The AWGN Red Alert Problem

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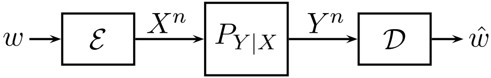
The Red Alert Problem



$w \in \{ \quad 1, 2, \ldots, 2^{nR} \}$

Minimize the probability of missed detection,

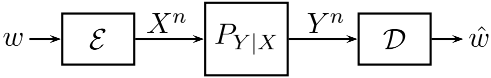
$$\Pr(\hat{w} \neq 0 | w = 0),$$

while keeping the probability of false alarm and the average probability of error between "standard" messages small,

$$\Pr(\hat{w} = 0 | w \neq 0) < \epsilon$$

$$\frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \Pr(\hat{w} \neq w | w \neq 0) < \epsilon \ .$$

Red Alert Message



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Red Alert Exponent

Red Alert Exponent = Exponent of the Probability of Missed Detection

$$E_{\mathsf{ALERT}}(R) = \lim_{n \to \infty} -\frac{1}{n} \log \Pr(\hat{w} \neq w | w = 0)$$

Previous Work:

- DMCs at capacity (**Borade-Nakiboğlu-Zheng IT '09**)

$$E_{\mathsf{ALERT}}(C) = \max_{x \in \mathcal{X}} D\big(p_Y^*(\cdot) || p_{Y|X}(\cdot|x)\big)$$

- BSCs at all rates (**Sahai-Draper ISIT '08**)

$$E_{\mathsf{ALERT}}(R) = \max_{q : h_B(q * p) - h_B(p) \leq R} D(q * p \| p)$$

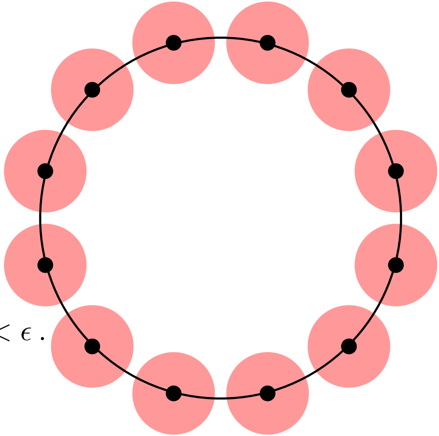
This Talk: AWGN channels at all rates

BSC at Capacity

- Generate $2^{n(C-\epsilon)}$ codewords independently in an i.i.d. Bernoulli($\frac{1}{2}$) manner.

- With high probability

$$\frac{1}{2^{n(C-\epsilon)}} \sum_{w=1}^{2^{n(C-\epsilon)}} \Pr(\hat{w} \neq w | w \neq 0) < \epsilon\,.$$



- Where should we place a "red alert" codeword?

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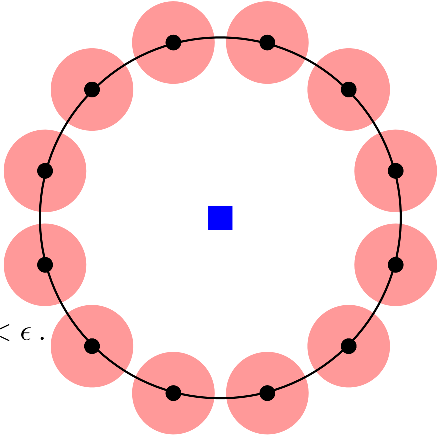

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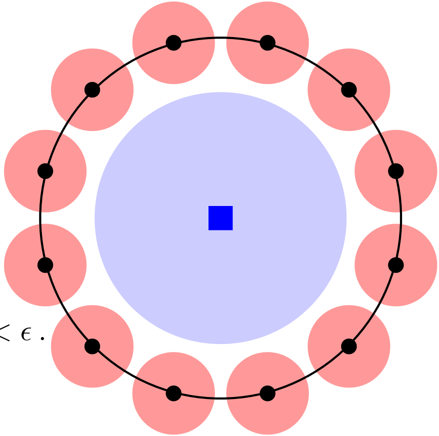
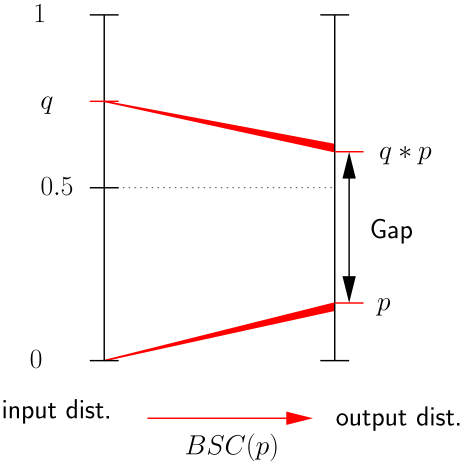


BSC at all Rates

- Red alert codeword $\mathbf{x}(0) = \mathbf{0}$
- Output looks $B(p)$

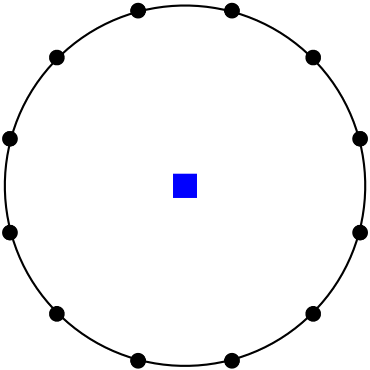
- Standard codewords i.i.d. $B(q)$
- Output looks $B(q * p)$ where $q * p = q(1-p) + (1-q)p$

- Hypothesis test between p and $q * p$
- Probability of missed detection via Sanov's, $E_{\text{ALERT}}(R) = D(q * p \| p)$



BSC Codebook Illustration

- Original view:
  - Looking down on sphere
  - Red alert codeword at north pole
- Rotate sphere to develop intuition for the AWGN case:
  - Codewords live on a parallel of the sphere
  - With higher peak energy, can place red alert codeword off the sphere



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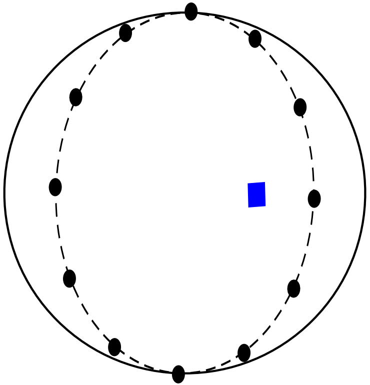

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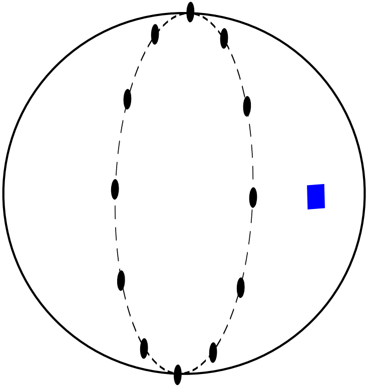

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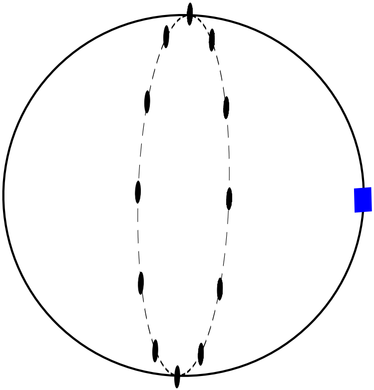

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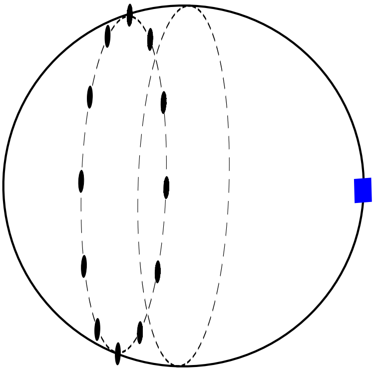

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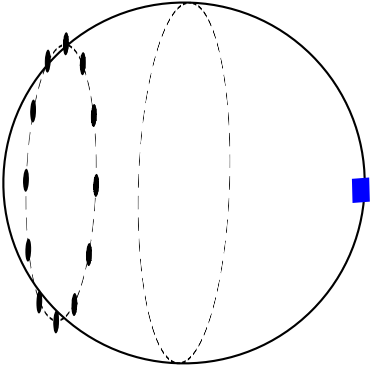

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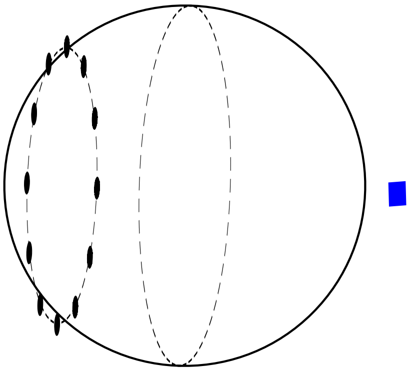

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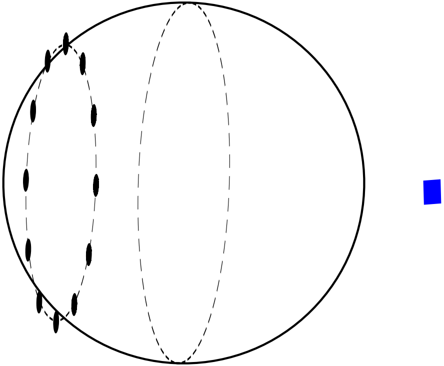
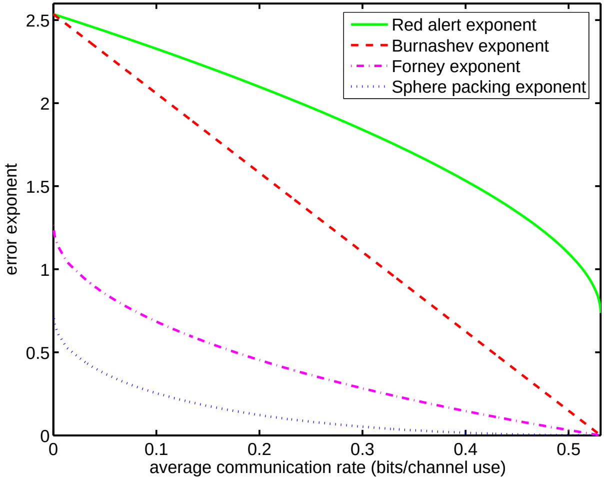


BSC Exponents



$$\lim_{n\to\infty} -\frac{1}{n}\log \Pr(\hat{w} \neq w | w = 0) = \sup_{q:H(q*p)-H(p)>R} D(q*p\|p)$$

Prior use of this type of UEP

- Kudryashov: streaming data systems with variable delay (PPI '79)
- Csiszár: message-wise UEP with many messages (Prob. Ctrl. Inform. Theory '80)
- Draper and Sahai: streaming with noisy feedback (Allerton '06) & Hallucination bound for BSC (ISIT '08)
- Borade, Nakiboğlu, and Zheng: msg- and bit-wise UEP with and w/out feedback (ISIT '08, T-IT '09)

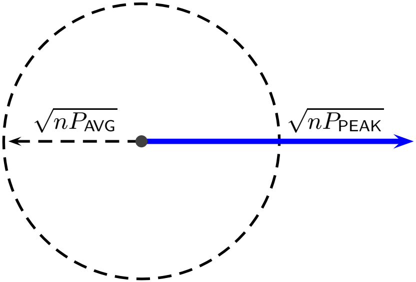
Problem Setup for AWGN channels

- 2^{nR} standard codewords and one special, red alert codeword.
- AWGN channel: $\mathbf{y} = \mathbf{x} + \mathbf{z}$
- Average power constraint:

$$\frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \|\mathbf{x}(w)\|^2 \leq nP_{\mathsf{AVG}}$$

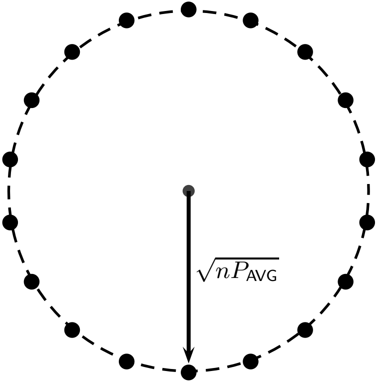
- Peak energy constraint:

$$\|\mathbf{x}(w)\|^2 \leq nP_{\mathsf{PEAK}}$$



Allerton 2010: Conical Codebook

- Generate 2^{nC} codewords randomly on the surface of the $\sqrt{nP_{\mathsf{AVG}}}$ sphere.



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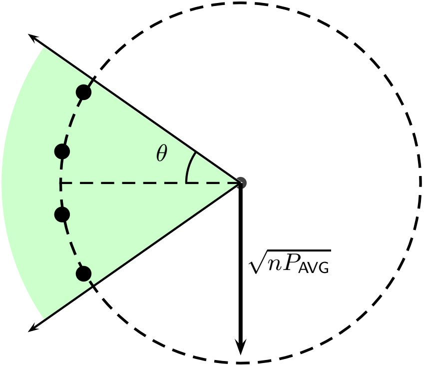
- Generate 2^{nC} codewords randomly on the surface of the $\sqrt{nP_{\mathsf{AVG}}}$ sphere.

- Keep those in the cone of half angle:

$$\theta = \sin^{-1}\left(2^{-C+R}\right)$$



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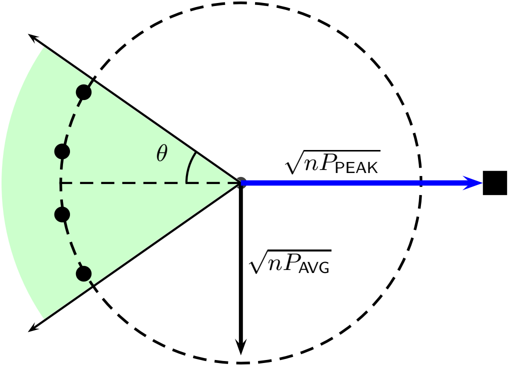
- Place red alert codeword at the limit of the peak power constraint.



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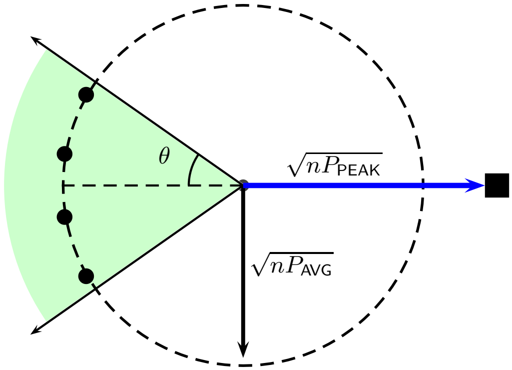

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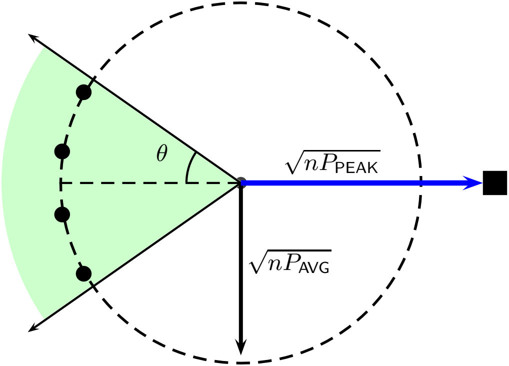
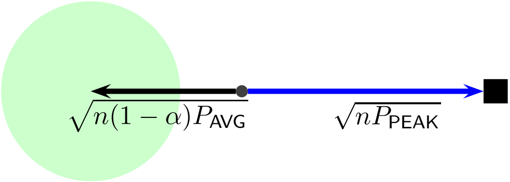


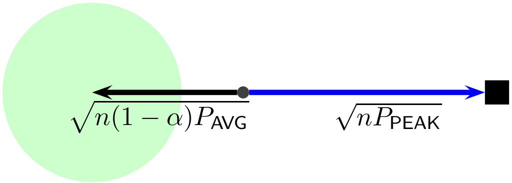
Does not yield optimal exponent.

Offset Codebook

- Red alert codeword at the limit of the peak power constraint.
- Choose α so that
$$R < \frac{1}{2}\log\left(1 + \frac{\alpha P_{\mathsf{AVG}} - \epsilon}{N}\right)$$
- Standard codewords generated i.i.d. $\mathcal{N}(0, \alpha P_{\mathsf{AVG}} - \epsilon)$.
- Use remaining power to shift standard codewords away from red alert codeword.

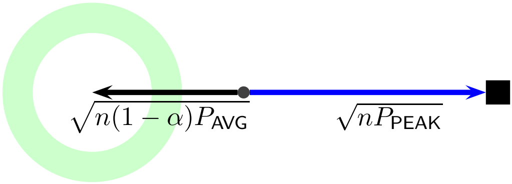


Standard Decoding Region



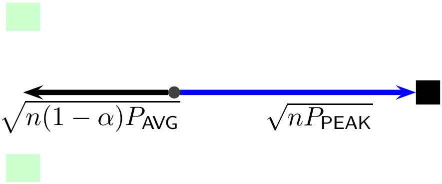
Standard Decoding Region

- Most codewords live near the surface of the sphere.



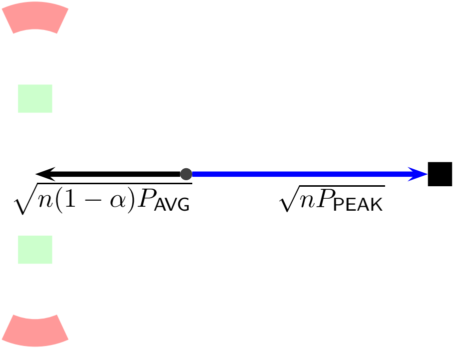
Standard Decoding Region

- Most codewords live near the surface of the sphere.
- Most codewords orthogonal to the red alert codeword.



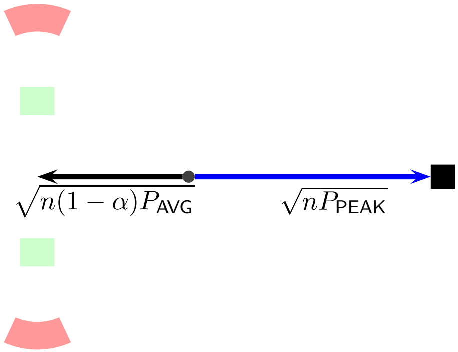
Standard Decoding Region

- Most codewords live near the surface of the sphere.
- Most codewords orthogonal to the red alert codeword.
- Noise pushes codewords further.



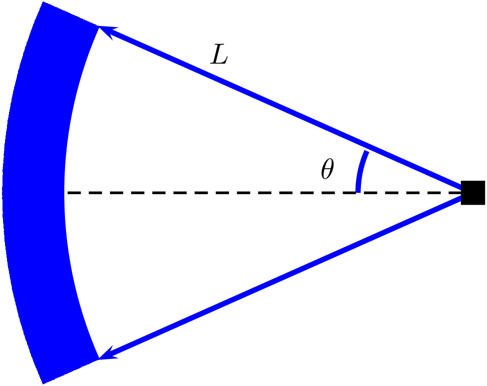
Standard Decoding Region

- Most codewords live near the surface of the sphere.
- Most codewords orthogonal to the red alert codeword.
- Noise pushes codewords further.
- Want to characterize the decoding region with respect to the red alert codeword



Missed Detection Bound

- Missed detection if noise pushes red alert codeword into standard decoding region.
- This only occurs if noise has magnitude larger than L and lies in cone of half-angle θ.
- Angle and magnitude of Gaussian noise are independent.



Distance Bound

$$\begin{aligned}
L^2 &= \| -\mathbf{x}(0) + \mathbf{x}(w) + \mathbf{z}\|^2 \\
&= \left\| \left(\sqrt{P_{\mathsf{PEAK}}} + \sqrt{(1-\alpha)P_{\mathsf{AVG}}} \right) \mathbf{1} + \mathbf{x}_\alpha(w) + \mathbf{z} \right\|^2 \\
&\leq n \left(\sqrt{P_{\mathsf{PEAK}}} + \sqrt{(1-\alpha)P_{\mathsf{AVG}}} \right)^2 + n\epsilon + n(\alpha P_{\mathsf{AVG}} + N + \epsilon) \\
&= n\big(P_{\mathsf{PEAK}} + P_{\mathsf{AVG}} + N + 2\sqrt{P_{\mathsf{PEAK}}(1-\alpha)P_{\mathsf{AVG}}} + 2\epsilon\big) \\
&= nN(1+\beta)
\end{aligned}$$

Standard large deviations bound on Chi-square random variables:

$$\Pr(\|\mathbf{z}\| \geq L) \leq 2\exp\left(-n\left(\frac{\beta}{2} - \frac{1}{2}\log(1+\beta) - \epsilon \right) \right)$$

Angle Bound

Angle of n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\measuredangle(\mathbf{a}, \mathbf{b}) = \cos^{-1}\left(\frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)$$

Shannon '59: The fraction of surface area of an n-dimensional sphere carved out by a cone centered at its origin with half-angle θ:

$$\frac{\text{Surface Area of Cone}(\theta)}{\text{Surface Area of Sphere}} = \frac{\sin^n \theta}{\sqrt{2\pi n}\sin\theta\cos\theta}\left(1 + O\left(\frac{1}{n}\right)\right)$$

Eventually, we get:

$$\Pr\big(\mathbf{z}\text{ lies in Cone}(\theta)\big) \leq \exp\left(-n\left(\frac{1}{2}\log(1+\beta) - R - \epsilon\right)\right)$$

Achievable Red Alert Exponent

Theorem

For any rate $0 \leq R \leq C$, the following red alert exponent is achievable

$$E_{\mathit{ALERT}}(R) = \frac{P_{\mathit{PEAK}} + P_{\mathit{AVG}} + 2\sqrt{P_{\mathit{PEAK}}(P_{\mathit{AVG}} + N(1 - e^{2R}))}}{2N} - R \ .$$

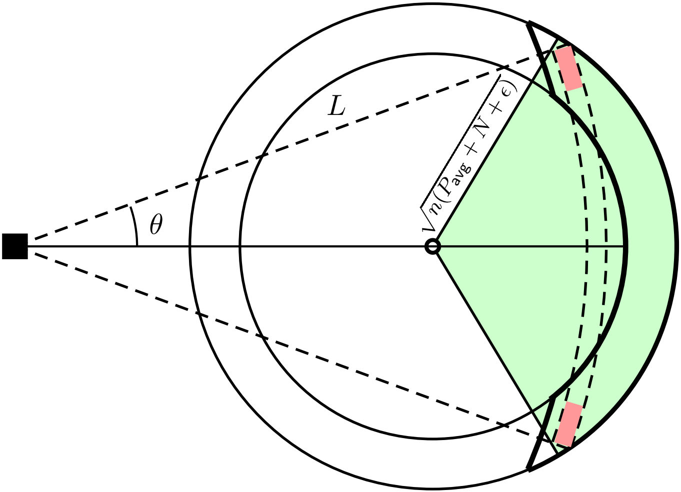
Proof:

$$\begin{aligned}
&\Pr(\text{Missed Detection}) \\
&\leq \Pr(\|\mathbf{z}\| \geq L)\Pr\big(\mathbf{z} \text{ lies in Cone}(\theta)\big) \\
&\leq 2\exp\left(-n\left(\frac{\beta}{2} - \frac{1}{2}\log(1+\beta) + \frac{1}{2}\log(1+\beta) - R - 2\epsilon\right)\right) \\
&\leq 2\exp\left(-n\left(\frac{\beta}{2} - R - 2\epsilon\right)\right)
\end{aligned}$$

Converse Ideas

- Constant fraction of codewords must nearly obey the average power constraint.
- Constant fraction of codewords are close to the average probability of error.
- Constant fraction of codewords live on some thin shell.
- Need at least the volume of a sphere of radius $\sqrt{nN}$ to decode one codeword.
- Pack the total volume into the shell (distorted by noise) such that it minimizes the probability of missed detection.

Converse Illustration



Optimal Red Alert Exponent

Applying Cramér's Theorem yields:

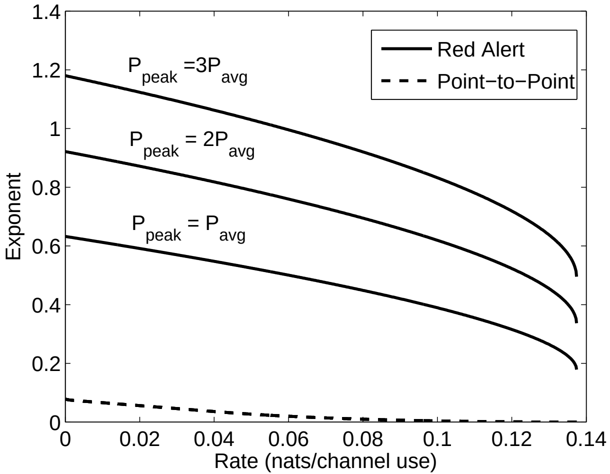
Theorem

For any rate $0 \leq R \leq C$, the red alert exponent is upper bounded by

$$E_{\mathsf{ALERT}}(R) = \frac{P_{\mathsf{PEAK}} + P_{\mathsf{AVG}} + 2\sqrt{P_{\mathsf{PEAK}}(P_{\mathsf{AVG}} + N(1 - e^{2R}))}}{2N} - R \; .$$

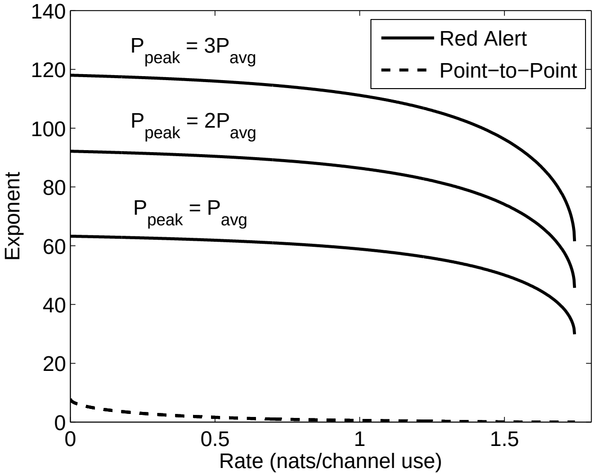
Plots

Red alert exponent for $P_{\mathsf{AVG}} = -5\text{dB}$.



Plots

Red alert exponent for $P_{\mathsf{AVG}} = 15\text{dB}$.



Future work

- Bounds for finite block lengths
- Code design
- Use in, and analysis of, streaming data systems
- Multiple special messages