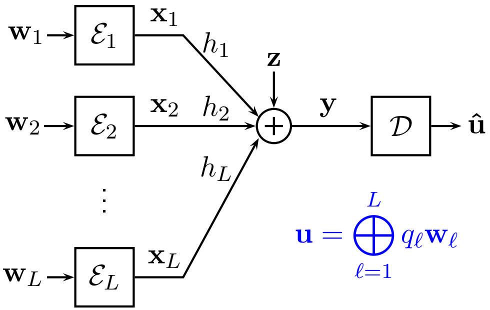
Expanding the Compute-and-Forward Framework

Bobak Nazer
Boston University

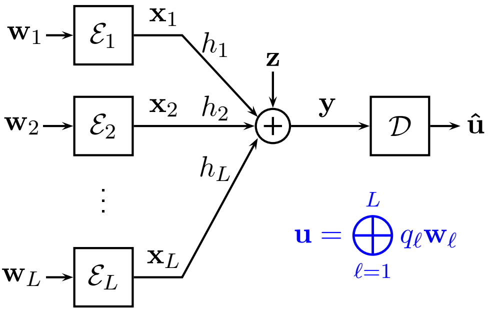
Joint work with Viveck Cadambe, Vasilis Ntranos, and Giuseppe Caire.

ITA Workshop
February 5, 2015

Compute-and-Forward: Single Receiver

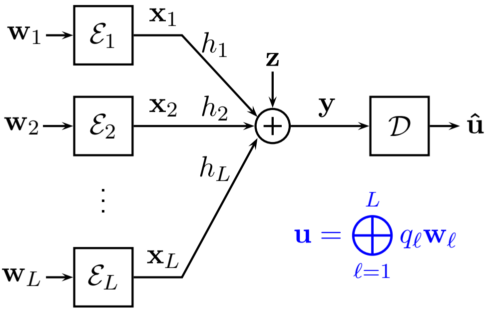


- Messages are finite field vectors, $\mathbf{w}_\ell \in \mathbb{Z}_p^k$, p prime.
- Real-valued inputs and outputs, $\mathbf{x}_\ell, \mathbf{y} \in \mathbb{R}^n$.
- Equal power constraint, $\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP$.
- Gaussian noise, $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.
- Equal rates: $R = \dfrac{k}{n} \log_2 p$



- Messages are finite field vectors, $\mathbf{w}_\ell \in \mathbb{Z}_p^k$, p prime.
- Real-valued inputs and outputs, $\mathbf{x}_\ell, \mathbf{y} \in \mathbb{R}^n$.
- Equal power constraint, $\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP$.
- Gaussian noise, $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.
- Equal rates: $R = \frac{k}{n} \log_2 p$

- Decoder wants a linear combination of the messages with vanishing probability of error $\lim_{n\to\infty} \mathbb{P}(\hat{\mathbf{u}} \neq \mathbf{u}) = 0$.



- Messages are finite field vectors, $\mathbf{w}_\ell \in \mathbb{Z}_p^k$, p prime.
- Real-valued inputs and outputs, $\mathbf{x}_\ell, \mathbf{y} \in \mathbb{R}^n$.
- Equal power constraint, $\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP$.
- Gaussian noise, $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.
- Equal rates: $R = \dfrac{k}{n} \log_2 p$

- Decoder wants a linear combination of the messages with vanishing probability of error $\lim_{n\to\infty} \mathbb{P}(\hat{\mathbf{u}} \neq \mathbf{u}) = 0$.
- What rates are achievable as a function of h_ℓ and q_ℓ?

- Want to characterize achievable rates as a function of $\mathbf{h} = [h_1 \ \cdots \ h_L]^\mathsf{T}$ and $\mathbf{q} = [q_1 \ \cdots \ q_L]^\mathsf{T}$.

Computation Rate

- Want to characterize achievable rates as a function of $\mathbf{h} = [h_1 \ \cdots \ h_L]^\mathsf{T}$ and $\mathbf{q} = [q_1 \ \cdots \ q_L]^\mathsf{T}$.
- Easier to think about integer rather than finite field coefficients.

Computation Rate

- Want to characterize achievable rates as a function of $\mathbf{h} = [h_1 \ \cdots \ h_L]^\mathsf{T}$ and $\mathbf{q} = [q_1 \ \cdots \ q_L]^\mathsf{T}$.
- Easier to think about integer rather than finite field coefficients.
- The linear combination with integer coefficient vector $\mathbf{a} = [a_1 \ a_2 \ \cdots \ a_L]^\mathsf{T} \in \mathbb{Z}^L$ corresponds to

$$\mathbf{u} = \bigoplus_{\ell=1}^{L} q_\ell \mathbf{w}_\ell \qquad \text{where } q_\ell = [a_\ell] \bmod p \ .$$

Computation Rate

- Want to characterize achievable rates as a function of $\mathbf{h} = [h_1 \ \cdots \ h_L]^\mathsf{T}$ and $\mathbf{q} = [q_1 \ \cdots \ q_L]^\mathsf{T}$.
- Easier to think about integer rather than finite field coefficients.
- The linear combination with integer coefficient vector $\mathbf{a} = [a_1 \ a_2 \ \cdots \ a_L]^\mathsf{T} \in \mathbb{Z}^L$ corresponds to

$$\mathbf{u} = \bigoplus_{\ell=1}^{L} q_\ell \mathbf{w}_\ell \qquad \text{where } q_\ell = [a_\ell] \bmod p \ .$$

- **Key Definition:** The computation rate region described by $R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a})$ is *achievable* if, for any $\epsilon > 0$ and n, p large enough, the receiver can decode any linear combination with integer coefficient vector $\mathbf{a} \in \mathbb{Z}^L$ with probability of error at most ϵ so long as the message rate R satisfies

$$R < R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a})$$

Compute-and-Forward: Effective Noise

$$
\begin{aligned}
\mathbf{y} &= \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z} \\
&= \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \sum_{\ell=1}^{L} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}
\end{aligned}
$$

Desired Codebook:

- Closed under integer linear combinations $\Longrightarrow$ lattice codebook.

Compute-and-Forward: Effective Noise

$$\mathbf{y} = \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z}$$

$$= \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{L} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}}_{\text{Effective Noise}}$$

Desired Codebook:

- Closed under integer linear combinations $\Longrightarrow$ lattice codebook.
- Independent effective noise $\Longrightarrow$ dithering.

Compute-and-Forward: Effective Noise

$$\mathbf{y} = \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z}$$

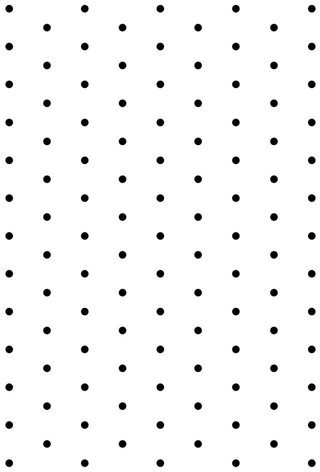
$$= \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{L} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}}_{\text{Effective Noise}} \xrightarrow{\text{Decode}} \bigoplus_{\ell=1}^{L} q_\ell \mathbf{w}_\ell$$

Desired Codebook:

- Closed under integer linear combinations $\implies$ lattice codebook.
- Independent effective noise $\implies$ dithering.
- Isomorphic to $\mathbb{Z}_p^k$ $\implies$ nested lattice codebook.

Nested Lattices

- A lattice is a discrete subgroup of $\mathbb{R}^n$.

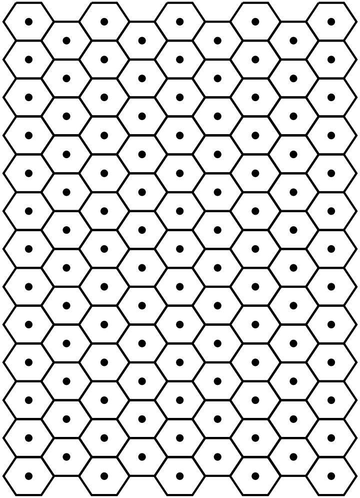


Nested Lattices

- A lattice is a discrete subgroup of $\mathbb{R}^n$.

- Nearest neighbor quantizer:

$$Q_\Lambda(\mathbf{x}) = \arg\min_{\boldsymbol{\lambda}\in\Lambda} \|\mathbf{x} - \boldsymbol{\lambda}\|_2$$



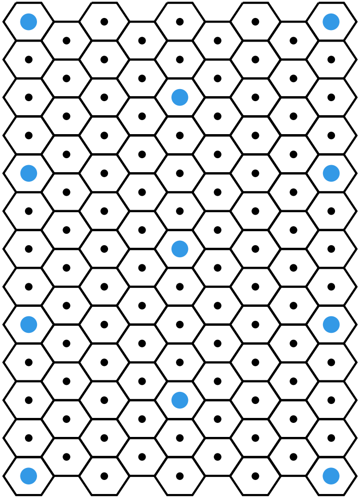
Nested Lattices

- A lattice is a discrete subgroup of $\mathbb{R}^n$.

- Nearest neighbor quantizer:

$$Q_\Lambda(\mathbf{x}) = \arg\min_{\boldsymbol{\lambda}\in\Lambda} \|\mathbf{x} - \boldsymbol{\lambda}\|_2$$

- Two lattices Λ_C and Λ_F are nested if $\Lambda \subset \Lambda_\mathsf{F}$



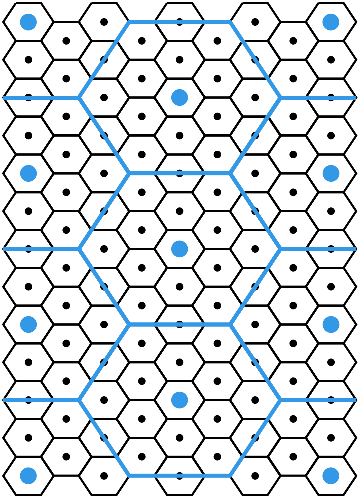
Nested Lattices

- A lattice is a discrete subgroup of $\mathbb{R}^n$.

- Nearest neighbor quantizer:

$$Q_\Lambda(\mathbf{x}) = \arg\min_{\boldsymbol{\lambda}\in\Lambda} \|\mathbf{x} - \boldsymbol{\lambda}\|_2$$

- Two lattices Λ_C and Λ_F are nested if $\Lambda \subset \Lambda_\mathsf{F}$



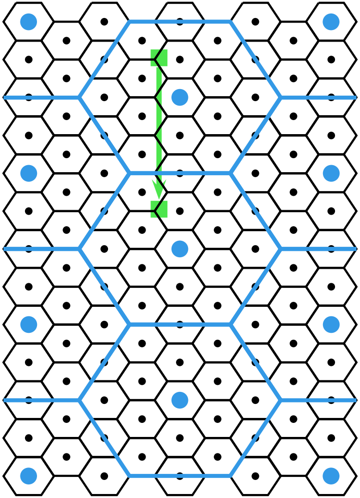
Nested Lattices

- A lattice is a discrete subgroup of $\mathbb{R}^n$.
- Nearest neighbor quantizer:

$$Q_\Lambda(\mathbf{x}) = \arg\min_{\boldsymbol{\lambda}\in\Lambda} \|\mathbf{x} - \boldsymbol{\lambda}\|_2$$

- Two lattices Λ_C and Λ_F are nested if $\Lambda \subset \Lambda_\mathsf{F}$
- Quantization error serves as modulo operation:

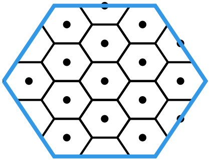
$$[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}) \ .$$



Distributive Law:

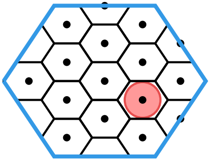
$$\big[\mathbf{x}_1 + a[\mathbf{x}_2] \bmod \Lambda\big] \bmod \Lambda = [\mathbf{x}_1 + a\mathbf{x}_2] \bmod \Lambda \quad \text{for all } a \in \mathbb{Z}.$$

- Nested Lattice Code: Formed by taking all elements of Λ_{F} that lie in the fundamental Voronoi region of Λ_{C}.



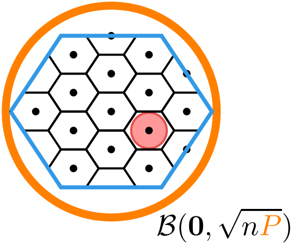
Nested Lattice Codes

- Nested Lattice Code: Formed by taking all elements of Λ_F that lie in the fundamental Voronoi region of Λ_C.
- Fine lattice Λ_F protects against noise.



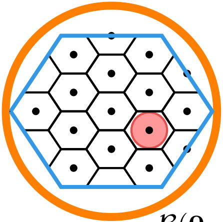
Nested Lattice Codes

- Nested Lattice Code: Formed by taking all elements of Λ_{F} that lie in the fundamental Voronoi region of Λ_{C}.
- Fine lattice Λ_{F} protects against noise.
- Coarse lattice Λ_{C} enforces the power constraint.



Nested Lattice Codes

- Nested Lattice Code: Formed by taking all elements of Λ_{F} that lie in the fundamental Voronoi region of Λ_{C}.
- Fine lattice Λ_{F} protects against noise.
- Coarse lattice Λ_{C} enforces the power constraint.
- Existence of good nested lattice codes: **Loeliger '97, Forney-Trott-Chung '00, Erez-Litsyn-Zamir '05, Ordentlich-Erez '12.**
- **Erez-Zamir '04:** Nested lattice codes can achieve the Gaussian capacity.
- **Zamir-Shamai-Erez '02:** Excellent framework for multi-terminal binning.



$\mathcal{B}(\mathbf{0}, \sqrt{nP})$

Achievability

- Construction A induces an isomorphism between the finite field $\mathbb{Z}_p^k$ and the nested lattice codebook $[\Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}]$.

Achievability

- Construction A induces an isomorphism between the finite field $\mathbb{Z}_p^k$ and the nested lattice codebook $[\Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}]$.
- Each encoder maps its message $\mathbf{w}_\ell$ to a lattice codeword $\boldsymbol{\lambda}_\ell$.

Achievability

- Construction A induces an isomorphism between the finite field $\mathbb{Z}_p^k$ and the nested lattice codebook $[\Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}]$.
- Each encoder maps its message $\mathbf{w}_\ell$ to a lattice codeword $\boldsymbol{\lambda}_\ell$.
- Decode integer-linear combination $\left[\sum_\ell a_\ell \boldsymbol{\lambda}_\ell\right] \bmod \Lambda_{\mathsf{c}}$ from

$$\beta \mathbf{y} = \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{L} (\beta h_\ell - a_\ell)\mathbf{x}_\ell + \beta \mathbf{z}}_{\text{effective noise}} \ .$$

Achievability

- Construction A induces an isomorphism between the finite field $\mathbb{Z}_p^k$ and the nested lattice codebook $[\Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}]$.
- Each encoder maps its message $\mathbf{w}_\ell$ to a lattice codeword $\boldsymbol{\lambda}_\ell$.
- Decode integer-linear combination $\left[\sum_\ell a_\ell \boldsymbol{\lambda}_\ell\right] \bmod \Lambda_{\mathsf{c}}$ from

$$\beta \mathbf{y} = \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{L} (\beta h_\ell - a_\ell)\mathbf{x}_\ell + \beta \mathbf{z}}_{\text{effective noise}} \ .$$

- Succeeds w.h.p. if noise tolerance σ^2_{eff} of the fine lattice satisfies

$$\sigma^2_{\mathsf{eff}} > \min_{\beta \in \mathbb{R}} \left(\beta^2 + P\|\beta \mathbf{h} - \mathbf{a}\|^2\right) = \left\|\left(P^{-1}\mathbf{I} + \mathbf{h}\mathbf{h}^{\mathsf{T}}\right)^{-1/2}\mathbf{a}\right\|^2 \ .$$

Achievability

- Construction A induces an isomorphism between the finite field $\mathbb{Z}_p^k$ and the nested lattice codebook $[\Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}]$.
- Each encoder maps its message $\mathbf{w}_\ell$ to a lattice codeword $\boldsymbol{\lambda}_\ell$.
- Decode integer-linear combination $\left[\sum_\ell a_\ell \boldsymbol{\lambda}_\ell\right] \bmod \Lambda_{\mathsf{c}}$ from

$$\beta \mathbf{y} = \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{L} (\beta h_\ell - a_\ell)\mathbf{x}_\ell + \beta \mathbf{z}}_{\text{effective noise}} \ .$$

- Succeeds w.h.p. if noise tolerance σ^2_{eff} of the fine lattice satisfies

$$\sigma^2_{\mathsf{eff}} > \min_{\beta \in \mathbb{R}} \left(\beta^2 + P\|\beta \mathbf{h} - \mathbf{a}\|^2\right) = \left\|\left(P^{-1}\mathbf{I} + \mathbf{h}\mathbf{h}^{\mathsf{T}}\right)^{-1/2}\mathbf{a}\right\|^2 \ .$$

- Map $\left[\sum_\ell a_\ell \boldsymbol{\lambda}_\ell\right] \bmod \Lambda_{\mathsf{c}}$ back to $\mathbb{Z}_p$ to get $\mathbf{u} = \bigoplus_\ell q_\ell \mathbf{w}_\ell$.

Achievability

- Construction A induces an isomorphism between the finite field $\mathbb{Z}_p^k$ and the nested lattice codebook $[\Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}]$.
- Each encoder maps its message $\mathbf{w}_\ell$ to a lattice codeword $\boldsymbol{\lambda}_\ell$.
- Decode integer-linear combination $\left[\sum_\ell a_\ell \boldsymbol{\lambda}_\ell\right] \bmod \Lambda_{\mathsf{c}}$ from

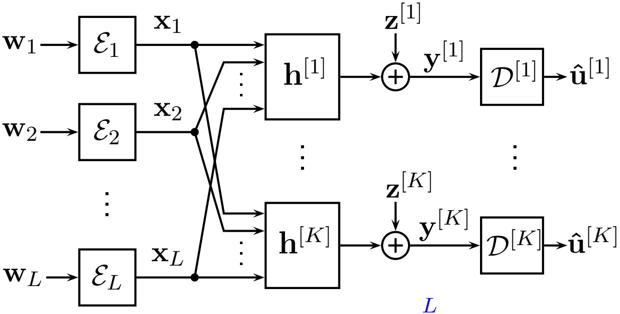
$$\beta \mathbf{y} = \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{L} (\beta h_\ell - a_\ell)\mathbf{x}_\ell + \beta \mathbf{z}}_{\text{effective noise}} \ .$$

- Succeeds w.h.p. if noise tolerance σ^2_{eff} of the fine lattice satisfies

$$\sigma^2_{\mathsf{eff}} > \min_{\beta \in \mathbb{R}} \left(\beta^2 + P\|\beta \mathbf{h} - \mathbf{a}\|^2\right) = \left\| \left(P^{-1}\mathbf{I} + \mathbf{h}\mathbf{h}^{\mathsf{T}}\right)^{-1/2} \mathbf{a} \right\|^2 \ .$$

- Map $\left[\sum_\ell a_\ell \boldsymbol{\lambda}_\ell\right] \bmod \Lambda_{\mathsf{c}}$ back to $\mathbb{Z}_p$ to get $\mathbf{u} = \bigoplus_\ell q_\ell \mathbf{w}_\ell$.
- **Nazer-Gastpar IT '11:** The achievable computation rate is

$$R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a}) = \frac{1}{2} \log^+ \left(\frac{P}{\left\| \left(P^{-1}\mathbf{I} + \mathbf{h}\mathbf{h}^{\mathsf{T}}\right)^{-1/2} \mathbf{a} \right\|^2} \right) \ .$$



$$\mathbf{u}^{[i]} = \bigoplus_{\ell=1}^{L} q_\ell^{[i]} \mathbf{w}_\ell \ ,$$

- Equal power constraints and Gaussian noise as before.
- Since some receivers will see better channels than others, it will be useful to allow for different rates $R_1, \ldots, R_L$. How can we retain the connection to $\mathbb{Z}_p$?

Different Message Rates

- Basic Idea: Keep linear combinations over $\mathbb{Z}_p^k$ and allow encoders to zeropad in order to reduce their rates.

Different Message Rates

- Basic Idea: Keep linear combinations over $\mathbb{Z}_p^k$ and allow encoders to zeropad in order to reduce their rates.
- Specifically, the message $\mathbf{w}_\ell$ of the ℓ^{th} encoder consists of k_ℓ information symbols over $\mathbb{Z}_p$ followed by $k - k_\ell$ zeros where $k \triangleq \max_\ell k_\ell$. The message rate is $R_\ell = \frac{k_\ell}{n} \log p$.

Different Message Rates

- Basic Idea: Keep linear combinations over $\mathbb{Z}_p^k$ and allow encoders to zeropad in order to reduce their rates.
- Specifically, the message $\mathbf{w}_\ell$ of the ℓ^{th} encoder consists of k_ℓ information symbols over $\mathbb{Z}_p$ followed by $k - k_\ell$ zeros where $k \triangleq \max_\ell k_\ell$. The message rate is $R_\ell = \frac{k_\ell}{n} \log p$.
- The computation rate region described by $R_{\text{comp}}(\mathbf{h}, \mathbf{a})$ is *achievable* if, for any $\epsilon > 0$ and n, p large enough, each receiver can decode any linear combination with integer coefficient vector $\mathbf{a}^{[1]}, \ldots, \mathbf{a}^{[K]} \in \mathbb{Z}^L$ with probability of error at most ϵ so long as

$$R_\ell < \min_{i : a_\ell^{[i]} \neq 0} R_{\text{comp}}(\mathbf{h}^{[i]}, \mathbf{a}^{[i]})$$

Achievability

- Using Construction A, generate L fine lattices $\Lambda_{\mathsf{F},1}, \ldots, \Lambda_{\mathsf{F},L}$ and a coarse lattice Λ_{C}, all nested in some order.

Achievability

- Using Construction A, generate L fine lattices $\Lambda_{\mathsf{F},1}, \ldots, \Lambda_{\mathsf{F},L}$ and a coarse lattice Λ_{C}, all nested in some order.
- The ℓ^{th} encoder employs the nested lattice codebook $\left[\Lambda_{\mathsf{F},\ell}/\Lambda_{\mathsf{C}}\right]$.

Achievability

- Using Construction A, generate L fine lattices $\Lambda_{\mathsf{F},1}, \ldots, \Lambda_{\mathsf{F},L}$ and a coarse lattice Λ_{C}, all nested in some order.
- The ℓ^{th} encoder employs the nested lattice codebook $\left[\Lambda_{\mathsf{F},\ell} / \Lambda_{\mathsf{C}}\right]$.
- Let Λ_{F} denote the densest fine lattice in the ensemble. Can show there is an isomorphism between $\mathbb{Z}_p^k$ and $[\Lambda_{\mathsf{F}} / \Lambda_{\mathsf{C}}]$.

Achievability

- Using Construction A, generate L fine lattices $\Lambda_{\mathsf{F},1}, \ldots, \Lambda_{\mathsf{F},L}$ and a coarse lattice Λ_{C}, all nested in some order.
- The ℓ^{th} encoder employs the nested lattice codebook $\left[\Lambda_{\mathsf{F},\ell}/\Lambda_{\mathsf{C}}\right]$.
- Let Λ_{F} denote the densest fine lattice in the ensemble. Can show there is an isomorphism between $\mathbb{Z}_p^k$ and $[\Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}]$.
- Furthermore, can show that if the last $k - k_\ell$ elements of a vector in $\mathbb{Z}_p^k$ are zero, then it will be mapped to $[\Lambda_{\mathsf{F},\ell}/\Lambda_{\mathsf{C}}]$.

Achievability

- Using Construction A, generate L fine lattices $\Lambda_{\mathsf{F},1},\ldots,\Lambda_{\mathsf{F},L}$ and a coarse lattice Λ_{C}, all nested in some order.
- The ℓ^{th} encoder employs the nested lattice codebook $\left[\Lambda_{\mathsf{F},\ell}/\Lambda_{\mathsf{C}}\right]$.
- Let Λ_{F} denote the densest fine lattice in the ensemble. Can show there is an isomorphism between $\mathbb{Z}_p^k$ and $[\Lambda_{\mathsf{F}}/\Lambda_{\mathsf{C}}]$.
- Furthermore, can show that if the last $k-k_\ell$ elements of a vector in $\mathbb{Z}_p^k$ are zero, then it will be mapped to $[\Lambda_{\mathsf{F},\ell}/\Lambda_{\mathsf{C}}]$.
- **Nazer-Gastpar IT '11:** Overall, we can combine these codebooks with the techniques for the single receiver case to get that the following computation rate region is achievable:

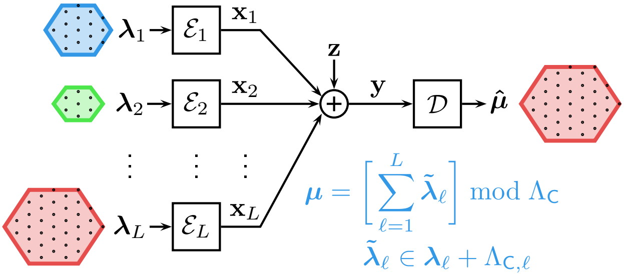
$$R_{\mathsf{comp}}(\mathbf{h},\mathbf{a}) = \frac{1}{2}\log^{+}\left(\frac{P}{\left\|\left(P^{-1}\mathbf{I}+\mathbf{h}\mathbf{h}^{\mathsf{T}}\right)^{-1/2}\mathbf{a}\right\|^{2}}\right) .$$

What's missing?

- Unequal power constraints:
$$\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP_\ell.$$

What's missing?

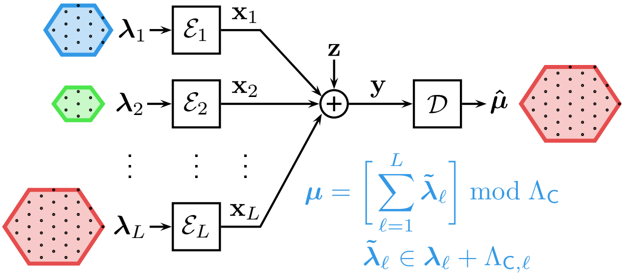
- Unequal power constraints:
  $$\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP_\ell.$$



What's missing?

- Unequal power constraints:

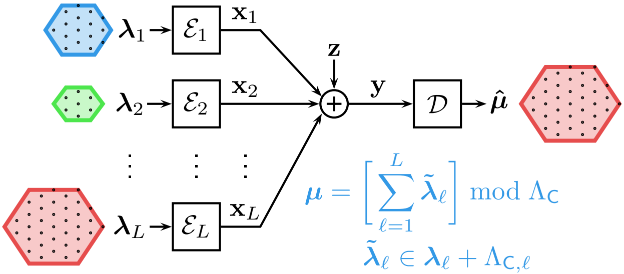
$$\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP_\ell.$$



- **Nam-Chung-Lee IT '11:** Proposed a natural way to incorporate unequal powers for the special case of equal channel gains.

What's missing?

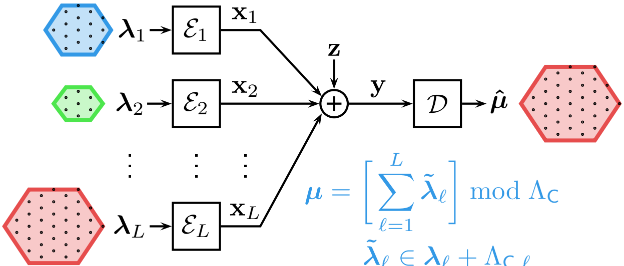
- Unequal power constraints:
  $\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP_\ell$.



- **Nam-Chung-Lee IT '11:** Proposed a natural way to incorporate unequal powers for the special case of equal channel gains.
- At each transmitter, use the same fine lattice and a different coarse lattice, chosen to meet the power constraint.

What's missing?

- Unequal power constraints:
  $\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP_\ell.$



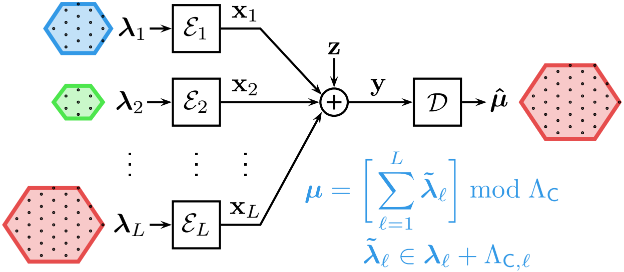
$$\boldsymbol{\mu} = \left[\sum_{\ell=1}^{L} \tilde{\boldsymbol{\lambda}}_\ell\right] \bmod \Lambda_{\mathsf{C}}$$

$$\tilde{\boldsymbol{\lambda}}_\ell \in \boldsymbol{\lambda}_\ell + \Lambda_{\mathsf{C},\ell}$$

- **Nam-Chung-Lee IT '11:** Proposed a natural way to incorporate unequal powers for the special case of equal channel gains.
- At each transmitter, use the same fine lattice and a different coarse lattice, chosen to meet the power constraint.
- Can we use this for the general compute-and-forward problem? While retaining the connection to the finite field?

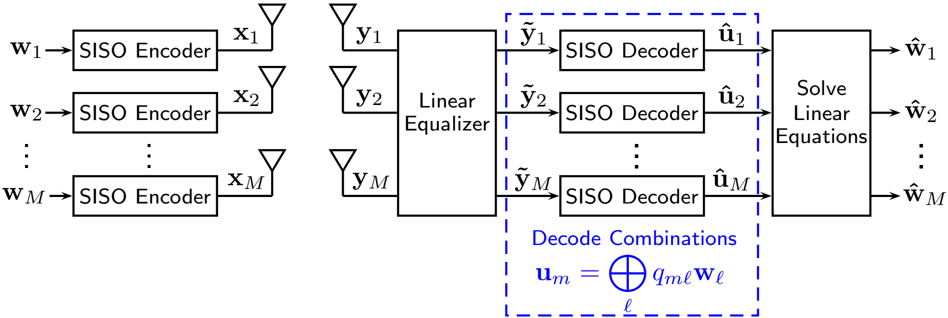
What's missing?

- Unequal power constraints:
  $\mathbb{E}\|\mathbf{x}_\ell\|^2 \le nP_\ell.$



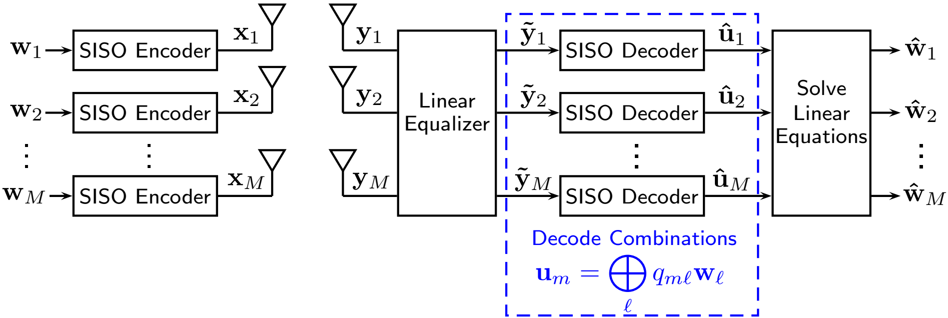
- **Nam-Chung-Lee IT '11:** Proposed a natural way to incorporate unequal powers for the special case of equal channel gains.
- At each transmitter, use the same fine lattice and a different coarse lattice, chosen to meet the power constraint.
- Can we use this for the general compute-and-forward problem? While retaining the connection to the finite field?
- **Zhu-Gastpar IZS '14:** Proposed a way to use this technique for compute-and-forward without a connection to the finite field.

What's missing?



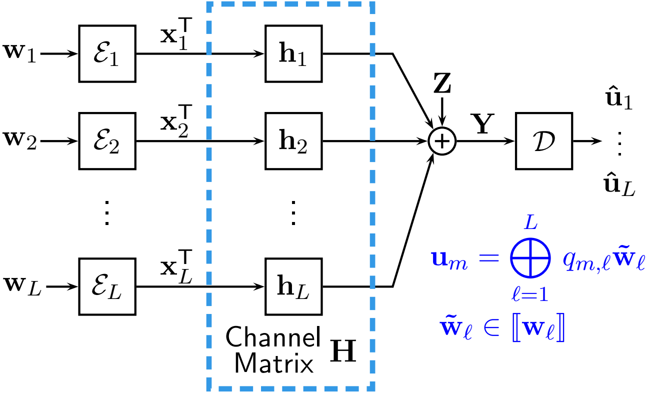
- Decoding multiple linear combinations at a single receiver is a useful technique for MIMO decoding (**Zhan-Nazer-Erez-Gastpar IT '14**) and interference alignment (**Ordentlich-Erez-Nazer IT '14, Ntranos-Cadambe-Nazer-Caire ISIT '13**).

What's missing?



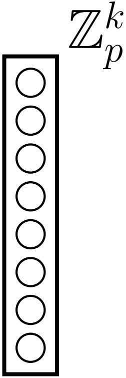
- Decoding multiple linear combinations at a single receiver is a useful technique for MIMO decoding (**Zhan-Nazer-Erez-Gastpar IT '14**) and interference alignment (**Ordentlich-Erez-Nazer IT '14, Ntranos-Cadambe-Nazer-Caire ISIT '13**).
- After a receiver has decoded one or more linear combinations, it can use these as side information to help decode the rest (**Nazer IZS '12, Ordentlich-Erez-Nazer IT '14, Ordentlich-Erez-Nazer Allerton '13**).

Expanding Compute-and-Forward: Single Receiver



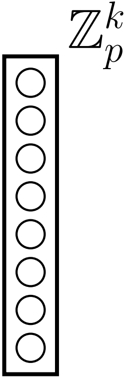
- Include unequal power constraints $\mathbb{E}\|\mathbf{x}_\ell\|^2 \leq nP_\ell$ and multiple antennas at the receiver.
- Relax to linear combinations of cosets.
- WLOG receiver wants L linear combinations (since we can set coefficients to 0).

- Each user is assigned an effective noise tolerance $\sigma^2_{\mathsf{eff},\ell}$ and power level P_ℓ.



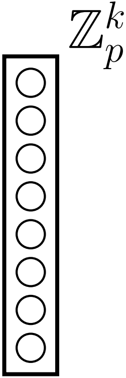
From Unequal Powers to Signal Levels

- Each user is assigned an effective noise tolerance $\sigma^2_{\text{eff},\ell}$ and power level P_ℓ.
- These can be handled by introducing two parameters $k_{\mathsf{C},\ell} \leq k_{\mathsf{F},\ell}$.



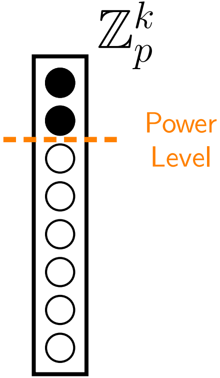
From Unequal Powers to Signal Levels

- Each user is assigned an effective noise tolerance $\sigma^2_{\mathsf{eff},\ell}$ and power level P_ℓ.
- These can be handled by introducing two parameters $k_{\mathsf{C},\ell} \leq k_{\mathsf{F},\ell}$.
- Total number of symbols is $k = k_{\mathsf{F}} - k_{\mathsf{C}}$ where $k_{\mathsf{F}} = \max_\ell k_{\mathsf{F},\ell}$ and $k_{\mathsf{C}} = \min_\ell k_{\mathsf{C},\ell}$.



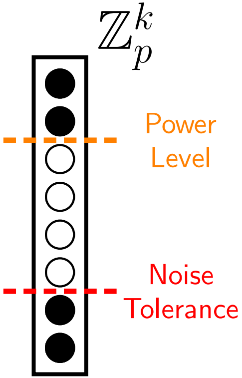
From Unequal Powers to Signal Levels

- Each user is assigned an effective noise tolerance $\sigma^2_{\mathsf{eff},\ell}$ and power level P_ℓ.
- These can be handled by introducing two parameters $k_{\mathsf{C},\ell} \leq k_{\mathsf{F},\ell}$.
- Total number of symbols is $k = k_{\mathsf{F}} - k_{\mathsf{C}}$ where $k_{\mathsf{F}} = \max_\ell k_{\mathsf{F},\ell}$ and $k_{\mathsf{C}} = \min_\ell k_{\mathsf{C},\ell}$.
- Transmitter sets top $k_{\mathsf{C},\ell} - k_{\mathsf{C}}$ symbols to zero to meet its power constraint.



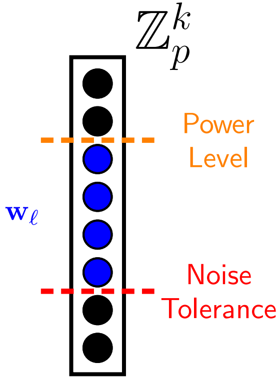
From Unequal Powers to Signal Levels

- Each user is assigned an effective noise tolerance $\sigma^2_{\mathsf{eff},\ell}$ and power level P_ℓ.
- These can be handled by introducing two parameters $k_{\mathsf{C},\ell} \leq k_{\mathsf{F},\ell}$.
- Total number of symbols is $k = k_{\mathsf{F}} - k_{\mathsf{C}}$ where $k_{\mathsf{F}} = \max_\ell k_{\mathsf{F},\ell}$ and $k_{\mathsf{C}} = \min_\ell k_{\mathsf{C},\ell}$.
- Transmitter sets top $k_{\mathsf{C},\ell} - k_{\mathsf{C}}$ symbols to zero to meet its power constraint.
- Transmitter sets bottom $k_{\mathsf{F}} - k_{\mathsf{F},\ell}$ symbols to zero to meet its noise tolerance constraint.



From Unequal Powers to Signal Levels

- Each user is assigned an effective noise tolerance $\sigma^2_{\textsf{eff},\ell}$ and power level P_ℓ.
- These can be handled by introducing two parameters $k_{\textsf{C},\ell} \leq k_{\textsf{F},\ell}$.
- Total number of symbols is $k = k_{\textsf{F}} - k_{\textsf{C}}$ where $k_{\textsf{F}} = \max_\ell k_{\textsf{F},\ell}$ and $k_{\textsf{C}} = \min_\ell k_{\textsf{C},\ell}$.
- Transmitter sets top $k_{\textsf{C},\ell} - k_{\textsf{C}}$ symbols to zero to meet its power constraint.
- Transmitter sets bottom $k_{\textsf{F}} - k_{\textsf{F},\ell}$ symbols to zero to meet its noise tolerance constraint.
- Remaining $k_{\textsf{F},\ell} - k_{\textsf{C},\ell}$ symbols carry information. Rate is $R_\ell = \dfrac{k_{\textsf{F},\ell} - k_{\textsf{C},\ell}}{n} \log p$.

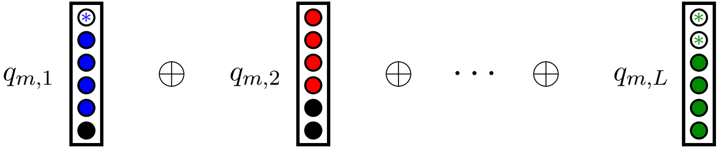


Linear Combinations of Cosets

- Receiver attempts to recover linear combinations of cosets:

$$\mathbf{u}_m = \bigoplus_{\ell=1}^{L} q_{m,\ell} \tilde{\mathbf{w}}_\ell$$

$$\tilde{\mathbf{w}}_\ell \in [\![\mathbf{w}_\ell]\!] \triangleq \left\{ \mathbf{w} \in \mathbb{Z}_p^k : \mathbf{w} = \begin{bmatrix} \mathbf{r} \\ \mathbf{w}_\ell \\ \mathbf{0}_{k_\mathsf{F} - k_{\mathsf{F},\ell}} \end{bmatrix} \text{ for some } \mathbf{r} \in \mathbb{Z}_p^{k_{\mathsf{C},\ell} - k_\mathsf{C}} \right\} .$$

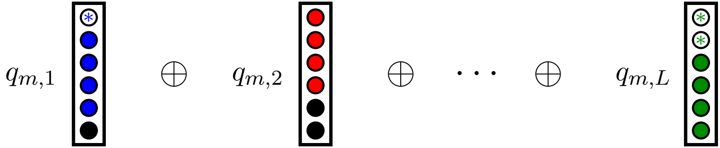


Linear Combinations of Cosets

- Receiver attempts to recover linear combinations of cosets:

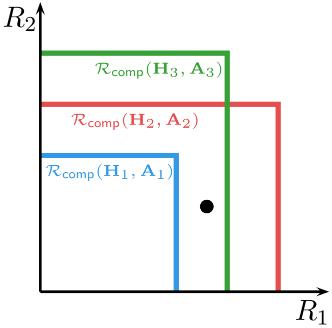
$$\mathbf{u}_m = \bigoplus_{\ell=1}^{L} q_{m,\ell}\tilde{\mathbf{w}}_\ell$$

$$\tilde{\mathbf{w}}_\ell \in [\![\mathbf{w}_\ell]\!] \triangleq \left\{ \mathbf{w} \in \mathbb{Z}_p^k : \mathbf{w} = \begin{bmatrix} \mathbf{r} \\ \mathbf{w}_\ell \\ \mathbf{0}_{k_\mathsf{F} - k_{\mathsf{F},\ell}} \end{bmatrix} \text{ for some } \mathbf{r} \in \mathbb{Z}_p^{k_{\mathsf{C},\ell} - k_\mathsf{C}} \right\} .$$



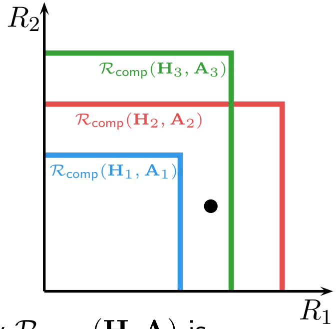
- As before, a linear combination with integer coefficient vector $\mathbf{a}_m$ is one that satisfies $[a_{m,\ell}] \bmod p = q_{m,\ell}$.

- We will specify the computation rate region via a set-valued function $\mathcal{R}_{\text{comp}}(\mathbf{H}, \mathbf{A})$ that maps each channel matrix $\mathbf{H} \in \mathbb{R}^{N_r \times L}$ and integer coefficient matrix $\mathbf{A} \in \mathbb{Z}^{L \times L}$ to a subset of $\mathbb{R}_+^L$.



Computation Rate Region

- We will specify the computation rate region via a set-valued function $\mathcal{R}_{\text{comp}}(\mathbf{H}, \mathbf{A})$ that maps each channel matrix $\mathbf{H} \in \mathbb{R}^{N_r \times L}$ and integer coefficient matrix $\mathbf{A} \in \mathbb{Z}^{L \times L}$ to a subset of $\mathbb{R}_+^L$.



- The computation rate region described by $\mathcal{R}_{\text{comp}}(\mathbf{H}, \mathbf{A})$ is *achievable* if, for every rate tuple $(R_1, R_2, \ldots, R_L) \in \mathbb{R}_+^L$, $\epsilon > 0$, and n large enough, we can select encoders and a decoder such that,
  - for all channel matrices $\mathbf{H} \in \mathbb{R}^{N_r \times L}$ and
  - every coefficient matrix $\mathbf{Q} \in \mathbb{Z}_p^{L \times L}$ for which there exists an integer matrix $\mathbf{A}$ satisfying $(R_1, R_2, \ldots, R_L) \in \mathcal{R}_{\text{comp}}(\mathbf{H}, \mathbf{A})$ and $[\mathbf{A}] \bmod p = \mathbf{Q}$,

  the probability of decoding error is at most ϵ.

- **Ordentlich-Erez Eilat '12:** Start with nested linear code in Construction A to get a "good" nested lattice code.

- **Ordentlich-Erez Eilat '12:** Start with nested linear code in Construction A to get a "good" nested lattice code.
- We use this idea to generate coarse lattices $\Lambda_{\mathsf{C},1}, \ldots, \Lambda_{\mathsf{C},L}$ and fine lattices $\Lambda_{\mathsf{F},1}, \ldots, \Lambda_{\mathsf{F},L}$, all nested in some order.

Codebook Construction

- **Ordentlich-Erez Eilat '12:** Start with nested linear code in Construction A to get a "good" nested lattice code.
- We use this idea to generate coarse lattices $\Lambda_{\mathsf{C},1}, \ldots, \Lambda_{\mathsf{C},L}$ and fine lattices $\Lambda_{\mathsf{F},1}, \ldots, \Lambda_{\mathsf{F},L}$, all nested in some order.
- Transmitter ℓ uses nested lattice code $\left[\Lambda_{\mathsf{F},\ell} / \Lambda_{\mathsf{C},\ell}\right]$.

Codebook Construction

- **Ordentlich-Erez Eilat '12:** Start with nested linear code in Construction A to get a "good" nested lattice code.
- We use this idea to generate coarse lattices $\Lambda_{\mathsf{C},1}, \ldots, \Lambda_{\mathsf{C},L}$ and fine lattices $\Lambda_{\mathsf{F},1}, \ldots, \Lambda_{\mathsf{F},L}$, all nested in some order.
- Transmitter ℓ uses nested lattice code $[\Lambda_{\mathsf{F},\ell}/\Lambda_{\mathsf{C},\ell}]$.
- Linear labeling idea from **Chen-Silva-Kschischang IT '13** allows us to build mapping between cosets and nested lattice codes.

"Parallel" Computation

- Separately decode each linear combination from the output $\mathbf{Y}$.

"Parallel" Computation

- Separately decode each linear combination from the output $\mathbf{Y}$.
- Equalization step:

$$\begin{aligned}\tilde{\mathbf{y}}_m^{\mathsf{T}} &= \mathbf{b}_m^{\mathsf{T}}\mathbf{Y} \\ &= \mathbf{a}_m^{\mathsf{T}}\mathbf{X} + \left(\mathbf{b}_m^{\mathsf{T}}\mathbf{H} - \mathbf{a}_m^{\mathsf{T}}\right)\mathbf{X} + \mathbf{b}_m^{\mathsf{T}}\mathbf{Z} \ .\end{aligned}$$

"Parallel" Computation

- Separately decode each linear combination from the output $\mathbf{Y}$.
- Equalization step:

$$\begin{aligned}\tilde{\mathbf{y}}_m^{\mathsf{T}} &= \mathbf{b}_m^{\mathsf{T}}\mathbf{Y} \\ &= \mathbf{a}_m^{\mathsf{T}}\mathbf{X} + \left(\mathbf{b}_m^{\mathsf{T}}\mathbf{H} - \mathbf{a}_m^{\mathsf{T}}\right)\mathbf{X} + \mathbf{b}_m^{\mathsf{T}}\mathbf{Z}\ .\end{aligned}$$

- Effective noise:

$$\begin{aligned}\sigma^2_{\mathsf{para}}(\mathbf{H}, \mathbf{a}_m) &= \min_{\mathbf{b}_m} \|\mathbf{b}_m\|^2 + \left\|\left(\mathbf{b}_m^{\mathsf{T}}\mathbf{H} - \mathbf{a}_m^{\mathsf{T}}\right)\mathbf{P}^{1/2}\right\|^2 \\ &= \left\|\left(\mathbf{P}^{-1} + \mathbf{H}^{\mathsf{T}}\mathbf{H}\right)^{-1/2}\mathbf{a}_m\right\|^2\end{aligned}$$

"Parallel" Computation

Theorem (Nazer-Cadambe-Ntranos-Caire '15)

For an AWGN network with L transmitters, a receiver, and power constraints $P_1, P_2, \ldots, P_L$, the following computation rate region is achievable,

$$\mathcal{R}^{(\textit{para})}_{\textit{comp}}(\mathbf{H}, \mathbf{A}) = \bigcup_{\substack{\tilde{\mathbf{A}} \in \mathbb{Z}^{L \times L} \\ \mathrm{rowspan}(\mathbf{A}) \subseteq \mathrm{rowspan}(\tilde{\mathbf{A}})}} \mathcal{R}_{\textit{para}}(\mathbf{H}, \tilde{\mathbf{A}})$$

$$\mathcal{R}_{\textit{para}}(\mathbf{H}, \tilde{\mathbf{A}}) = \Bigg\{ (R_1, \ldots, R_L) \in \mathbb{R}_+^L \ : \quad R_\ell \le \frac{1}{2} \log^+ \left(\frac{P_\ell}{\sigma^2_{\textit{para}}(\mathbf{H}, \tilde{\mathbf{a}}_m)} \right) \forall (m, \ell) \textit{ s.t. } \tilde{a}_{m,\ell} \neq 0 \Bigg\}$$

"Successive" Computation

- Can rebuild real sum $\mathbf{a}_m^{\mathsf{T}}\mathbf{X}$ from $\mathbf{u}_m$ and $\mathbf{Y}$.

"Successive" Computation

- Can rebuild real sum $\mathbf{a}_m^{\mathsf{T}}\mathbf{X}$ from $\mathbf{u}_m$ and $\mathbf{Y}$.
- This allows us to build a better effective channel output:

$$\begin{aligned}\tilde{\mathbf{y}}_m^{\mathsf{T}} &= \mathbf{b}_m^{\mathsf{T}}\mathbf{Y} + \mathbf{c}_m^{\mathsf{T}}\mathbf{A}_{m-1}\mathbf{X} \\ &= \mathbf{a}_m^{\mathsf{T}}\mathbf{X} + \left(\mathbf{b}_m^{\mathsf{T}}\mathbf{H} + \mathbf{c}_m^{\mathsf{T}}\mathbf{A}_{m-1} - \mathbf{a}_m^{\mathsf{T}}\right)\mathbf{X} + \mathbf{b}_m^{\mathsf{T}}\mathbf{Z} \ .\end{aligned}$$

"Successive" Computation

- Can rebuild real sum $\mathbf{a}_m^\mathsf{T}\mathbf{X}$ from $\mathbf{u}_m$ and $\mathbf{Y}$.
- This allows us to build a better effective channel output:

$$\begin{aligned}\tilde{\mathbf{y}}_m^\mathsf{T} &= \mathbf{b}_m^\mathsf{T}\mathbf{Y} + \mathbf{c}_m^\mathsf{T}\mathbf{A}_{m-1}\mathbf{X} \\ &= \mathbf{a}_m^\mathsf{T}\mathbf{X} + \left(\mathbf{b}_m^\mathsf{T}\mathbf{H} + \mathbf{c}_m^\mathsf{T}\mathbf{A}_{m-1} - \mathbf{a}_m^\mathsf{T}\right)\mathbf{X} + \mathbf{b}_m^\mathsf{T}\mathbf{Z}\ .\end{aligned}$$

- Effective noise:

$$\begin{aligned}\sigma^2_{\mathsf{succ}}(\mathbf{H},\mathbf{a}_m|\mathbf{A}_{m-1}) &= \min_{\mathbf{b}_m,\mathbf{c}_m} \|\mathbf{b}_m\|^2 + \left\|\left(\mathbf{b}_m^\mathsf{T}\mathbf{H} + \mathbf{c}_m^\mathsf{T}\mathbf{A}_{m-1} - \mathbf{a}_m^\mathsf{T}\right)\mathbf{P}^{1/2}\right\|^2 \\ &= \left\|\mathbf{N}_{m-1}\left(\mathbf{P}^{-1} + \mathbf{H}^\mathsf{T}\mathbf{H}\right)^{-1/2}\mathbf{a}_m\right\|^2\end{aligned}$$

where $\mathbf{N}_{m-1}$ is the nullspace projection corresponding to $\mathbf{A}_{m-1}$.

“Successive” Computation

- Algebraic Successive Cancellation: We can use $\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}$ to eliminate certain codewords from our observation, and thus remove the constraints on them.

"Successive" Computation

- Algebraic Successive Cancellation: We can use $\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}$ to eliminate certain codewords from our observation, and thus remove the constraints on them.

- Let $\mathcal{I} \subset \{1, \ldots, L\} \times \{1, \ldots, L\}$ denote a set of index pairs. We say that $\mathcal{I}$ is an *admissible mapping* for $\mathbf{A}$ if there exists a real-valued, lower unitriangular matrix $\mathbf{L} \in \mathbb{R}^{L \times L}$ such that the (m, ℓ)th entry of $\mathbf{LA}$ is equal to zero for all $(m, \ell) \notin \mathcal{I}$.

“Successive” Computation

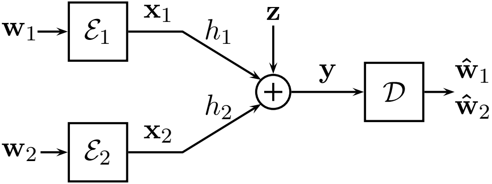
Theorem (Nazer-Cadambe-Ntranos-Caire ’15)

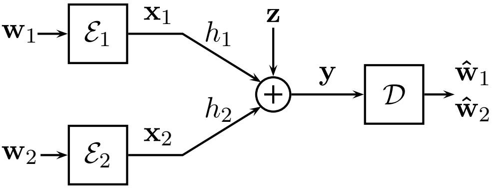
For an AWGN network with L transmitters, a receiver, and power constraints $P_1, P_2, \ldots, P_L$, the following computation rate region is achievable,

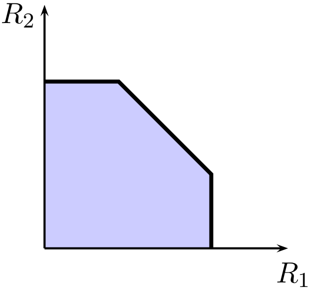
$$\mathcal{R}_{\textit{comp}}^{(\textit{succ})}(\mathbf{H}, \mathbf{A}) = \bigcup_{\substack{\tilde{\mathbf{A}} \in \mathbb{Z}^{L \times L} \\ \mathrm{rowspan}(\mathbf{A}) \subseteq \mathrm{rowspan}(\tilde{\mathbf{A}}) \\ \mathcal{I} \textit{ admissible}}} \mathcal{R}_{\textit{succ}}(\mathbf{H}, \tilde{\mathbf{A}}, \mathcal{I})$$

$$\mathcal{R}_{\textit{succ}}(\mathbf{H}, \tilde{\mathbf{A}}, \mathcal{I}) = \Bigg\{ (R_1, \ldots, R_L) \in \mathbb{R}_+^L \ : \ R_\ell \leq \frac{1}{2} \log^+ \left(\frac{P_\ell}{\sigma^2_{\textit{succ}}(\mathbf{H}, \tilde{\mathbf{a}}_m | \tilde{\mathbf{A}}_{m-1})} \right) \ \forall (m, \ell) \in \mathcal{I} \Bigg\}$$

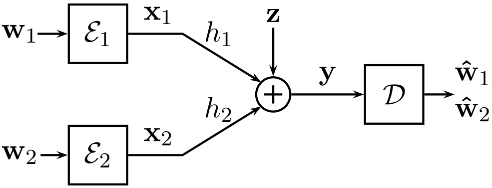
Multiple-Access via Computation

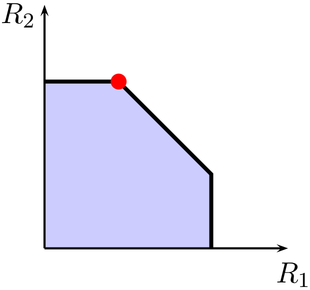




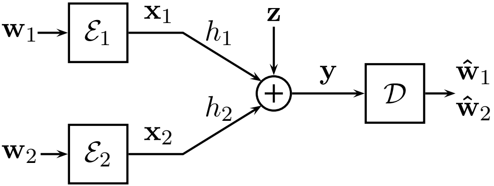


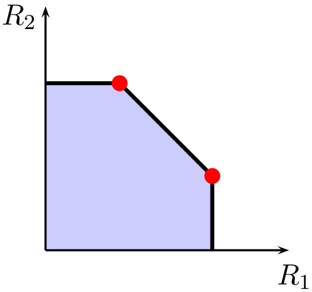
- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.



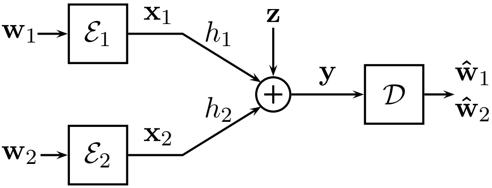


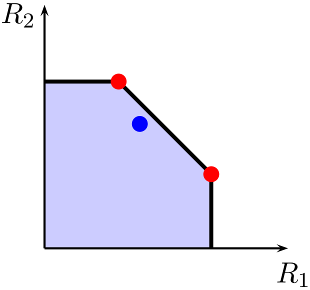
- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.





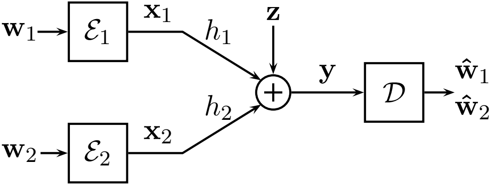
- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.

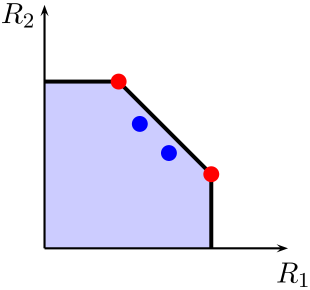




- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.
- Parallel computation can achieve additional rate tuples near the sum rate boundary.

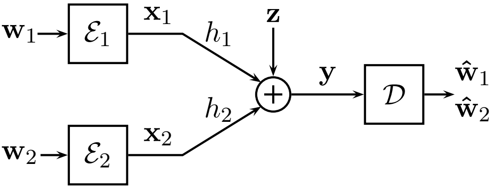
Multiple-Access via Computation

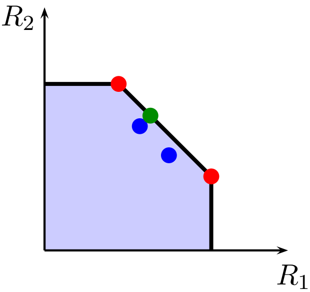




- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.
- Parallel computation can achieve additional rate tuples near the sum rate boundary.

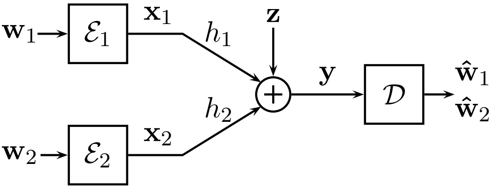
Multiple-Access via Computation

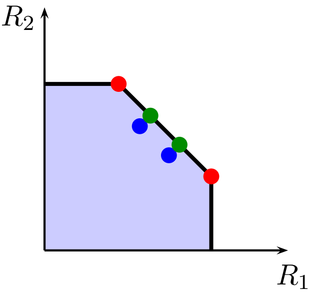




- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.
- Parallel computation can achieve additional rate tuples near the sum rate boundary.
- Successive computation can move these to the exact sum rate.

Multiple-Access via Computation





- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.
- Parallel computation can achieve additional rate tuples near the sum rate boundary.
- Successive computation can move these to the exact sum rate.

Multiple Receivers

- Under this framework, rate regions naturally generalize to multiple receivers:

$$\mathcal{R}^{(\mathsf{para})}_{\mathsf{comp}}(\mathbf{H}^{[1]}, \dots, \mathbf{H}^{[K]}, \mathbf{A}^{[1]}, \dots, \mathbf{A}^{[K]}) = \bigcap_{i=1}^{K} \mathcal{R}^{(\mathsf{para})}_{\mathsf{comp}}(\mathbf{H}^{[i]}, \mathbf{A}^{[i]})$$

$$\mathcal{R}^{(\mathsf{succ})}_{\mathsf{comp}}(\mathbf{H}^{[1]}, \dots, \mathbf{H}^{[K]}, \mathbf{A}^{[1]}, \dots, \mathbf{A}^{[K]}) = \bigcap_{i=1}^{K} \mathcal{R}^{(\mathsf{succ})}_{\mathsf{comp}}(\mathbf{H}^{[i]}, \mathbf{A}^{[i]})$$

Applications and Future Directions

- **He-Nazer-Shamai ISIT '14:** Using this framework, we have found an uplink-downlink duality relationship for compute-and-forward. Allows us to build a connection to the work of **Hong-Caire IT '13.**
- **Ntranos-Cadambe-Nazer-Caire ISIT '13:** Used these ideas for integer-forcing interference alignment.
- **Nazer-Gastpar ITW '14:** Used the problem statement to bring compute-and-forward to the discrete memoryless setting.
- Can the algebraic perspective of **Chen-Silva-Kschischang IT '13** be applied to the expanded problem?
- Currently trying to bring in more sophisticated multi-user techniques.