

Normalized Second Moment: $\quad G(\Lambda) = \dfrac{\sigma_\Lambda^2}{(\mathsf{Vol}(\mathcal{V}))^{2/n}}$

$$\implies \frac{1}{n}\log(\mathsf{Vol}(\mathcal{V})) = \frac{1}{2}\log\left(\frac{\sigma_\Lambda^2}{G(\Lambda)}\right) = \frac{1}{2}\log\left(\frac{P}{G(\Lambda)}\right)$$

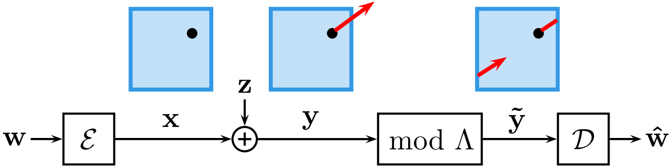
mod Λ *AWGN Channel Capacity*



- Random i.i.d. coding over the Voronoi region $\mathcal{V}$ can achieve:

$$\begin{aligned} C &\geq \frac{1}{n}\log(\mathsf{Vol}(\mathcal{V})) - \frac{1}{2}\log(2\pi e N) \\ &= \frac{1}{2}\log\left(\frac{P}{G(\Lambda)}\right) - \frac{1}{2}\log(2\pi e N) \\ &= \frac{1}{2}\log\left(\frac{P}{N}\right) - \frac{1}{2}\log(2\pi e G(\Lambda)) \end{aligned}$$

What is $G(\Lambda)$?



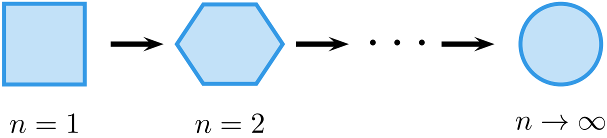
- The normalized second moment $G(\Lambda)$ is a dimensionless quantity that captures the shaping gain.
- Integer lattice is not so bad, $G(\mathbb{Z}^n) = 1/12$.
- Capacity under $\mod \mathbb{Z}^n$ is at least

$$
\begin{aligned}
C &\geq \frac{1}{2}\log\left(\frac{P}{N}\right) - \frac{1}{2}\log\left(\frac{2\pi e}{12}\right) \\
&\approx \frac{1}{2}\log\left(\frac{P}{N}\right) - 0.255
\end{aligned}
$$

Asymptotically Good $G(\Lambda)$

Theorem (Zamir-Feder-Poltyrev '94)

There exists a sequence of lattices $\Lambda^{(n)}$ *such that* $\lim_{n\to\infty} G(\Lambda^{(n)}) = \frac{1}{2\pi e}$.



- Best possible normalized second moment is that of a sphere.
- Using a sequence $\Lambda^{(n)}$ with an asymptotically good $G(\Lambda^{(N)})$ allows to approach

$$
\begin{aligned}
R &= \frac{1}{2}\log\left(\frac{P}{N}\right) - \frac{1}{2}\log\left(\frac{2\pi e}{2\pi e}\right) \\
&= \frac{1}{2}\log\left(\frac{P}{N}\right)
\end{aligned}
$$

Asymptotically Good $G(\Lambda)$

- Can actually get this with a linear code tiled over $\mathbb{Z}^n$ (see, for instance, **Erez-Litsyn-Zamir '05**.)
- Many works looking at this from different perspectives.
- We will just assume existence.

Recall the two key properties of random linear codes $\mathbf{G}$ from earlier:

Codeword Properties

1. Marginally uniform over $\mathbb{F}_q^n$. For a given message $\mathbf{w} \neq \mathbf{0}$, the codeword $\mathbf{x} = \mathbf{G}\mathbf{w}$ looks like an i.i.d. uniform sequence.

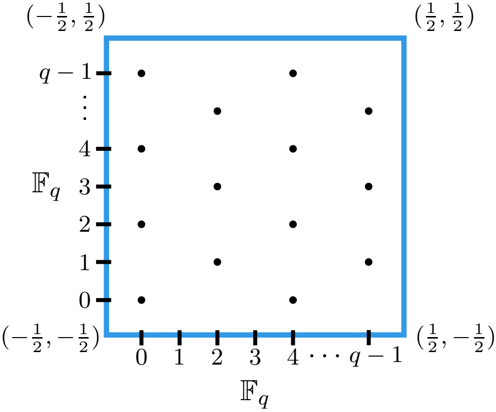
$$\mathbb{P}\{\mathbf{x} = \mathsf{x}\} = \frac{1}{q^n} \quad \text{for all } \mathsf{x} \in \mathbb{F}_q^n$$

2. Pairwise independent. For $\mathbf{w}_1, \mathbf{w}_2 \neq \mathbf{0}$, $\mathbf{w}_1 \neq \mathbf{w}_2$, codewords $\mathbf{x}_1, \mathbf{x}_2$ are independent.

$$\mathbb{P}\{\mathbf{x_1} = \mathsf{x}_1, \mathbf{x_2} = \mathsf{x}_2\} = \frac{1}{q^{2n}} = \mathbb{P}\{\mathbf{x}_1 = \mathsf{x}_1\}\mathbb{P}\{\mathbf{x}_2 = \mathsf{x}_2\}$$

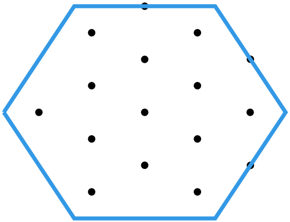
Linear Codes for $\bmod \Lambda$ *Channels*

- Instead of an "inner" random codes, we can use a q-ary linear code.
- This is exactly a nested lattice.
- Each codeword has a uniform marginal distribution over the grid.
- Rate loss due to finite constellation which goes to 0 as $q \to \infty$.
- Codewords are pairwise independent so we can apply the union bound.



$$\mathbf{x} = [\gamma \mathbf{G} \mathbf{w}] \mod \mathbb{Z}^n$$

- General coarse lattice $\Lambda = \mathbf{B}\mathbb{Z}^n$.
- First, apply generator matrix for linear code $\mathbf{G}\mathbf{w}$. Then scale down by γ and tile over $\mathbb{Z}^n$.
- Multiply by $\mathbf{B}$ and apply $\bmod \Lambda$ to get codebook.
- As q gets large, each codeword's marginal distribution looks uniform over $\mathcal{V}$.
- Codewords are pairwise independent so we can apply the union bound.



$$\mathbf{x} = [\mathbf{B}\gamma\mathbf{G}\mathbf{w}] \mod \Lambda$$

MMSE Scaling

- **Erez-Zamir '04:** Prior to taking $\bmod \Lambda$, scale by α.

$$\begin{aligned}\tilde{\mathbf{y}} &= [\alpha\mathbf{y}] \bmod \Lambda \\ &= [\alpha\mathbf{x} + \alpha\mathbf{z}] \bmod \Lambda \\ &= [\mathbf{x} + \underbrace{\alpha\mathbf{z} - (1-\alpha)\mathbf{x}}_{\text{Effective Noise}}] \bmod \Lambda\end{aligned}$$

- For now, ignore that the effective noise is not independent of the codeword. Effective noise variance $N_{\mathsf{EFFEC}} = \alpha^2 N + (1-\alpha)^2 P$.
- Optimal choice of α is the MMSE coefficient $\alpha_{\mathsf{MMSE}} = \dfrac{P}{N+P}$.

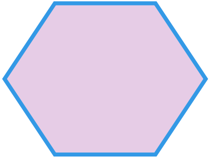
$$N_{\mathsf{EFFEC}} = \alpha_{\mathsf{MMSE}}^2 N + (1-\alpha_{\mathsf{MMSE}})^2 P = \frac{PN}{N+P}$$

$$C = \frac{1}{2}\log\left(\frac{P}{N_{\mathsf{EFFEC}}}\right) = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

- Now the noise is dependent on the codeword.
- Dithering can solve this problem (just as in the discrete case).
- Map message $\mathbf{w}$ to a lattice codeword $\mathbf{t}$.
- Generate a random dither vector $\mathbf{d}$ uniformly over $\mathcal{V}$.
- Transmitter sends a dithered codeword:

$$\mathbf{x} = [\mathbf{t} + \mathbf{d}] \bmod \Lambda$$

- $\mathbf{x}$ is now independent of the codeword $\mathbf{t}$.



Dithering

- Now the noise is dependent on the codeword.
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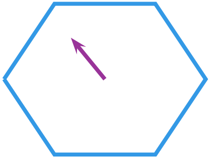


Decoding – Remove Dither First

- Transmitter sends dithered codeword $\mathbf{x} = [\mathbf{t} + \mathbf{d}] \bmod \Lambda$.
- After scaling the channel output $\mathbf{y}$ by α, the decoder subtracts the dither $\mathbf{d}$.

$$\begin{aligned}
\tilde{\mathbf{y}} &= [\alpha\mathbf{y} - \mathbf{d}] \bmod \Lambda \\
&= [\alpha\mathbf{x} + \alpha\mathbf{z} - \mathbf{d}] \bmod \Lambda \\
&= [\mathbf{x} - \mathbf{d} + \alpha\mathbf{z} - (1-\alpha)\mathbf{x}] \bmod \Lambda \\
&= \Big[[\mathbf{t} + \mathbf{d}] \bmod \Lambda - \mathbf{d} + \alpha\mathbf{z} - (1-\alpha)\mathbf{x}\Big] \bmod \Lambda \\
&= [\mathbf{t} + \alpha\mathbf{z} - (1-\alpha)\mathbf{x}] \bmod \Lambda \qquad \text{Distributive Law}
\end{aligned}$$

- Effective noise is now independent from the codeword $\mathbf{t}$.
- By the probabilistic method, (at least) one good fixed dither exists. No common randomness necessary.

Summary

- Linear code embedded in the integer lattice:

$$R = \frac{1}{2}\log\left(\frac{P}{N}\right) - \frac{1}{2}\log\left(\frac{2\pi e}{12}\right)$$

- Linear code embedded in the integer lattice, MMSE scaling:

$$R = \frac{1}{2}\log\left(1+\frac{P}{N}\right) - \frac{1}{2}\log\left(\frac{2\pi e}{12}\right)$$

- Linear code embedded in a good shaping lattice, MMSE scaling:

$$R = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

Theorem (Erez-Zamir '04)

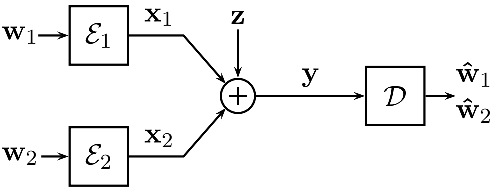
Nested lattice codes can achieve the AWGN capacity.

Rate Region

$$R_1 < \frac{1}{2}\log\left(1+\frac{P_1}{N}\right)$$

$$R_2 < \frac{1}{2}\log\left(1+\frac{P_2}{N}\right)$$

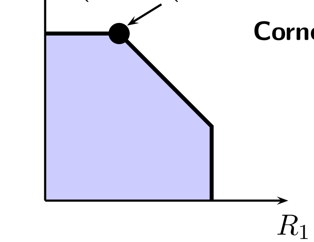
$$R_1 + R_2 < \frac{1}{2}\log\left(1+\frac{P_1+P_2}{N}\right)$$



Power constraints P_1, P_2. Noise variance N.

Successive Cancellation

$$\left(\frac{1}{2}\log\left(1+\frac{P_1}{N+P_2}\right),\ \frac{1}{2}\log\left(1+\frac{P_2}{N}\right)\right)$$



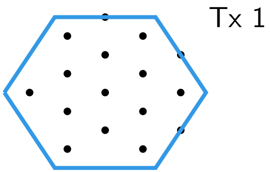
Corner Point

1. Decode $\mathbf{x}_1$, treating $\mathbf{x}_2$ as noise.
2. Subtract $\mathbf{x}_1$ from $\mathbf{y}$.
3. Decode $\mathbf{x}_2$.

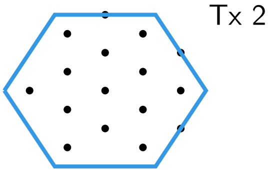
Codebook Generation

Select a nested lattice code:

- Coarse lattice $\Lambda = \mathbf{B}\mathbb{Z}^n$ for shaping.
- Fine lattice from q-ary linear code $\mathbf{G}$ for coding.



Encoding



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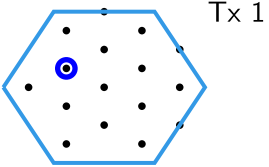
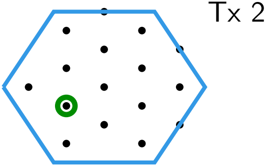


$$\mathbf{t}_1 = [\mathbf{B}\gamma\mathbf{G}\mathbf{w_1}] \bmod \Lambda$$

Encoding

- Map messages $\mathbf{w}_1, \mathbf{w}_2$ to lattice points $\mathbf{t}_1, \mathbf{t}_2$.



$$\mathbf{t}_2 = [\mathbf{B}\gamma\mathbf{G}\mathbf{w_2}] \bmod \Lambda$$

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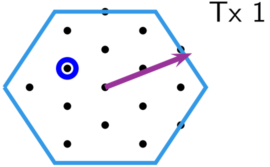

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- Map messages $\mathbf{w}_1, \mathbf{w}_2$ to lattice points $\mathbf{t}_1, \mathbf{t}_2$.
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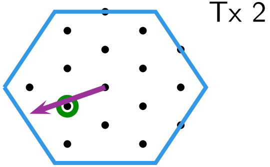

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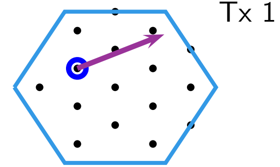
- Add dithers to lattice points and take $\bmod \Lambda$ to get transmitted signals $\mathbf{x}_1, \mathbf{x}_2$.



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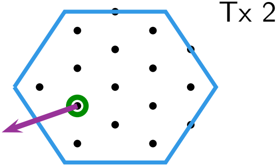

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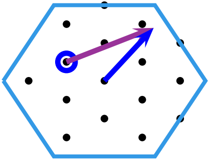
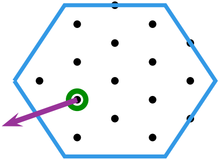
Tx 1



$$\mathbf{t}_1 = [\mathbf{B}\gamma\mathbf{G}\mathbf{w_1}] \bmod \Lambda$$
$$\mathbf{x}_1 = [\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda$$

Tx 2



$$\mathbf{t}_2 = [\mathbf{B}\gamma\mathbf{G}\mathbf{w_2}] \bmod \Lambda$$
$$\mathbf{x}_2 = [\mathbf{t}_1 + \mathbf{d}_2] \bmod \Lambda$$

Codebook Generation

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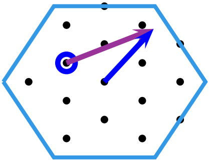
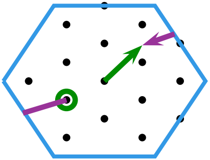


$$\mathbf{t}_1 = [\mathbf{B}\gamma\mathbf{G}\mathbf{w_1}] \bmod \Lambda$$
$$\mathbf{x}_1 = [\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda$$

Tx 2

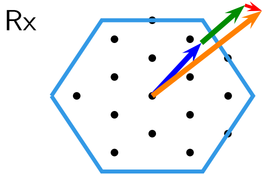


$$\mathbf{t}_2 = [\mathbf{B}\gamma\mathbf{G}\mathbf{w_2}] \bmod \Lambda$$
$$\mathbf{x}_2 = [\mathbf{t}_1 + \mathbf{d}_2] \bmod \Lambda$$

Lattice Achievability "Recipe" – Multiple-Access Corner Point

Receiver observes $\mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}$.

Decoding



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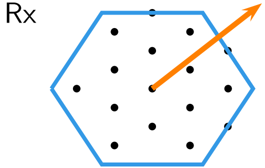
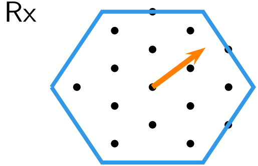


Lattice Achievability "Recipe" – Multiple-Access Corner Point

Receiver observes $\mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}$.

Decoding

- Scale by α.



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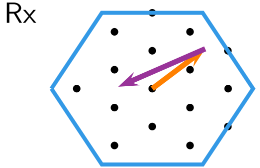
Decoding

- Scale by α.
- Subtract dither $\mathbf{d}_1$.

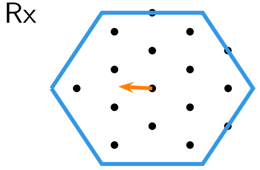


Lattice Achievability "Recipe" – Multiple-Access Corner Point

Receiver observes $\mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}$.

Decoding

- Scale by α.
- Subtract dither $\mathbf{d}_1$.
- Take $\mod \Lambda$.



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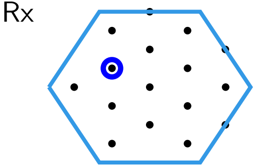
Decoding



- Scale by α.
- Subtract dither $\mathbf{d}_1$.
- Take $\bmod \Lambda$.
- Decode to nearest codeword.

$$
\begin{aligned}
&[\alpha\mathbf{y} - \mathbf{d}_1] \bmod \Lambda \\
&= [\alpha(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}) - \mathbf{d}_1] \bmod \Lambda \\
&= [\mathbf{x}_1 - \mathbf{d}_1 + \alpha\mathbf{z} + \alpha\mathbf{x}_2 - (1-\alpha)\mathbf{x}_1] \bmod \Lambda \\
&= \Big[[\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda - \mathbf{d}_1 + \alpha\mathbf{z} + \alpha\mathbf{x}_2 - (1-\alpha)\mathbf{x}_1\Big] \bmod \Lambda \\
&= [\mathbf{t}_1 + \underbrace{\alpha\mathbf{z} + \alpha\mathbf{x}_2 - (1-\alpha)\mathbf{x}_1}_{\text{Effective Noise}}]
\end{aligned}
$$

Lattice Achievability "Recipe" – Multiple-Access Corner Point

- Effective noise after scaling is $N_{\mathsf{EFFEC}} = \alpha^2(N + P_2) + (1-\alpha)^2 P_1$.
- Minimized by setting α to be the MMSE coefficient:

$$\alpha_{\mathsf{MMSE}} = \frac{P_1}{N + P_1 + P_2}$$

- Plugging in, we get

$$N_{\mathsf{EFFEC}} = \frac{(N + P_2)P_1}{N + P_1 + P_2}$$

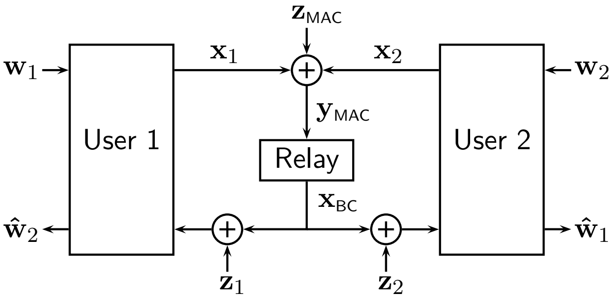
- Resulting rate is

$$R = \frac{1}{2}\log\left(\frac{P_1}{N_{\mathsf{EFFEC}}}\right) = \frac{1}{2}\log\left(1 + \frac{P_1}{N + P_2}\right)$$

- To obtain different rates for $\mathbf{x}_1$ and $\mathbf{x}_2$, use nested linear codes $\mathbf{G}_1$ and $\mathbf{G}_2$ inside Voronoi region $\mathcal{V}$.

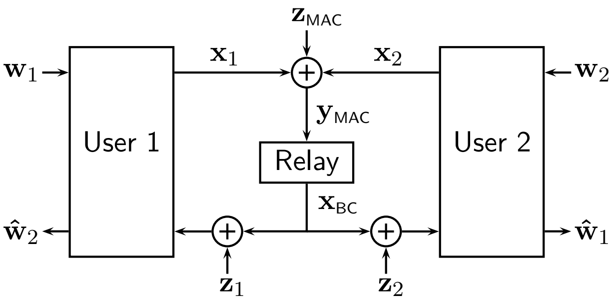
AWGN Two-Way Relay Channel – Symmetric Rates

AWGN Two-Way Relay Channel – Symmetric Rates



- Equal power constraints P.
- Equal noise variances N.
- Equal rates R.

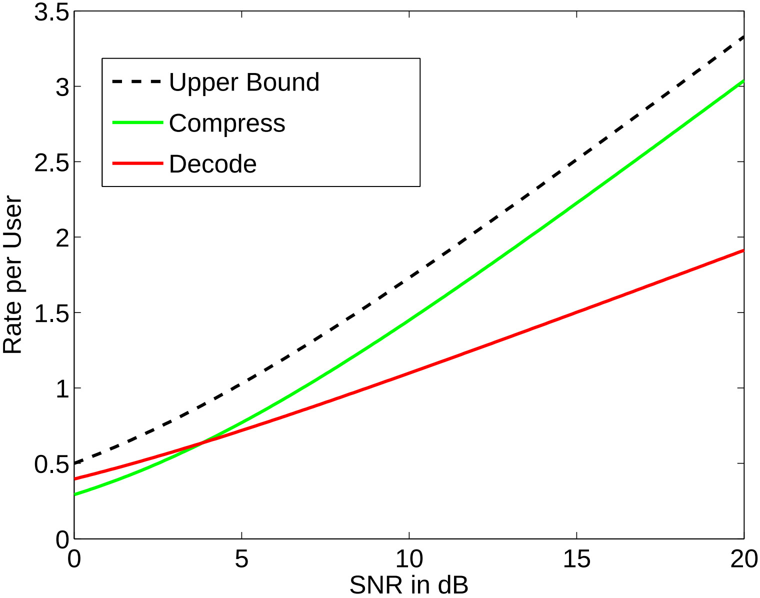
AWGN Two-Way Relay Channel – Symmetric Rates



- Equal power constraints P.
- Equal noise variances N.
- Equal rates R.

- Upper Bound:
  $R \leq \frac{1}{2} \log \left(1 + \frac{P}{N}\right)$
- Decode-and-Forward: Relay decodes $\mathbf{w}_1, \mathbf{w}_2$ and transmits $\mathbf{w}_1 \oplus \mathbf{w}_2$.
  $R = \frac{1}{4} \log \left(1 + \frac{2P}{N}\right)$
- Compress-and-Forward: Relay transmits quantized $\mathbf{y}$.
  $R = \frac{1}{2} \log \left(1 + \frac{P}{N} \frac{P}{3P + N}\right)$

AWGN Two-Way Relay Channel – Symmetric Rates

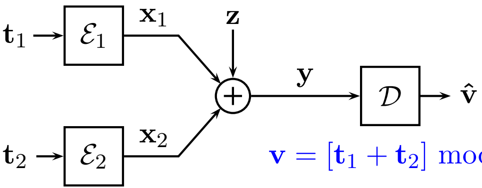


Decoding the Sum of Lattice Codewords

Encoders use the same nested lattice codebook.

Transmit lattice codewords:

$$\mathbf{x}_1 = \mathbf{t}_1$$
$$\mathbf{x}_2 = \mathbf{t}_2$$



$$\mathbf{v} = [\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda$$

Decoder recovers modulo sum.

$$\begin{aligned} &[\mathbf{y}] \bmod \Lambda \\ &= [\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}] \bmod \Lambda \\ &= [\mathbf{t}_1 + \mathbf{t}_2 + \mathbf{z}] \bmod \Lambda \\ &= \Big[[\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda + \mathbf{z} \Big] \bmod \Lambda \qquad \text{Distributive Law} \\ &= [\mathbf{v} + \mathbf{z}] \bmod \Lambda \end{aligned}$$

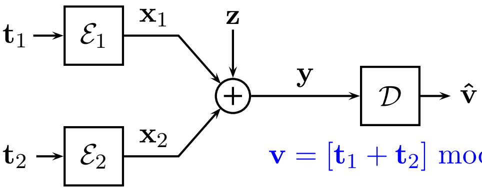
$$R = \frac{1}{2} \log \left(\frac{P}{N} \right)$$

Decoding the Sum of Lattice Codewords – MMSE Scaling

Encoders use the same nested lattice codebook.

Transmit dithered codewords:

$$\mathbf{x}_1 = [\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda$$
$$\mathbf{x}_2 = [\mathbf{t}_2 + \mathbf{d}_2] \bmod \Lambda$$



$$\mathbf{v} = [\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda$$

Decoder scales by α, removes dithers, recovers modulo sum.

$$
\begin{aligned}
&[\alpha\mathbf{y} - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= [\alpha(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}) - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= [\mathbf{x}_1 + \mathbf{x}_2 - (1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \alpha\mathbf{z} - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= \Big[[\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda - (1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \alpha\mathbf{z}\Big] \bmod \Lambda \\
&= [\mathbf{v} \underbrace{- (1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \alpha\mathbf{z}}_{\text{Effective Noise}}] \bmod \Lambda
\end{aligned}
$$

$$N_{\mathsf{EFFEC}} = (1-\alpha)^2 2P + \alpha^2 N$$

Decoding the Sum of Lattice Codewords – MMSE Scaling

- Effective noise after scaling is $N_{\textsf{EFFEC}} = (1-\alpha)^2 2P + \alpha^2 N$.
- Minimized by setting α to be the MMSE coefficient:

$$\alpha_{\textsf{MMSE}} = \frac{2P}{N + 2P}$$

- Plugging in, we get

$$N_{\textsf{EFFEC}} = \frac{2NP}{N + 2P}$$

- Resulting rate is

$$R = \frac{1}{2}\log\left(\frac{P}{N_{\textsf{EFFEC}}}\right) = \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{N}\right)$$

- Getting the full "one plus" term is an open challenge. Does not seem possible with nested lattices.

From Messages to Lattice Points and Back

- Map messages to lattice points

$$\mathbf{t}_1 = \phi(\mathbf{w}_1) = [\mathbf{B}\gamma\mathbf{G}\mathbf{w}_1] \bmod \Lambda$$
$$\mathbf{t}_2 = \phi(\mathbf{w}_2) = [\mathbf{B}\gamma\mathbf{G}\mathbf{w}_2] \bmod \Lambda$$

- Mapping between finite field messages and lattice codewords preserves linearity:

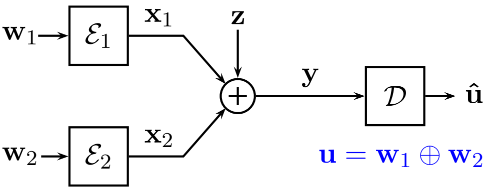
$$\phi^{-1}\Big([\mathbf{t}_1 + \mathbf{t}_2] \mod \Lambda\Big) = \mathbf{w}_1 \oplus \mathbf{w}_2$$

- This means that after decoding a $\bmod \Lambda$ equation of lattice points we can immediately recover the finite field equation of the messages. See **Nazer-Gastpar '11** for more details.

Finite Field Computation over a Gaussian MAC

Map messages to lattice points:

$$\mathbf{t}_1 = \phi(\mathbf{w}_1)$$
$$\mathbf{t}_2 = \phi(\mathbf{w}_2)$$



Transmit dithered codewords:

$$\mathbf{x}_1 = [\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda$$
$$\mathbf{x}_2 = [\mathbf{t}_2 + \mathbf{d}_2] \bmod \Lambda$$

- If decoder can recover $[\mathbf{t}_1 + \mathbf{t}_2] \bmod \Lambda$, it also can get the sum of the messages

$$\mathbf{w}_1 \oplus \mathbf{w}_2 = \phi^{-1}\Big([\mathbf{t}_1 + \mathbf{t}_2] \mod \Lambda\Big) .$$

- Achievable rate $R = \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{N}\right)$.

AWGN Two-Way Relay Channel – Symmetric Rates

Has $\mathbf{w}_1$

Wants $\mathbf{w}_2$

Relay

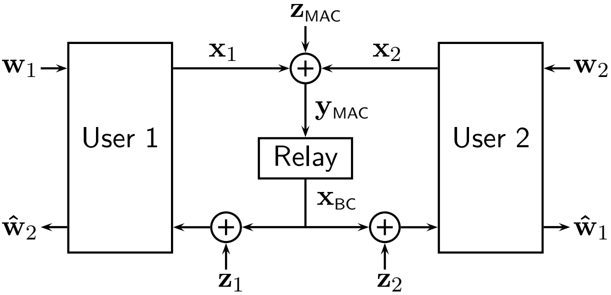
Has $\mathbf{w}_2$

Wants $\mathbf{w}_1$

- Equal power constraints P.
- Equal noise variances N.
- Equal rates R.

- Upper Bound:
  $$R \leq \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$
- Compute-and-Forward: Relay decodes $\mathbf{w}_1 \oplus \mathbf{w}_2$ and retransmits.
  $$R = \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{N}\right)$$
- **Wilson-Narayanan-Pfister-Sprintson '10:** Applies nested lattice codes to the two-way relay channel.

AWGN Two-Way Relay Channel – Symmetric Rates



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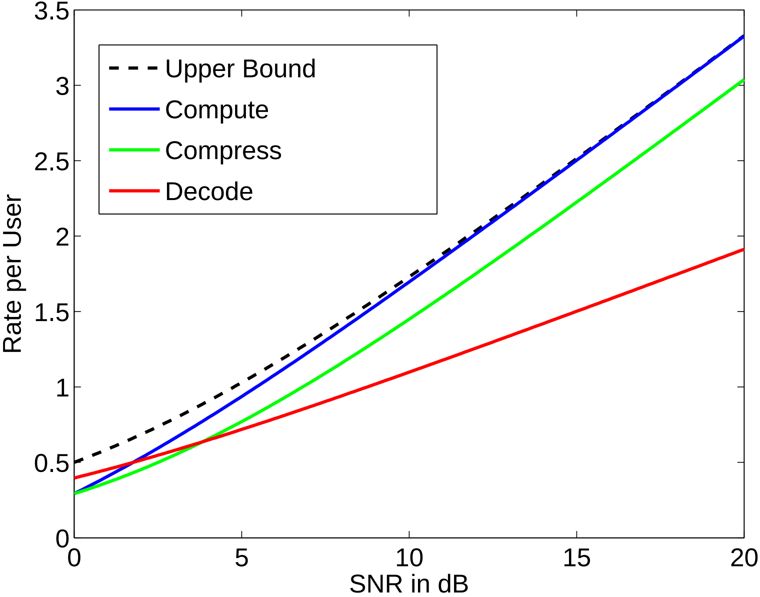
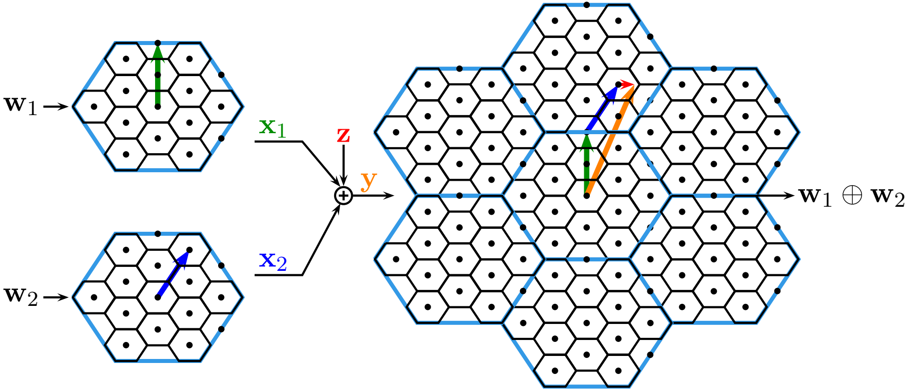


Compute-and-Forward Illustration



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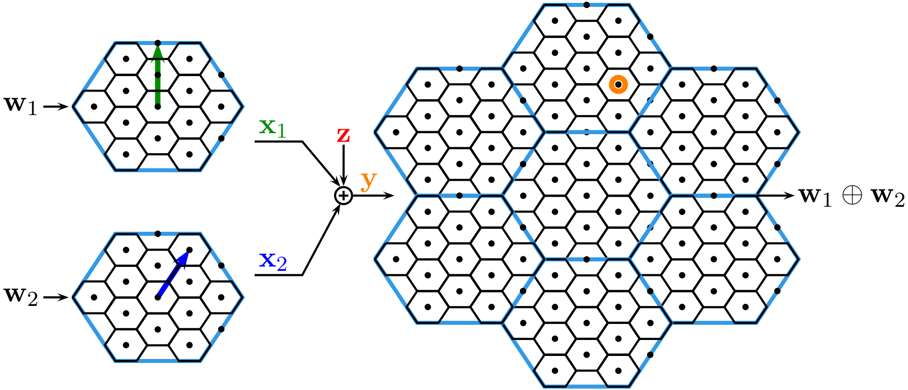
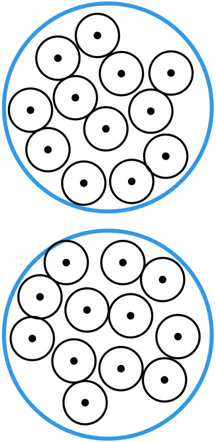


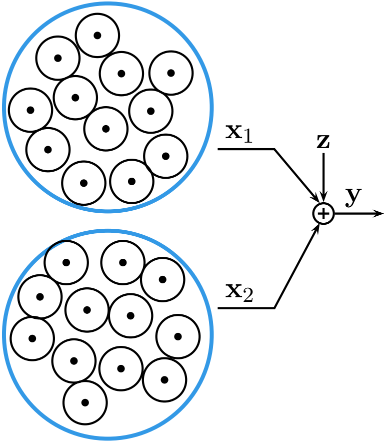
Random i.i.d. codes are not good for computation



2^{nR} codewords each.

2^{n2R} possible sums of codewords.

Random i.i.d. codes are not good for computation



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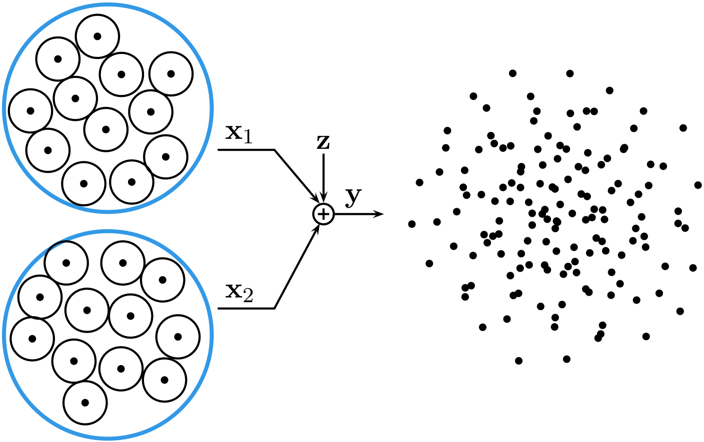

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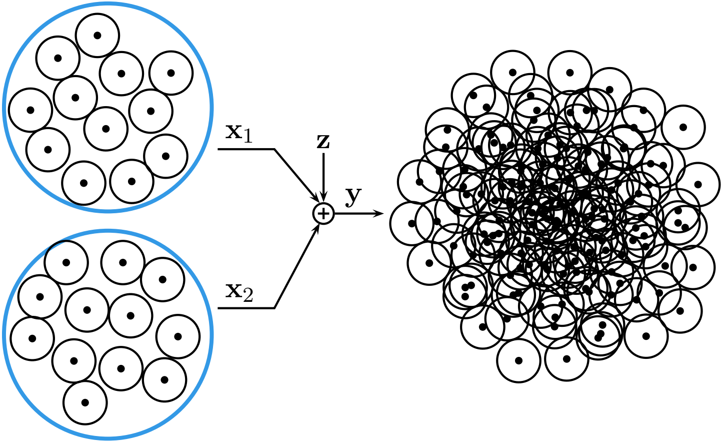

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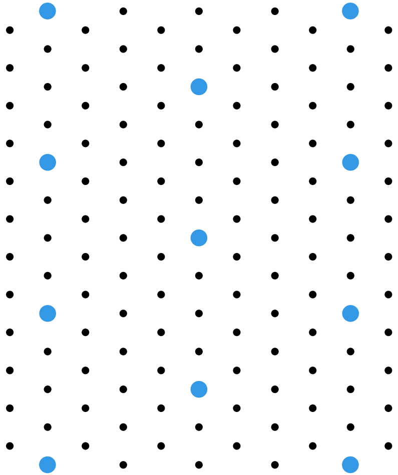
Unequal Power Constraints – Double Nesting

- What if the power constraints are not equal?
- Idea from **Nam-Chung-Lee '10**:
- Draw the codewords from the same fine lattice Λ_{FINE}.
- Use two nested coarse lattices Λ_1 and Λ_2 to enforce the power constraints P_1 and P_2.



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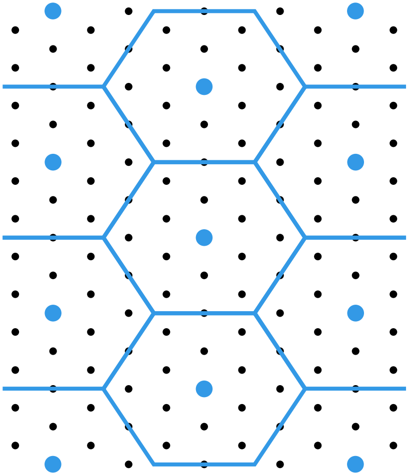

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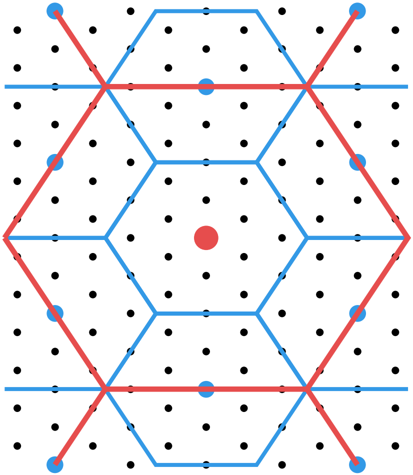

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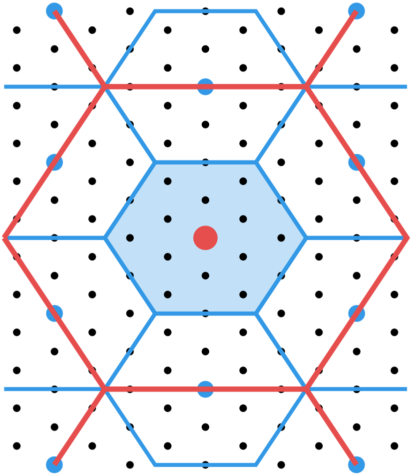

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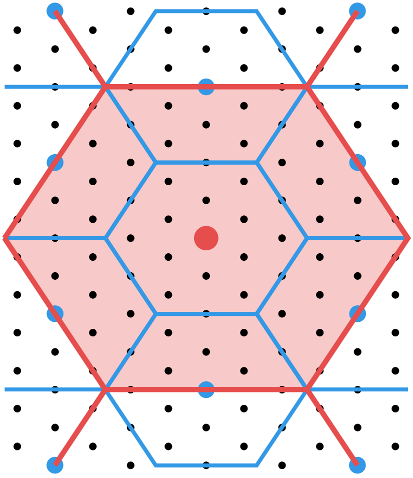

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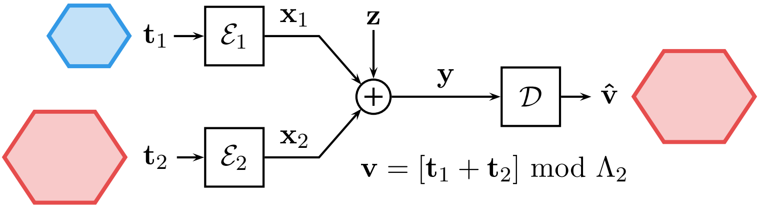
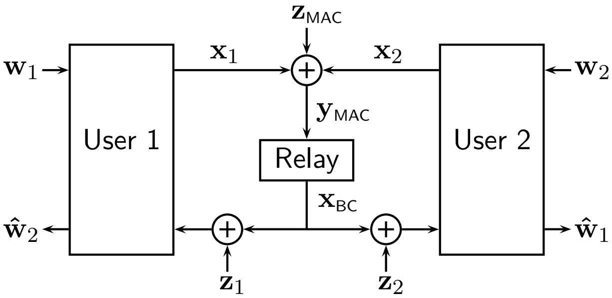


- Encoder 1 sends $\mathbf{x}_1 = [\mathbf{t}_1 + \mathbf{d}_1] \bmod \Lambda_1$. Coarse lattice Λ_1 has second moment P_1.
- Encoder 2 sends $\mathbf{x}_2 = [\mathbf{t}_2 + \mathbf{d}_2] \bmod \Lambda_2$. Coarse lattice Λ_2 has second moment $P_2 > P_1$.
- Decoder performs MMSE scaling, remove dithers, recovers $\bmod \Lambda_2$ sum.

$$R_1 = \frac{1}{2} \log \left(\frac{P_1}{P_1 + P_2} + \frac{P_1}{N} \right) \qquad R_2 = \frac{1}{2} \log \left(\frac{P_2}{P_1 + P_2} + \frac{P_2}{N} \right)$$

AWGN Two-Way Relay Channel



- User powers P_1, P_2.
- MAC noise variance N_{MAC}.
- Relay power P_{BC}.
- Broadcast noise variances N_1, N_2.

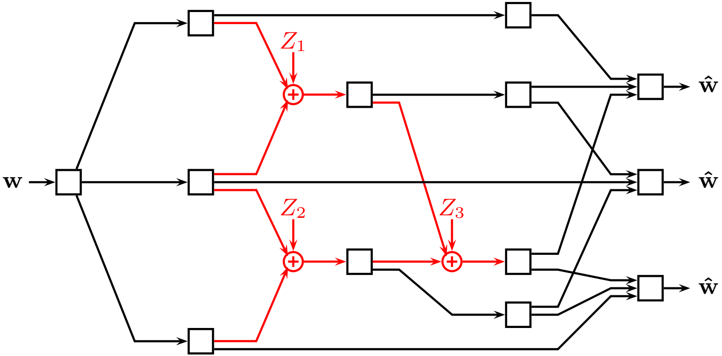
Theorem (Nam-Chung-Lee '10)

Capacity region is within $1/2$ *bit of:*

$$R_1 \leq \min\left(\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2}+\frac{P_1}{N_{MAC}}\right),\ \frac{1}{2}\log\left(1+\frac{P_{BC}}{N_2}\right)\right)$$
$$R_2 \leq \min\left(\frac{1}{2}\log\left(\frac{P_2}{P_1+P_2}+\frac{P_2}{N_{MAC}}\right),\ \frac{1}{2}\log\left(1+\frac{P_{BC}}{N_1}\right)\right)$$

Moreover, "constant gap" goes to zero as powers increase.

Multiple-Access Networks



- Multicast demands
- Multi-access interference
- No broadcast constraints

- Compute-and-forward is well-suited for multicasting over multiple-access networks.
- Equal transmitter powers: **Nazer-Gastpar '07.**
  Unequal transmitter powers: **Nam-Chung-Lee '09.**

I. Discrete Alphabets

II. AWGN Channels

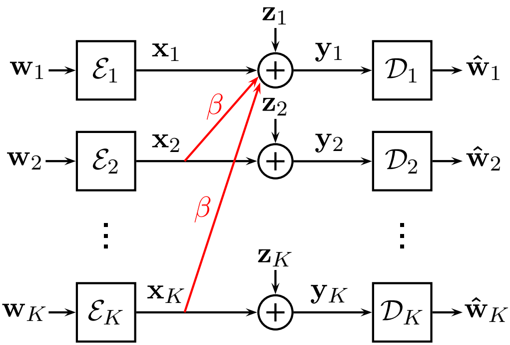
III. Network Applications

Many-to-One Interference Channel – Symmetric Very Strong Case

- Equal rates R.
- Only receiver 1 sees interference:

$$\mathbf{y}_1 = \mathbf{x}_1 + \beta \sum_{\ell=2}^{K} \mathbf{x}_\ell + \mathbf{z}_1$$

- How big does β have to be to achieve $R = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$? (i.e. "very strong" case)



- Scheme A: Decode $\mathbf{w}_2, \ldots, \mathbf{w}_K$ at receiver 1 and remove prior to decoding $\mathbf{w}_1$.

$$R \leq \frac{1}{2(K-1)} \log\left(1 + \frac{\beta^2 (K-1) P}{N+P}\right)$$

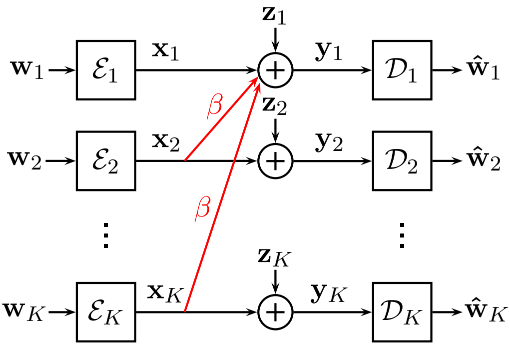
- Scheme B: Decode $\mathbf{w}_2 \oplus \cdots \oplus \mathbf{w}_K$ at receiver 1 and remove prior to decoding $\mathbf{w}_1$.

Many-to-One Interference Channel – Symmetric Very Strong Case

Encoders use the same nested lattice codebook.

Transmit dithered codewords:

$$\mathbf{x}_\ell = [\mathbf{t}_\ell + \mathbf{d}_\ell] \bmod \Lambda$$



Decoder scales by β^{-1}, removes dithers, recovers modulo sum.

$$\left[\beta^{-1}\mathbf{y}_1 - \sum_{\ell=2}^{K} \mathbf{d}_\ell\right] \bmod \Lambda = \left[\sum_{\ell=2}^{K}(\mathbf{x}_\ell - \mathbf{d}_\ell) + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right] \bmod \Lambda$$

(Distributive Law) $$= \left[\left[\sum_{\ell=2}^{K} \mathbf{t}_\ell\right] \bmod \Lambda + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right] \bmod \Lambda$$

$$\left[\beta^{-1}\mathbf{y}_1 - \sum_{\ell=2}^{K}\mathbf{d}_\ell\right] \bmod \Lambda = \left[\left[\sum_{\ell=2}^{K}\mathbf{t}_\ell\right] \bmod \Lambda + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right] \bmod \Lambda$$

- Effective noise variance $N_{\mathsf{EFFEC}} = \beta^{-2}(P+N)$.
- Can decode $\bmod \Lambda$ sum of lattice points at rate $R = \frac{1}{2}\log\left(\frac{\beta^2 P}{P+N}\right)$.
- Setting equal to "very strong" condition $R = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$ we get

$$\beta^2 = \frac{(P+N)^2}{PN}$$

- How can we recover $\mathbf{w}_1$?
- We need to first subtract the real sum of the codewords. So far, we only have the modulo-sum.

Successive Cancellation of Sums

- First, add back in dithers to get modulo sum of codewords:

$$\Bigg[\Big[\sum_{\ell=2}^{K}\mathbf{t}_\ell\Big] \bmod \Lambda + \Big[\sum_{\ell=2}^{K}\mathbf{d}_\ell\Big] \bmod \Lambda\Bigg] \bmod \Lambda = \Big[\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big] \bmod \Lambda$$

Successive Cancellation of Sums

- First, add back in dithers to get modulo sum of codewords:

$$\left[\left[\sum_{\ell=2}^{K} \mathbf{t}_\ell\right] \bmod \Lambda + \left[\sum_{\ell=2}^{K} \mathbf{d}_\ell\right] \bmod \Lambda\right] \bmod \Lambda = \left[\sum_{\ell=2}^{K} \mathbf{x}_\ell\right] \bmod \Lambda$$

- Subtract from $\mathbf{y}_1$ to expose the coarse lattice point nearest to the real sum $\sum_{\ell=2}^{K} \mathbf{x}_\ell$:

$$\beta^{-1}\mathbf{y}_1 - \left[\sum_{\ell=2}^{K} \mathbf{x}_\ell\right] \bmod \Lambda = Q_\Lambda\Big(\sum_{\ell=2}^{K} \mathbf{x}_\ell\Big) + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)$$

- Coarse lattice point easier to decode than fine lattice point:

$$Q_\Lambda\left(Q_\Lambda\Big(\sum_{\ell=2}^{K} \mathbf{x}_\ell\Big) + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right) = Q_\Lambda\Big(\sum_{\ell=2}^{K} \mathbf{x}_\ell\Big) \quad \text{w.h.p.}$$

Successive Cancellation of Sums

- First, add back in dithers to get modulo sum of codewords:

$$\left[\left[\sum_{\ell=2}^{K}\mathbf{t}_\ell\right] \bmod \Lambda + \left[\sum_{\ell=2}^{K}\mathbf{d}_\ell\right] \bmod \Lambda\right] \bmod \Lambda = \left[\sum_{\ell=2}^{K}\mathbf{x}_\ell\right] \bmod \Lambda$$

- Subtract from $\mathbf{y}_1$ to expose the coarse lattice point nearest to the real sum $\sum_{\ell=2}^{K}\mathbf{x}_\ell$:

$$\beta^{-1}\mathbf{y}_1 - \left[\sum_{\ell=2}^{K}\mathbf{x}_\ell\right] \bmod \Lambda = Q_\Lambda\Big(\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big) + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)$$

- Coarse lattice point easier to decode than fine lattice point:

$$Q_\Lambda\left(Q_\Lambda\Big(\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big) + \beta^{-1}(\mathbf{x}_1 + \mathbf{z}_1)\right) = Q_\Lambda\Big(\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big) \quad \text{w.h.p.}$$

- Finally, get back the real sum

$$\left[\sum_{\ell=2}^{K}\mathbf{x}_\ell\right] \bmod \Lambda + Q_\Lambda\Big(\sum_{\ell=2}^{K}\mathbf{x}_\ell\Big) = \sum_{\ell=2}^{K}\mathbf{x}_\ell$$

Successive Cancellation of Sums

- We now have the sum of interfering codewords and can cancel them out:

$$\mathbf{y}_1 - \beta \sum_{\ell=2}^{K} \mathbf{x}_\ell = \mathbf{x}_1 + \mathbf{z}_1$$

- Can apply standard MMSE lattice decoding to recover lattice point $\mathbf{t}_1$ and then map back to $\mathbf{w}_1$.
- Overall, structured coding permits

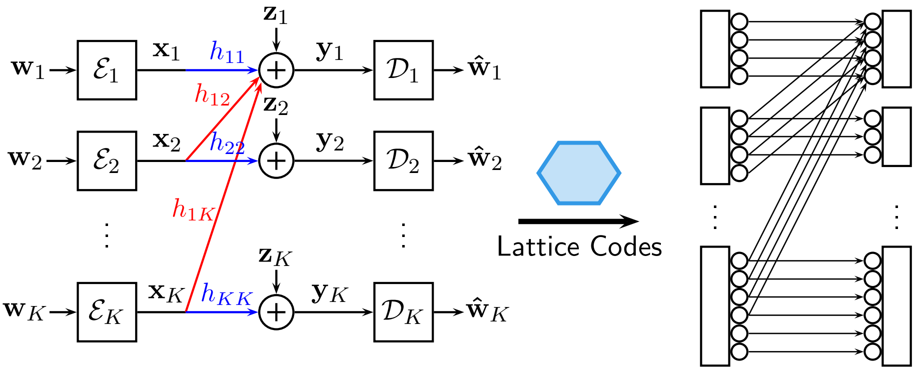
$$\beta^2 \geq \frac{(P+N)^2}{PN}$$

- Compare to decoding interfering codewords in their entirety:

$$\beta^2 \geq \frac{\left((1+\frac{P}{N})^{K-1} - 1\right)(N+P)}{(K-1)P}$$

- Originally shown in **Sridharan-Jafarian-Vishwanath-Jafar '08** using spherical shaping region. Nested lattice scheme from **Nazer '11**.

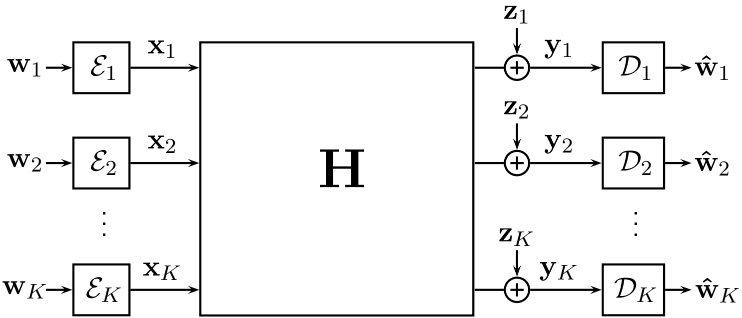
Many-to-One Interference Channel – Approximate Capacity



- Deterministic model by **Avestimehr-Diggavi-Tse '11** shows how to decompose by signal scale.

Theorem (Bresler-Parekh-Tse '10)

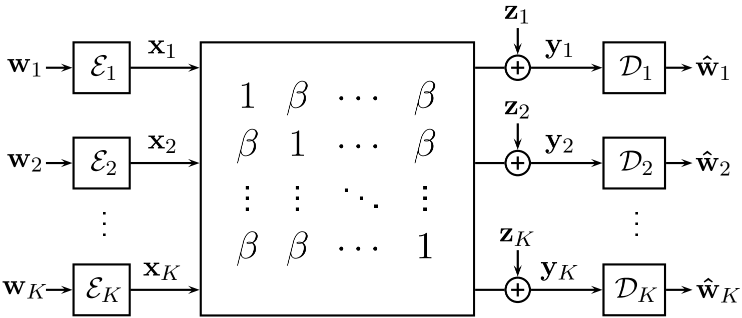
Lattices codes combined with the deterministic model can approach the capacity region to within $(3K+3)(1+\log(K+1))$ *bits per user.*



- Equal rates R. How big does β have to be to achieve $R = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$? (i.e. "very strong" case)
- Can use the many-to-one decoder at every receiver to get

$$\beta^2 \geq \frac{(P+N)^2}{PN}$$

- What about asymmetric interference channels?



- Equal rates R. How big does β have to be to achieve $R = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$? (i.e. "very strong" case)
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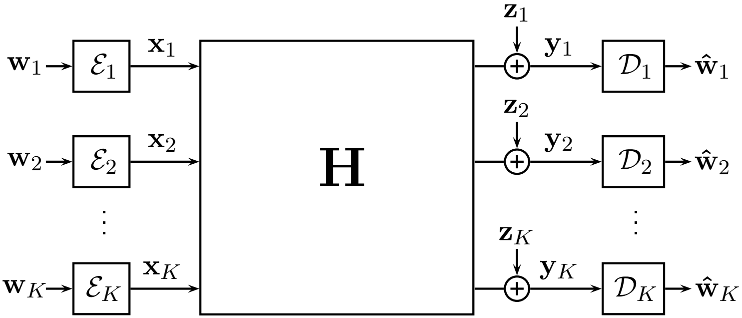
- What about asymmetric interference channels?

Interference Channel



- Not clear how to map to a deterministic model using lattices.
- "Real" interference alignment scheme of **Motahari et al. '08** uses a lattice structure to get $K/2$ DoF (up to a set of measure one)
- Some special cases at finite SNR: **Jafarian-Viswanath '09,'10, Ordentlich-Erez '11**
- Much more known for time-varying channels: **Cadambe-Jafar '08, Nazer et al. '11, much more**

Summary

- So far we have seen that lattices are very effective for scenarios where there is a single interference bottleneck.
- Also effective for multiple bottlenecks but less is known.
- We have so far assumed that the fading coefficients are known at the transmitters.
- In general, transmitters may not have access to channel state information.

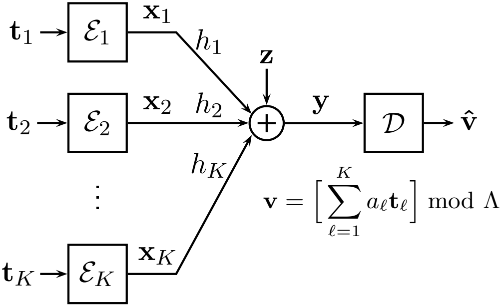
Computation over Fading Channels

Transmitters do not know channel realization.

Encoders use the same nested lattice codebook.

Transmit dithered codewords:

$$\mathbf{x}_\ell = [\mathbf{t}_\ell + \mathbf{d}_\ell] \bmod \Lambda$$



- Decoder removes dithers and recovers integer combination

$$\mathbf{v} = \left[\sum_{\ell=1}^{K} a_\ell \mathbf{t}_\ell\right] \bmod \Lambda$$

- Receiver can use its knowledge of the channel gains to match the equation coefficients a_ℓ to the channel coefficients h_ℓ.

- Distributive Law also holds for integer combinations. Let $a, b \in \mathbb{Z}$.

$$\begin{aligned}
&\Big[a[\mathbf{x}_1] \bmod \Lambda + b[\mathbf{x}_2] \bmod \Lambda\Big] \bmod \Lambda \\
&= \Big[a\Big(\mathbf{x}_1 - Q_\Lambda(\mathbf{x}_1)\Big) + b\Big(\mathbf{x}_2 - Q_\Lambda(\mathbf{x}_2)\Big)\Big] \bmod \Lambda \\
&= \Big[a\mathbf{x}_1 + b\mathbf{x}_2 - aQ_\Lambda(\mathbf{x}_1) - bQ_\Lambda(\mathbf{x}_2)\Big] \bmod \Lambda \\
&= [a\mathbf{x}_1 + b\mathbf{x}_2] \bmod \Lambda
\end{aligned}$$

- Last step follows since since $aQ_\Lambda(\mathbf{x}_1)$ and $bQ_\Lambda(\mathbf{x}_2)$ are elements of the lattice Λ.

- Transmit dithered codewords $\mathbf{x}_\ell = [\mathbf{t}_\ell + \mathbf{d}_\ell] \bmod \Lambda$
- Decoder removes dithers and recovers integer combination

$$
\begin{aligned}
&\Big[\mathbf{y} - \sum_{\ell=1}^{K} a_\ell \mathbf{d}_\ell\Big] \bmod \Lambda \\
&= \Big[\sum_{\ell=1}^{K} h_\ell \mathbf{x}_\ell + \mathbf{z} - \sum_{\ell=1}^{K} a_\ell \mathbf{d}_\ell\Big] \bmod \Lambda \\
&= \Big[\sum_{\ell=1}^{K} a_\ell (\mathbf{x}_\ell - \mathbf{d}_\ell) + \sum_{\ell=1}^{K} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}\Big] \bmod \Lambda \\
&= \Big[\Big[\sum_{\ell=1}^{K} a_\ell \mathbf{t}_\ell\Big] \bmod \Lambda + \underbrace{\sum_{\ell=1}^{K} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}}_{\text{Effective Noise}}\Big] \bmod \Lambda \quad \text{Distributive Law}
\end{aligned}
$$

Computation over Fading Channels – Effective Noise

- Effective noise due to mismatch between channel coefficients $\mathbf{h} = [h_1 \cdots h_K]^T$ and equation coefficients $\mathbf{a} = [a_1 \cdots a_K]^T$.

$$
\begin{aligned}
N_{\mathsf{EFFEC}} &= N + P\|\mathbf{h} - \mathbf{a}\|^2 \\
R &= \frac{1}{2}\log\left(\frac{P}{N + P\|\mathbf{h} - \mathbf{a}\|^2}\right)
\end{aligned}
$$

Computation over Fading Channels – Effective Noise

- Effective noise due to mismatch between channel coefficients $\mathbf{h} = [h_1 \cdots h_K]^T$ and equation coefficients $\mathbf{a} = [a_1 \cdots a_K]^T$.

$$\begin{aligned} N_{\mathsf{EFFEC}} &= N + P\|\mathbf{h} - \mathbf{a}\|^2 \\ R &= \frac{1}{2}\log\left(\frac{P}{N + P\|\mathbf{h} - \mathbf{a}\|^2}\right) \end{aligned}$$

- Can do better with MMSE scaling.

$$\begin{aligned} N_{\mathsf{EFFEC}} &= \alpha^2 N + P\|\alpha\mathbf{h} - \mathbf{a}\|^2 \\ R &= \max_{\alpha} \frac{1}{2}\log\left(\frac{P}{\alpha^2 N + P\|\alpha\mathbf{h} - \mathbf{a}\|^2}\right) \\ &= \frac{1}{2}\log\left(\frac{N + P\|\mathbf{h}\|^2}{N\|\mathbf{a}\|^2 + P(\|\mathbf{h}\|^2\|\mathbf{a}\|^2 - (\mathbf{h}^T\mathbf{a})^2)}\right) \end{aligned}$$

- See **Nazer-Gastpar '11** for more details.

- The rate expression simplifies in some special cases.

$$R = \frac{1}{2} \log \left(\frac{N + P\|\mathbf{h}\|^2}{N\|\mathbf{a}\|^2 + P(\|\mathbf{h}\|^2\|\mathbf{a}\|^2 - (\mathbf{h}^T\mathbf{a})^2)} \right)$$

- Integer channels: $\mathbf{h} = \mathbf{a}$.

$$R = \frac{1}{2} \log \left(\frac{1}{\|\mathbf{a}\|^2} + \frac{P}{N} \right)$$

- Recovering a single message: Set $\mathbf{a} = \delta_m$, the m^{th} unit vector.

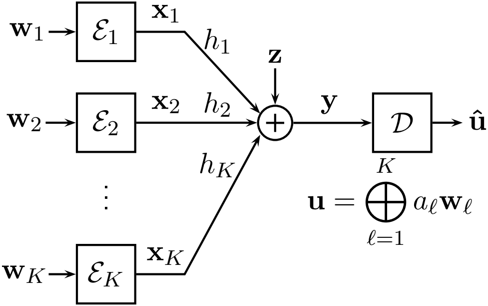
$$R = \frac{1}{2} \log \left(1 + \frac{h_m^2 P}{N + P \sum_{\ell \neq m} h_\ell^2} \right)$$

Transmitters do not know channel realization.

Encoders use the same nested lattice codebook.

Transmit dithered codewords:

$$\mathbf{x}_\ell = [\mathbf{t}_\ell + \mathbf{d}_\ell] \bmod \Lambda$$

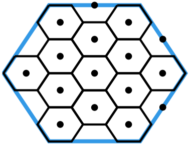


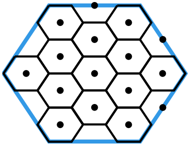
- Recall that mapping $\mathbf{t}_\ell = \phi(\mathbf{w}_\ell)$ between messages and lattice points preserves linearity.

$$\phi^{-1}\left(\left[\sum_{\ell=1}^{K} a_\ell \mathbf{t}_\ell\right] \bmod \Lambda\right) = \left[\sum_{\ell=1}^{K} a_\ell \mathbf{w}_\ell\right] \bmod q = \bigoplus_{\ell=1}^{K} a_\ell \mathbf{w}_\ell$$

- Digital interface that fits well with **network coding**.

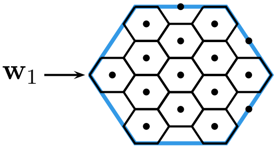
All users pick the same nested lattice code:

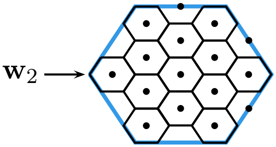




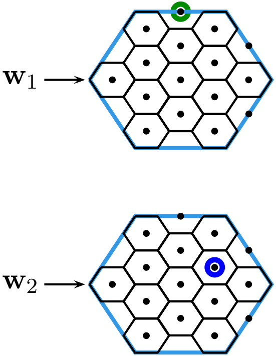
Computation Coding

Choose messages over field $\mathbf{w}_\ell \in \mathbb{F}_q^k$:

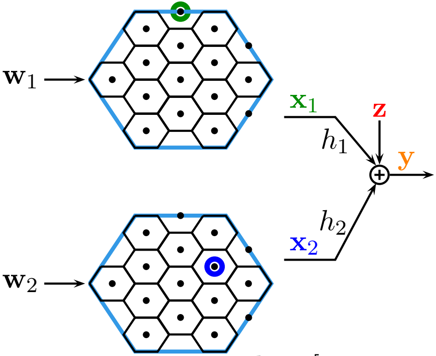




Map $\mathbf{w}_\ell$ to lattice point $\mathbf{t}_\ell = \phi(\mathbf{w}_\ell)$:



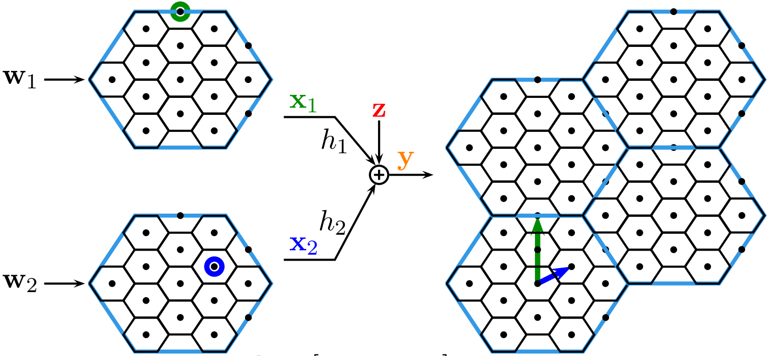
Transmit lattice points over the channel:



$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

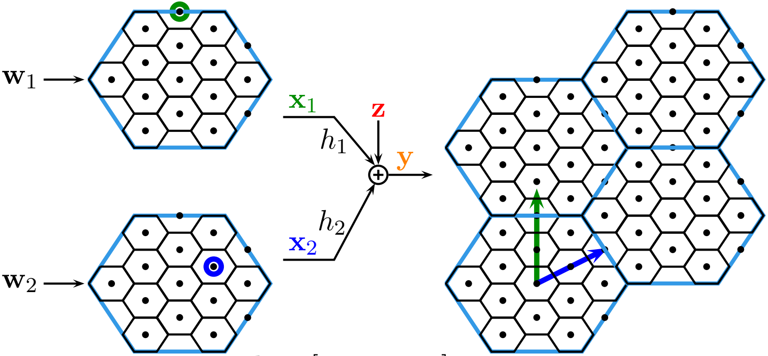
Transmit lattice points over the channel:



$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

Lattice codewords are scaled by channel coefficients:

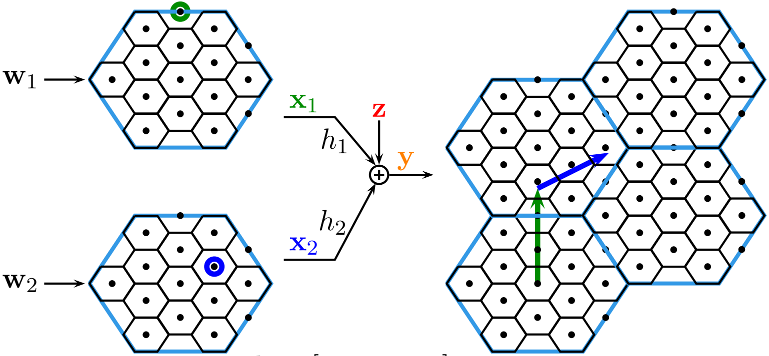


$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

Computation Coding

Scaled codewords added together plus noise:



$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

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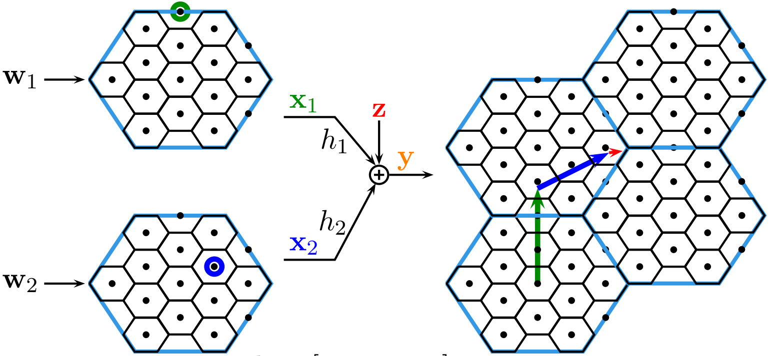
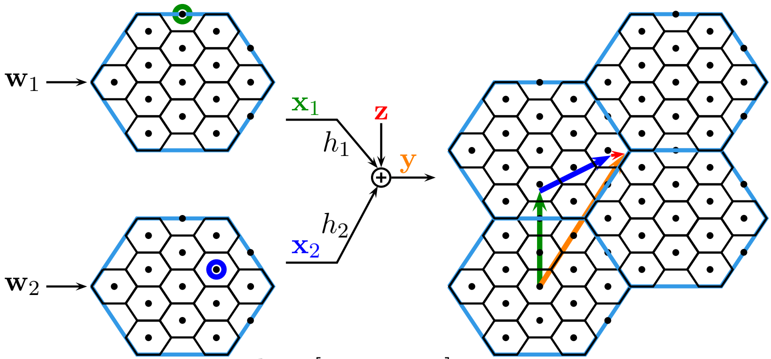
Scaled codewords added together plus noise:



$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

Extra noise penalty for non-integer channel coefficients:

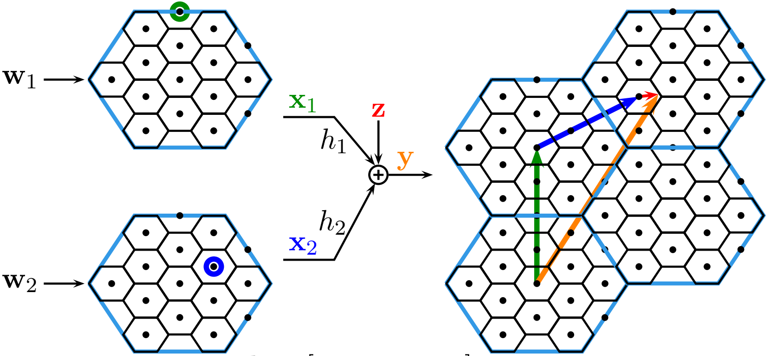


$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\mathbf{a} = [\ \ 2 \quad\ \ 3\ \ ]$$

Effective noise: $N + P\|\mathbf{h} - \mathbf{a}\|^2$

Scale output by α to reduce non-integer noise penalty:



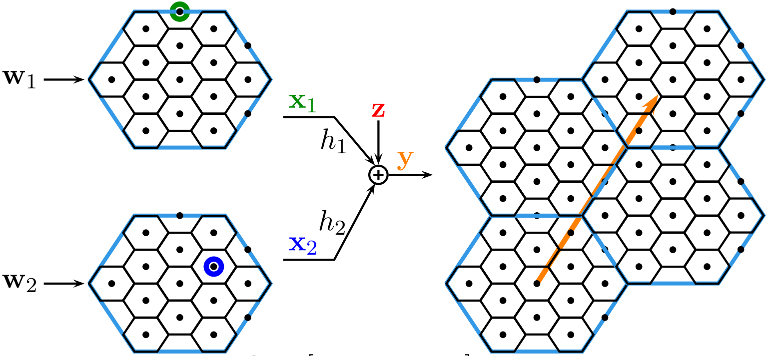
$$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad\quad 3\ ]$$

Effective noise: $\alpha^2 N + P\|\alpha\mathbf{h} - \mathbf{a}\|^2$

Computation Coding

Scale output by α to reduce non-integer noise penalty:

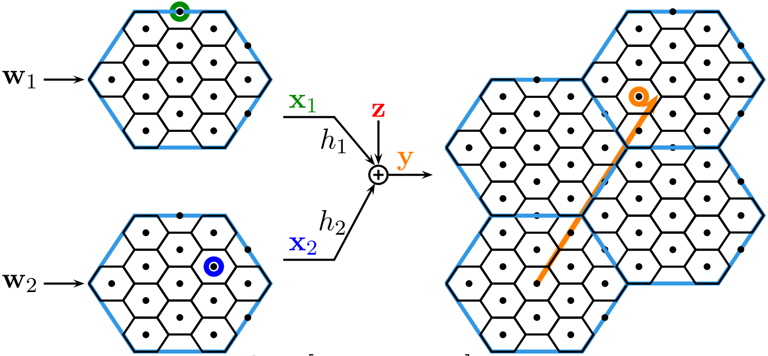


$$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

Effective noise: $\alpha^2 N + P\|\alpha\mathbf{h} - \mathbf{a}\|^2$

Decode to closest lattice point:

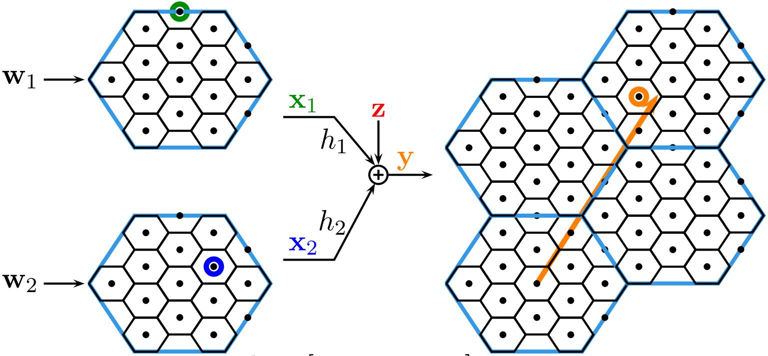


$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$

$\mathbf{a} = [\ 2 \quad 3\ ]$

Effective noise: $\alpha^2 N + P\|\alpha\mathbf{h} - \mathbf{a}\|^2$

Compute sum of lattice points modulo the coarse lattice:



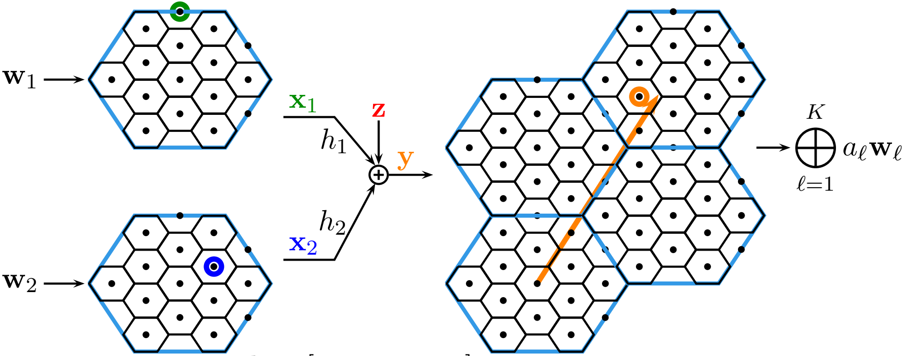
$$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

Effective noise: $\alpha^2 N + P\|\alpha\mathbf{h} - \mathbf{a}\|^2$

Computation Coding

Map back to equation of message symbols over the field:

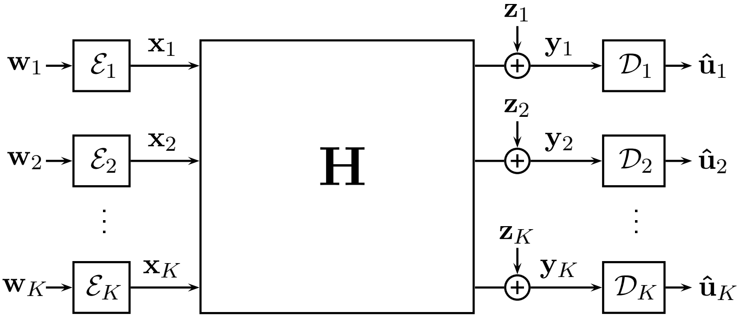


$$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

Effective noise: $\alpha^2 N + P\|\alpha\mathbf{h} - \mathbf{a}\|^2$

Computation over Fading Channels – Multiple Receivers



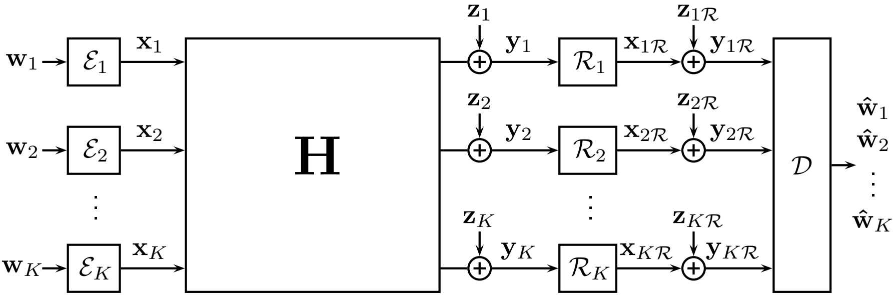
- Equal rates R. No channel state information (CSI) at transmitters.
- Receivers use their CSI to select coefficients, decode linear equation

$$\mathbf{u}_k = \bigoplus_{\ell=1}^{K} a_{k\ell}\mathbf{w}_\ell$$

- Reliable decoding possible if

$$R < \min_{k: a_{k\ell} \neq 0} \frac{1}{2}\log\left(\frac{N + P\|\mathbf{h}_k\|^2}{N\|\mathbf{a}_k\|^2 + P(\|\mathbf{h}_k\|^2\|\mathbf{a}_k\|^2 - (\mathbf{h}_k^T\mathbf{a}_k)^2)}\right)$$

Case Study – Hadamard Relay Network



- Equal rates R. $\mathbf{H}$ is a Hadamard matrix, $\mathbf{H}\mathbf{H}^T = K\mathbf{I}$

Upper Bound

$$\frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

Compute-and-Forward

$$\frac{1}{2}\log\left(\frac{1}{K}+\frac{P}{N}\right)$$

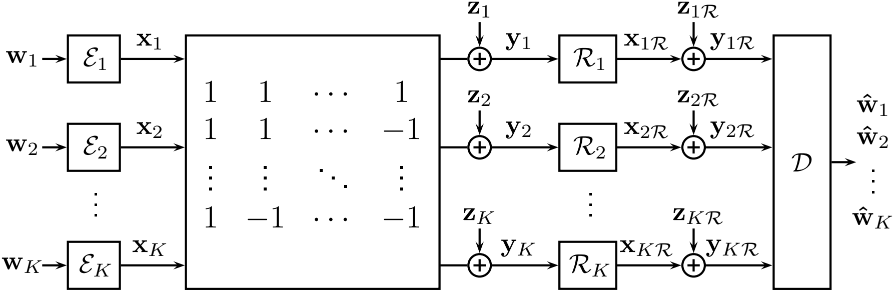
Compress-and-Forward

$$\frac{1}{2}\log\left(1+\frac{P}{N}\frac{P}{N+KP}\right)$$

Decode-and-Forward

$$\frac{1}{2K}\log\left(1+\frac{KP}{N}\right)$$

Case Study – Hadamard Relay Network



- Equal rates R. $\mathbf{H}$ is a Hadamard matrix, $\mathbf{H}\mathbf{H}^T = K\mathbf{I}$

Upper Bound

$$\frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

Compute-and-Forward

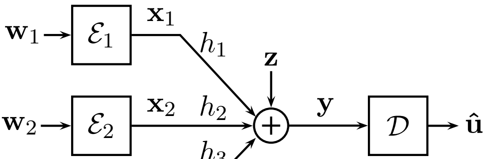
$$\frac{1}{2}\log\left(\frac{1}{K} + \frac{P}{N}\right)$$

Compress-and-Forward

$$\frac{1}{2}\log\left(1 + \frac{P}{N}\frac{P}{N + KP}\right)$$

Decode-and-Forward

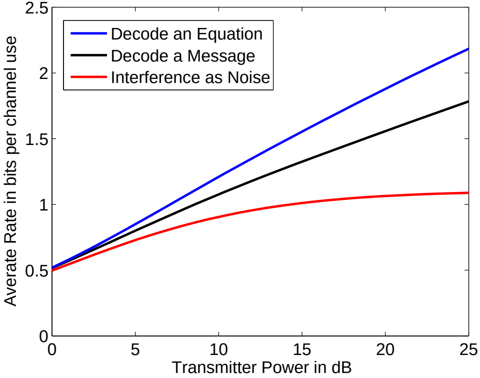
$$\frac{1}{2K}\log\left(1 + \frac{KP}{N}\right)$$



$$\mathbf{u} = \bigoplus_{\ell=1}^{K} a_\ell \mathbf{w}_\ell$$

Relay either decodes some linear function of messages or an individual message.

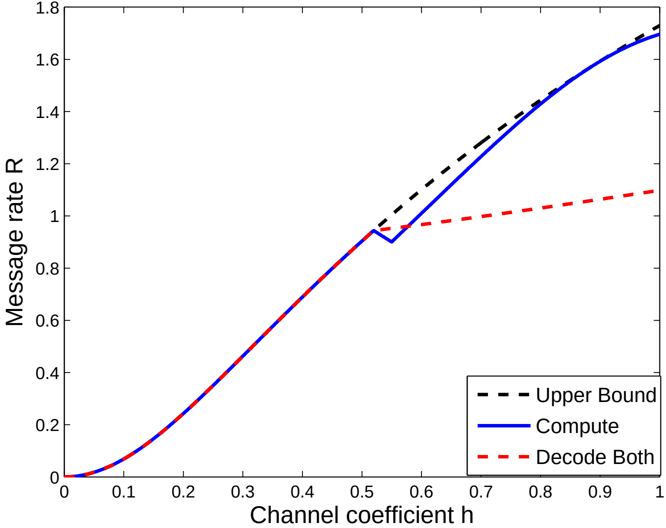
- Three transmitters that do not know the fading coefficients.
- Average rate plotted for i.i.d. Gaussian fading.



Computation over Fading Channels – No CSIT

- Receiver observes $\mathbf{y} = \mathbf{x}_1 + h\mathbf{x}_2 + \mathbf{z}$.
- Recovers $a\mathbf{w}_1 \oplus b\mathbf{w}_2$ for $a, b \neq 0$.

10dB



Computation over Fading Channels – No CSIT

- Receiver observes $\mathbf{y} = \mathbf{x}_1 + h\mathbf{x}_2 + \mathbf{z}$.
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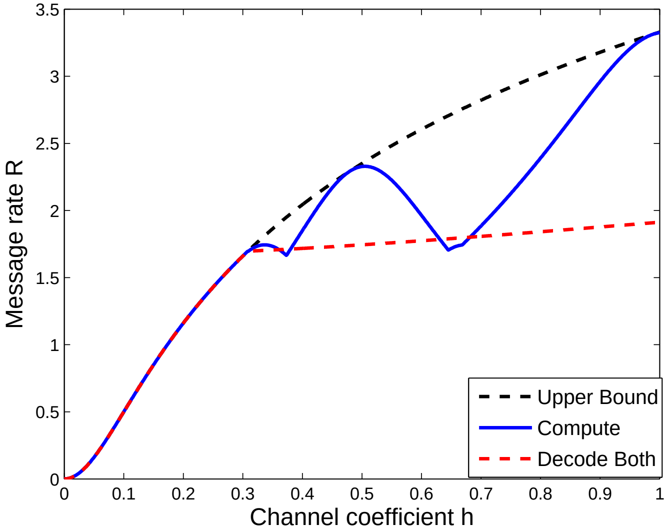
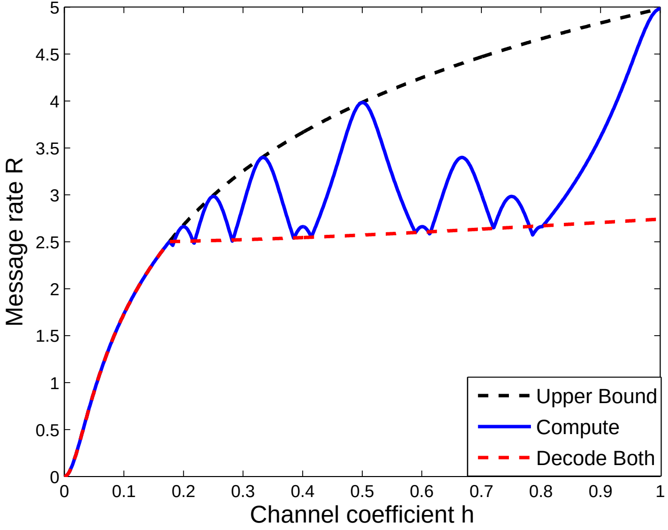
20dB



Computation over Fading Channels – No CSIT

- Receiver observes $\mathbf{y} = \mathbf{x}_1 + h\mathbf{x}_2 + \mathbf{z}$.
- Recovers $a\mathbf{w}_1 \oplus b\mathbf{w}_2$ for $a, b \neq 0$.

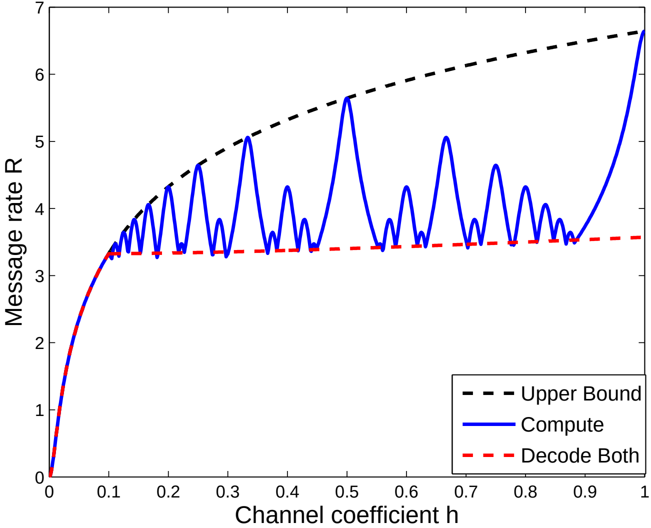
30dB



Computation over Fading Channels – No CSIT

- Receiver observes $\mathbf{y} = \mathbf{x}_1 + h\mathbf{x}_2 + \mathbf{z}$.
- Recovers $a\mathbf{w}_1 \oplus b\mathbf{w}_2$ for $a, b \neq 0$.

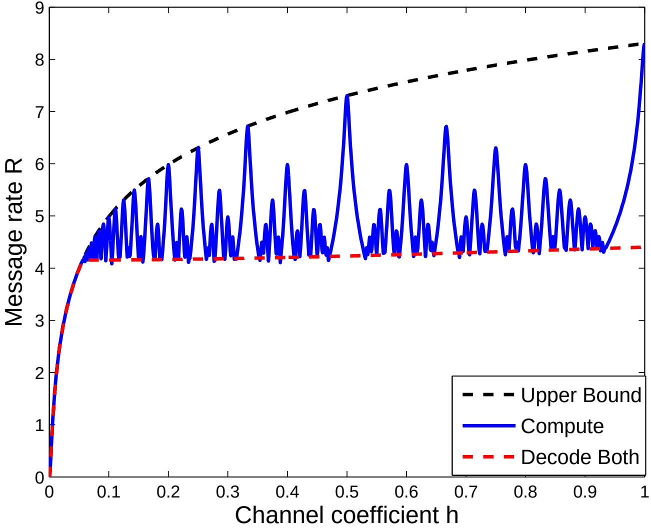
40dB



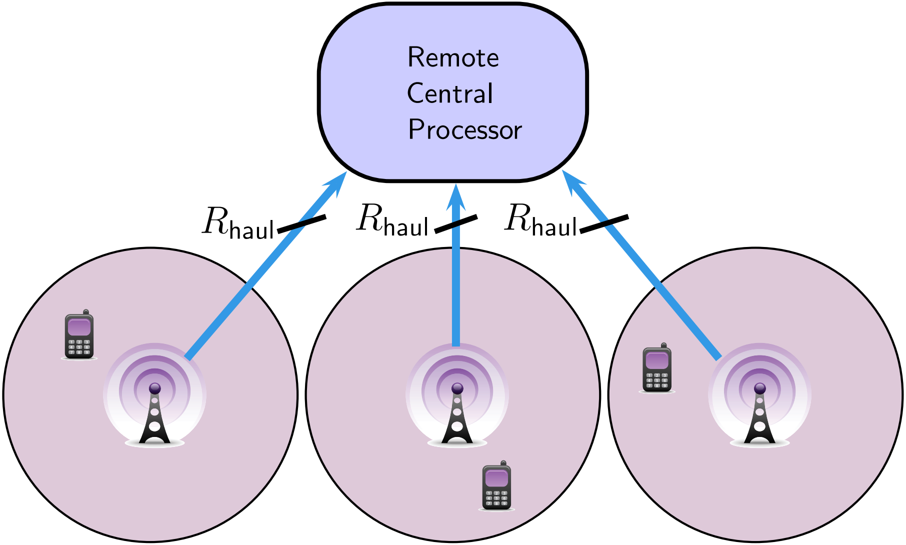
Computation over Fading Channels – No CSIT

- Receiver observes $\mathbf{y} = \mathbf{x}_1 + h\mathbf{x}_2 + \mathbf{z}$.
- Recovers $a\mathbf{w}_1 \oplus b\mathbf{w}_2$ for $a, b \neq 0$.

50dB



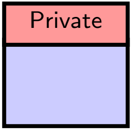
Rate-Constrained Cellular Backhaul



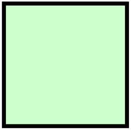
- Well-studied cellular model: **Wyner '94, Shamai-Wyner '97, Sanderovich et al. '09**

Structured Superposition

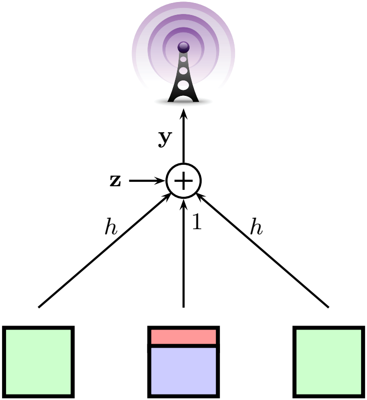
Structured Superposition



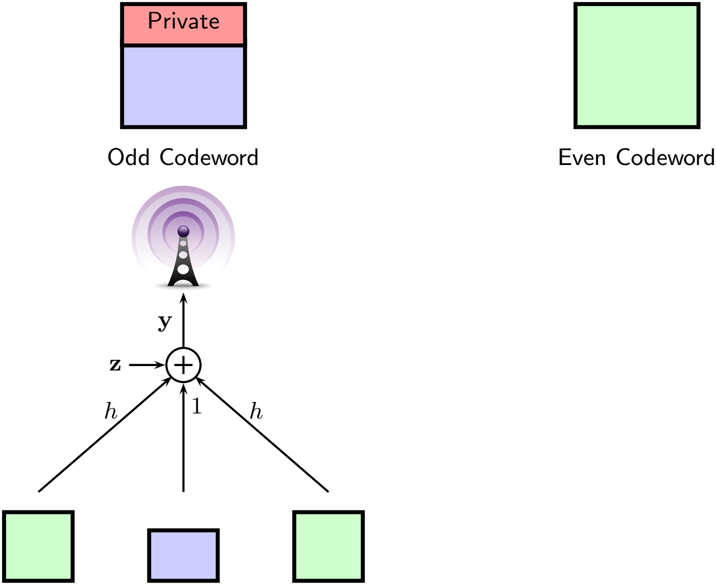
Odd Codeword



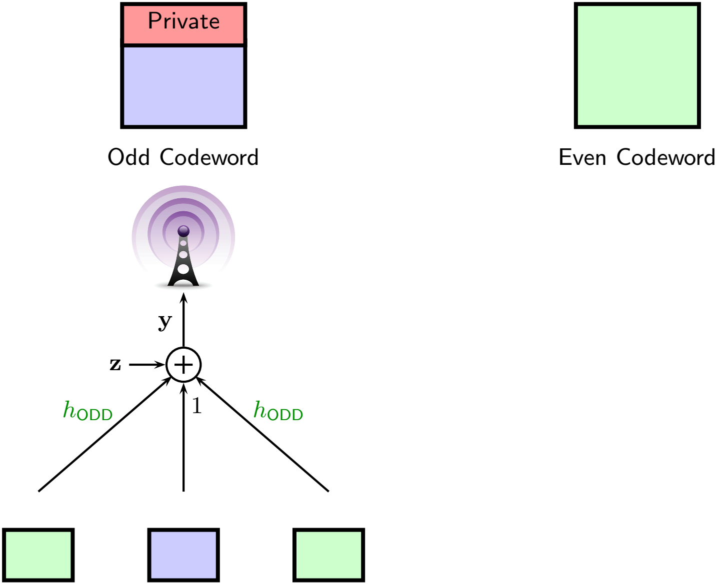
Even Codeword



Structured Superposition



Structured Superposition



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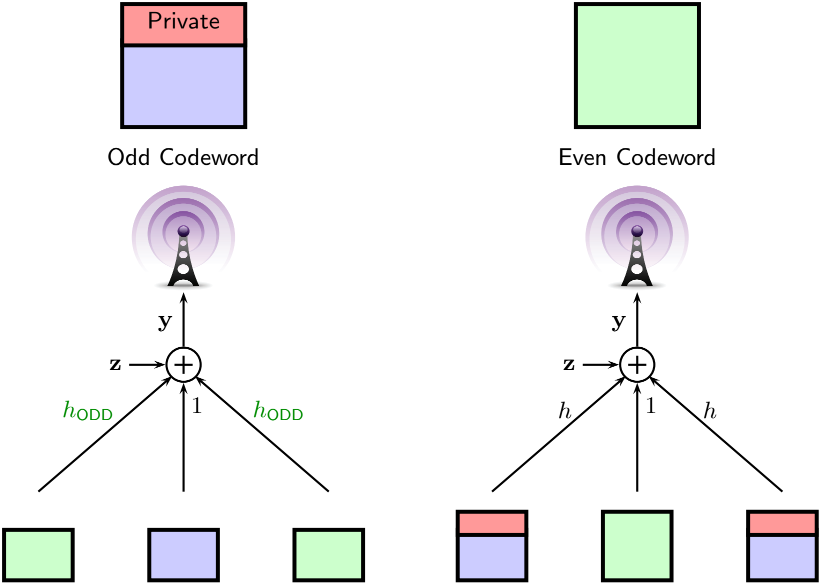

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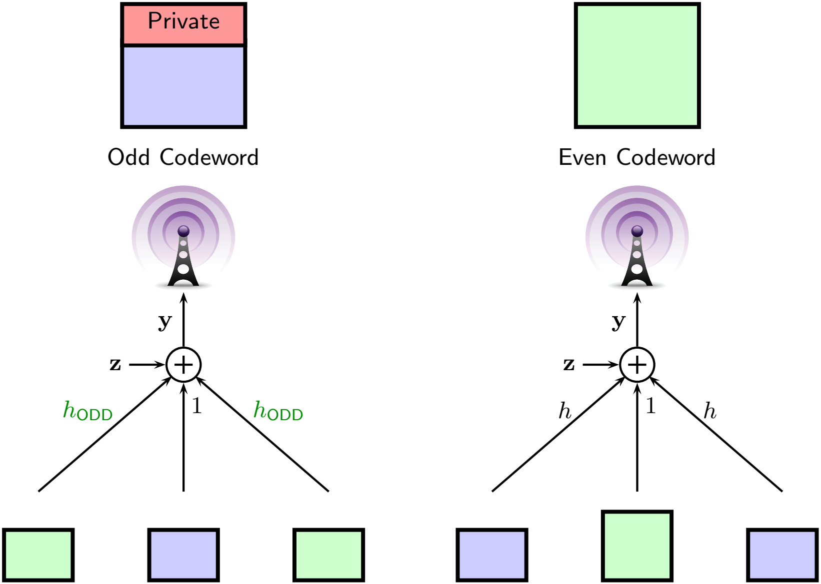
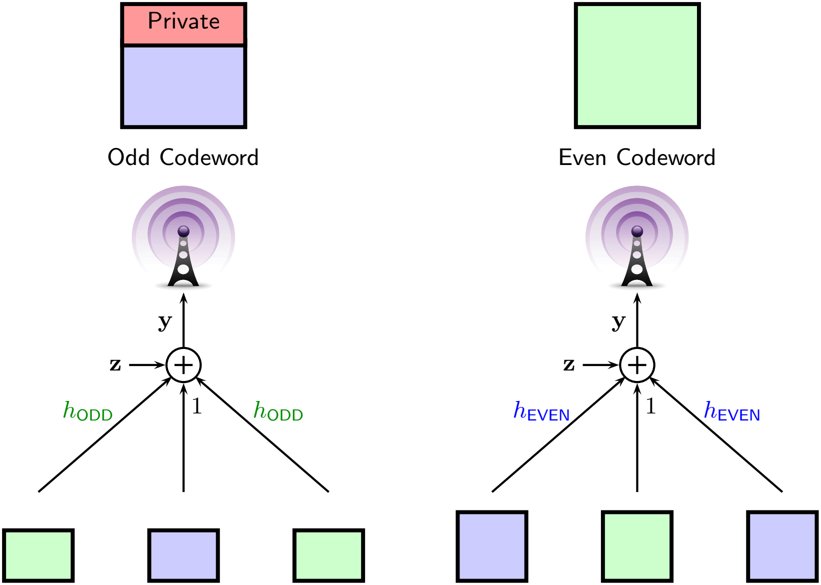


Structured Superposition



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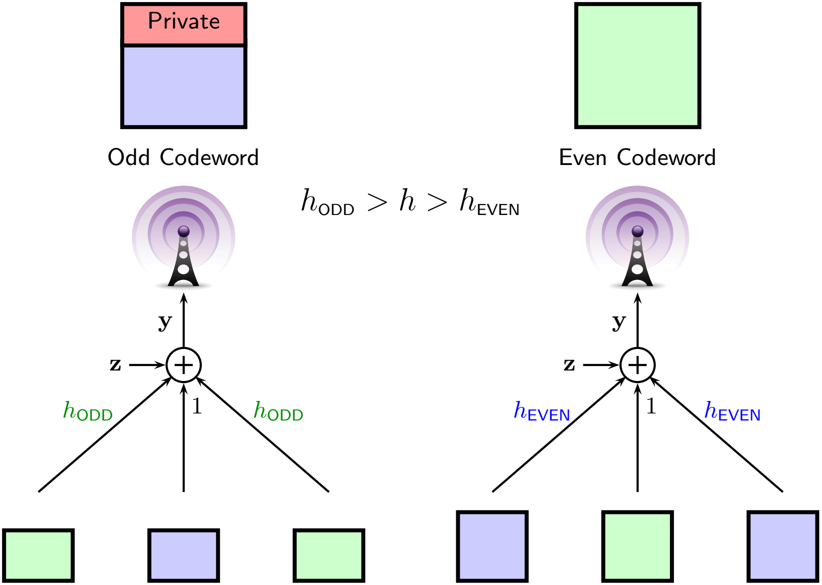

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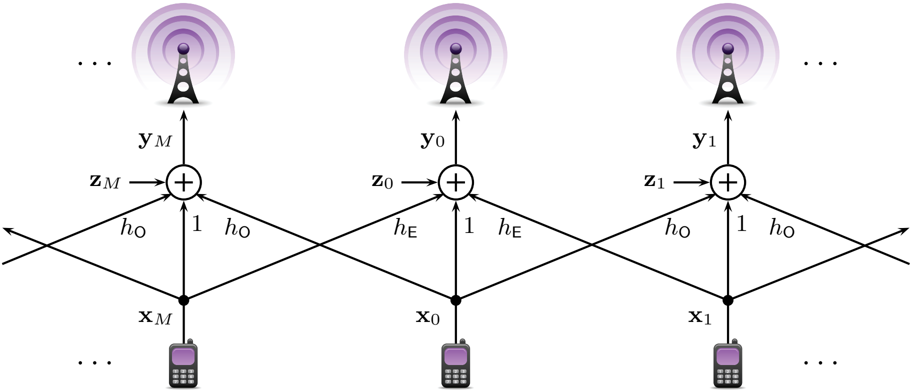
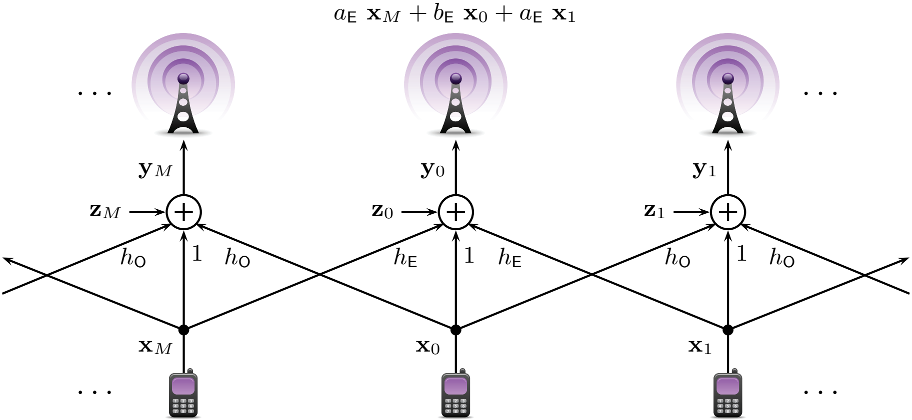


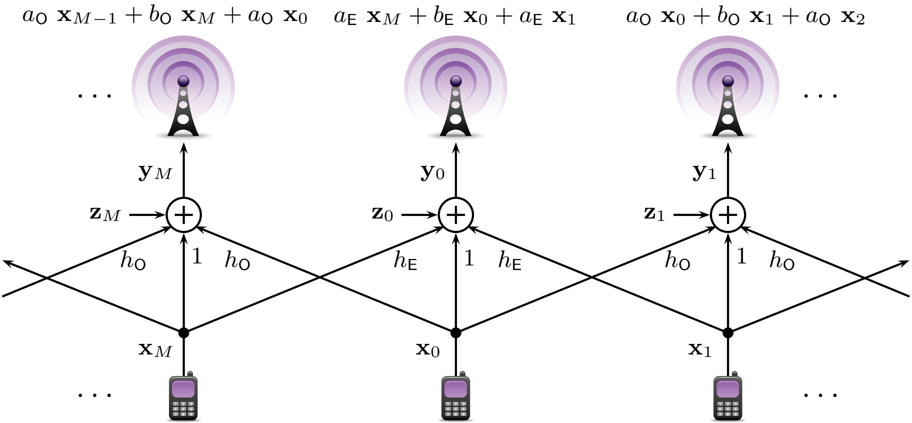
Nazer et al. '09: Each cell-site sees either h_E or h_O which is strictly better than h.

Structured Superposition



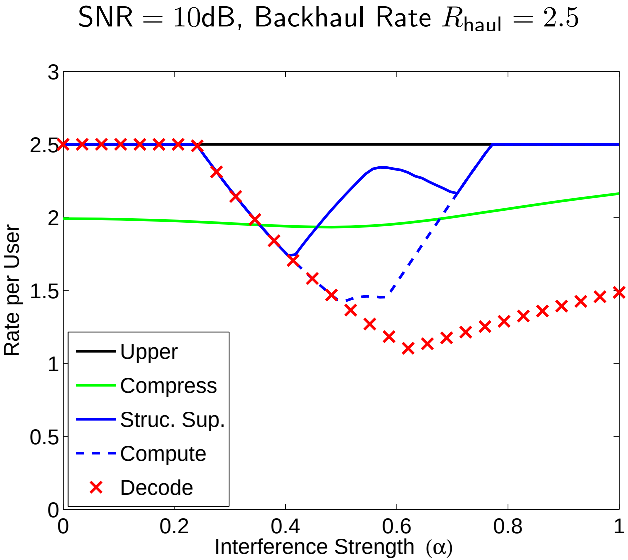
Nazer et al. '09: Each cell-site sees either h_{E} or h_{O} which is strictly better than h.

Structured Superposition



Nazer et al. '09: Each cell-site sees either h_{E} or h_{O} which is strictly better than h.

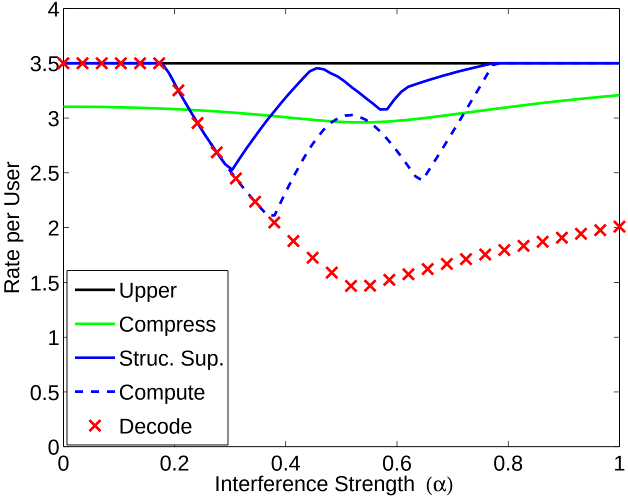
Structured Superposition: Performance



- Compress-and-forward rate taken from **Sanderovich et al. '09**
- Layering can reduce "non-integer loss."

Structured Superposition: Performance

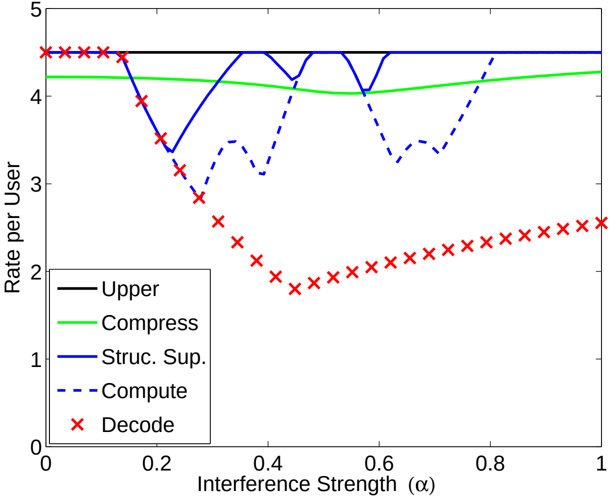
SNR $= 15$dB, Backhaul Rate $R_{\text{haul}} = 3.5$



- Compress-and-forward rate taken from **Sanderovich et al. '09**
- Layering can reduce "non-integer loss."

Structured Superposition: Performance

SNR $= 20$dB, Backhaul Rate $R_{\mathsf{haul}} = 4.5$



- Compress-and-forward rate taken from **Sanderovich et al. '09**
- Layering can reduce "non-integer loss."

- Choose equation coefficients to maximize rate:

$$R_{\mathsf{COMP}} = \max_{\mathbf{a} \in \mathbb{Z}^K} \max_{\alpha} \frac{1}{2} \log \left(\frac{P}{\alpha^2 N + P \|\alpha \mathbf{h} - \mathbf{a}\|^2} \right)$$

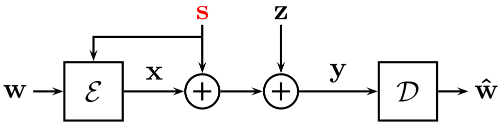
- Equivalently $\min_{\mathbf{a} \in \mathbb{Z}^K} \min_{\alpha} \alpha^2 N + P \|\alpha \mathbf{h} - \mathbf{a}\|^2$.

- Closely connected to Diophantine approximation, i.e. approximating irrationals with rationals.

- **Niesen-Whiting '11** shows that DoF $= \lim_{P \to \infty} \dfrac{R_{\mathsf{COMP}}}{\frac{1}{2}\log(1+P)} \leq 2$

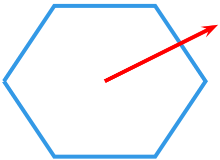
- Also shows that by combining compute-and-forward with interference alignment can get DoF to K.

$\mathbf{s}$ is interference known noncausally to the encoder.



Assume $\mathbf{s}$ i.i.d. Gaussian, very large variance P_S.

Erez-Shamai-Zamir '05:
Encoder subtracts $\alpha\mathbf{s}$, dithers, and takes $\mod \Lambda$.



$$\mathbf{x} = [\mathbf{t} - \alpha\mathbf{s} + \mathbf{d}] \mod \Lambda$$

Decoder scales by α, removes dither, takes $\mod \Lambda$, and recovers $\mathbf{t}$.
Interference is cancelled.

$$\begin{aligned}[\alpha\mathbf{y} - \mathbf{d}] \bmod \Lambda &= [\mathbf{x} + \alpha\mathbf{s} - \mathbf{d} + \mathbf{z} - (1-\alpha)\mathbf{x}] \bmod \Lambda \\ &= \Big[[\mathbf{t} - \alpha\mathbf{s} + \mathbf{d}] \bmod \Lambda + \alpha\mathbf{s} - \mathbf{d} + \mathbf{z} - (1-\alpha)\mathbf{x}\Big] \bmod \Lambda \\ &= \Big[\mathbf{t} + \mathbf{z} - (1-\alpha)\mathbf{x}\Big] \bmod \Lambda\end{aligned}$$

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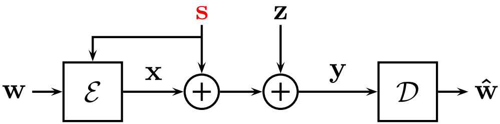

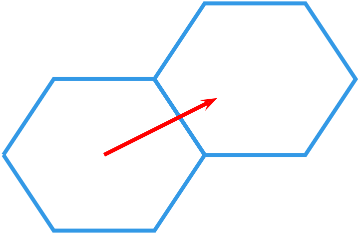

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\end{aligned}$$

Dirty Paper Coding

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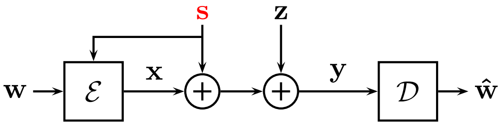

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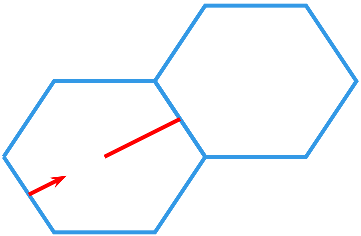

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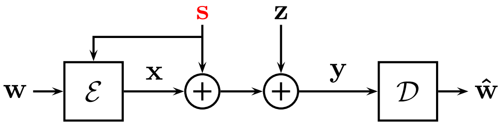

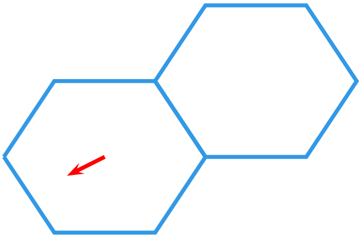

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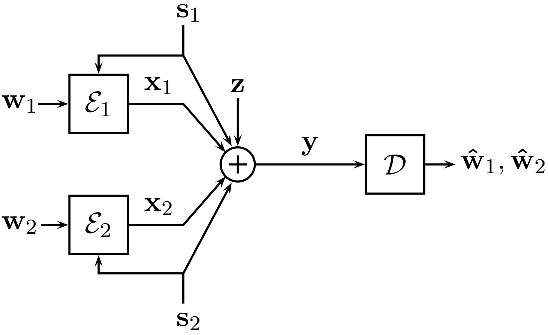


Philosof-Zamir-Erez-Khisti '11:

- Encoder 1 knows interference $\mathbf{s}_1$.
- Encoder 2 knows interference $\mathbf{s}_2$.
- Need to cancel out interference in a distributed fashion.
- Assume i.i.d. Gaussian interference with very large variance P_S. Random i.i.d. methods yield rate that goes to 0 as P_S goes to infinity.

Dirty Gaussian Multiple-Access Channel

Subtract (part of) the interference signals ahead of time:

$$\mathbf{x}_1 = [\mathbf{t}_1 - \alpha\mathbf{s}_1 + \mathbf{d}_1] \bmod \Lambda$$
$$\mathbf{x}_2 = [\mathbf{t}_2 - \alpha\mathbf{s}_2 + \mathbf{d}_2] \bmod \Lambda$$

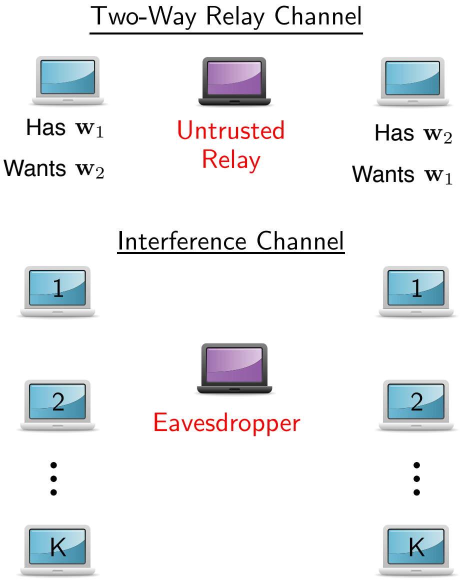
Decoder removes dithers:

$$\begin{aligned}
&[\alpha\mathbf{y} - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= [\alpha(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{z}) - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= [\mathbf{x}_1 + \mathbf{x}_2 + \alpha(\mathbf{s}_1 + \mathbf{s}_2) - (1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \alpha\mathbf{z}) - \mathbf{d}_1 - \mathbf{d}_2] \bmod \Lambda \\
&= \Big[\mathbf{t}_1 + \mathbf{t}_2 + (1-\alpha)(\mathbf{x}_1 + \mathbf{x}_2) + \alpha\mathbf{z}\Big] \bmod \Lambda
\end{aligned}$$

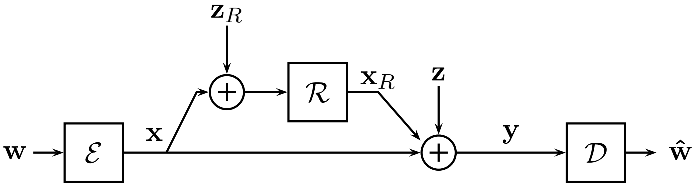
Select $\alpha = 2P/(2P+N)$ to obtain

$$R_1 + R_2 \leq \left[\frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{N}\right)\right]^+$$

- **He-Yener '09:** Lattice codes are useful for physical-layer secrecy.
- Random i.i.d. codes achieve 0 secure-degrees-of-freedom.
- Basic result: Random lattice codes achieve positive secure-degrees-of-freedom.



Relaying



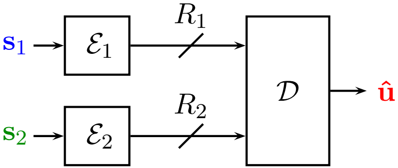
What can we prove with lattice codes for the AWGN relay channel?

- The full decode-and-forward rate can be achieved.
  See **Song-Devroye '10, Nockleby-Aazhang '11.**
- The full compress-and-forward rate can be achieved.
  See **Song-Devroye '11.**

- Correlated Gaussian sources.

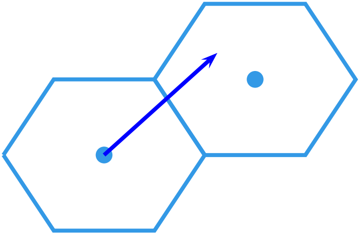
$$\begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

- Decoder wants the difference.
- Nested lattices are also good for Gaussian source coding.



$$\mathbf{u} = \mathbf{s}_1 - \mathbf{s}_2$$

$$D = \frac{1}{n}\mathbb{E}\|\hat{\mathbf{u}} - \mathbf{u}\|^2$$



Distributed Source Coding: "Gaussian Körner-Marton Problem"

- Correlated Gaussian sources.

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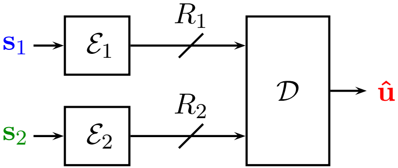
- Nested lattices are also good for Gaussian source coding.
- **Krithivasan-Pradhan '09:** with high probability, $\mathbf{s}_1$ and $\mathbf{s}_2$ will land near the same coarse lattice point.



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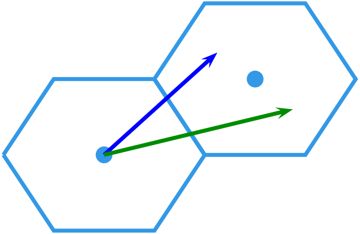

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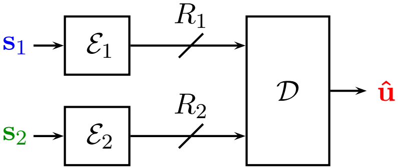

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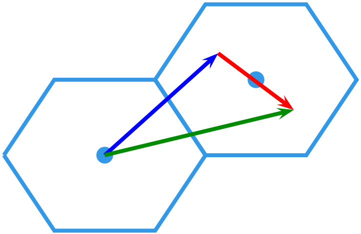

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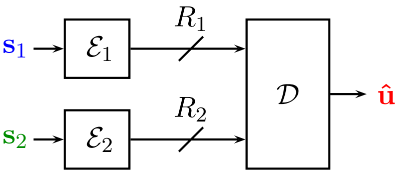
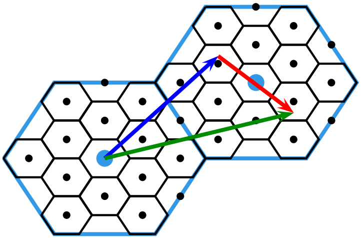


$$\mathbf{u} = \mathbf{s}_1 - \mathbf{s}_2$$

$$D = \frac{1}{n}\mathbb{E}\|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

- **Krithivasan-Pradhan '09:** with high probability, $\mathbf{s}_1$ and $\mathbf{s}_2$ will land near the same coarse lattice point.
- Only need to send:

$$\mathbf{t}_1 = \left[Q_{\Lambda_{\mathsf{FINE}}}(\mathbf{s}_1)\right] \bmod \Lambda$$
$$\mathbf{t}_2 = \left[Q_{\Lambda_{\mathsf{FINE}}}(\mathbf{s}_2)\right] \bmod \Lambda$$



Three-User Gaussian Distributed Source Coding

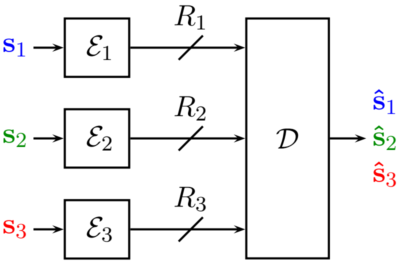
- Correlated Gaussian sources.

$$\begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

- Third source is the difference:

$$\mathbf{s}_3 = \mathbf{s}_1 - \mathbf{s}_3$$

- Structured codes make new rate points accessible in distributed Gaussian source coding.



$$D_1 = \frac{1}{n}\mathbb{E}\|\hat{\mathbf{s}}_1 - \mathbf{s}_1\|^2$$
$$D_2 = \frac{1}{n}\mathbb{E}\|\hat{\mathbf{s}}_2 - \mathbf{s}_2\|^2$$
$$D_3 = \frac{1}{n}\mathbb{E}\|\hat{\mathbf{s}}_3 - \mathbf{s}_3\|^2$$

- Example: Set $R_1 = 0$ and $R_2 = 0$.

- See **Tavildar-Wagner-Viswanath '10, Krithivasan-Pradhan '09, Maddah-Ali–Tse '10.**

Practical Implementations of Compute-and-Forward

- **Feng-Silva-Kschischang '10** develop practical nested lattice codes that work quite well for blocklengths as small as 100.
- **Hern and Narayanan '10** develop multi-level codes to use fields of size 2^k.
- **Ordentlich and Erez '10** propose mapping by set partitioning to go from binary codewords to higher order constellations.
- Further emerging work includes **Osmane and Belfiore '11**

Concluding Remarks

- Codes with algebraic structure lead to the highest known achievable rates for some communication scenarios of great interest.
- This applies to *source coding*, *channel coding*, and also *joint source-channel coding.*
- We have discussed a set of tools to apply and analyze *random linear* and *random lattice* codes to communication network scenarios.
- However, there is currently no general unified theory of how to generally use algebraic structure in the context of network information theory.

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