Integer-Forcing

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Joint work with Jiening Zhan (UC - Berkeley), Uri Erez (Tel Aviv), Michael Gastpar (EPFL / UC - Berkeley)

ITA 2012

Classical Decoding Strategies:

- Joint Decoding
- Zero-Forcing
- Successive Interference Cancellation

A Simple Example

- $\mathbf{Y} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \mathbf{Z}$

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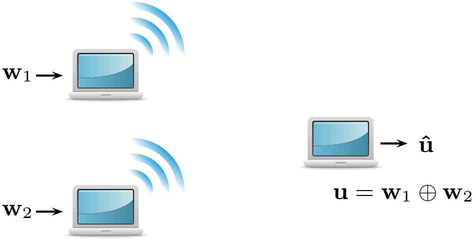
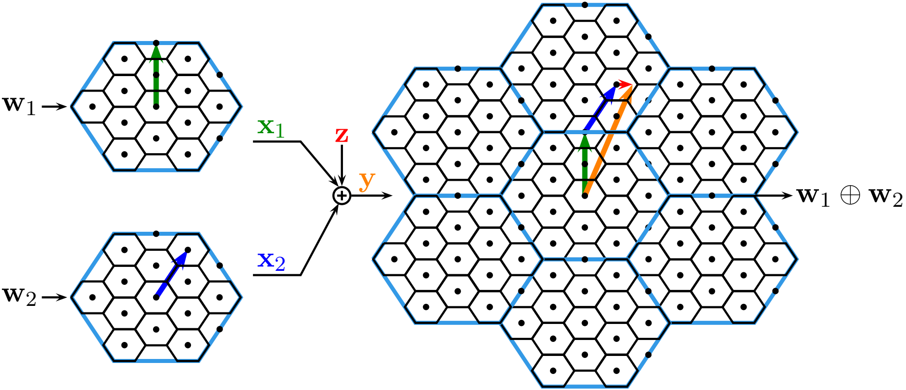
- Effective noise variances: $\sigma_1^2 = 5$ and $\sigma_2^2 = 2$.

- Integer-Forcing: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{Y} = \begin{bmatrix} \mathbf{x}_1 + 2\mathbf{x}_2 \\ \mathbf{x}_1 + \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{Z}$

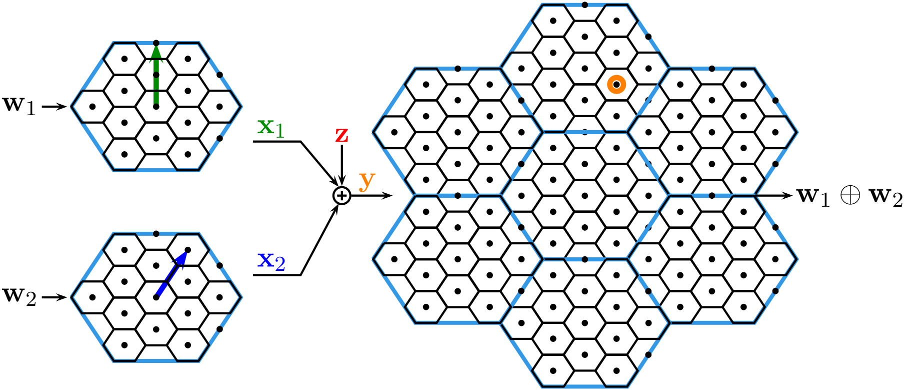
- Effective noise variances: $\sigma_1^2 = 1$ and $\sigma_2^2 = 1$.

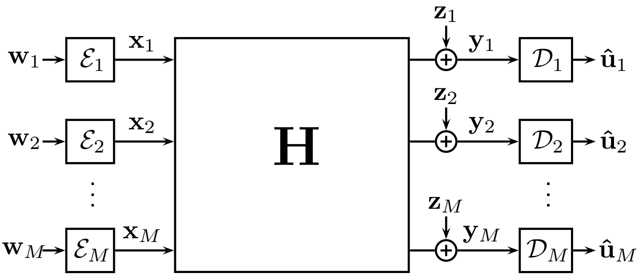


- Possible to decode linear functions of the transmitted messages. (**Nazer-Gastpar '11**)
- Provided that the codebooks share a common algebraic structure.


$\mathbf{w}_1$
$\mathbf{w}_2$
$\mathbf{x}_1$
$\mathbf{x}_2$
$\mathbf{z}$
$\mathbf{y}$
$\mathbf{w}_1 \oplus \mathbf{w}_2$

Compute-and-Forward





- Finite field messages, $\mathbf{w}_\ell \in \mathbb{F}_q^k$.
- Gaussian channel model, $\|\mathbf{x}_\ell\|^2 \leq nP$ and $\mathbf{z}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.
- Equal rates $R = \frac{k}{n} \log_2 q$.
- Each receiver wants a linear combination, $\mathbf{u}_m = \bigoplus_{\ell=1}^{M} a_{m\ell} \mathbf{w}_\ell$
- Vanishing probability of error, $\lim_{n \to \infty} \mathbb{P}(\cup_m \{\hat{\mathbf{u}}_m \neq \mathbf{u}_m\}) = 0$.

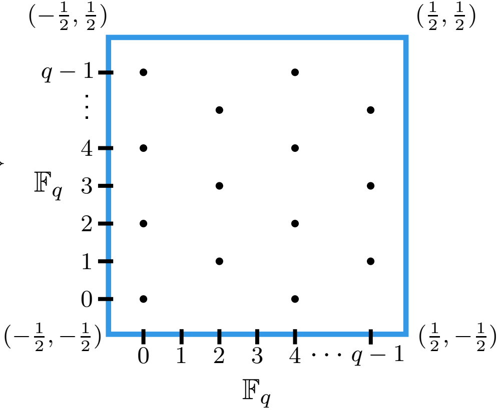
Effective Noise

$$\mathbf{y}_m = \sum_{\ell=1}^{M} h_{m\ell}\mathbf{x}_\ell + \mathbf{z}_m$$
$$= \sum_{\ell=1}^{M} a_{m\ell}\mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{M}(h_{m\ell} - a_{m\ell})\mathbf{x}_\ell + \mathbf{z}_m}_{\text{Effective Noise}}$$

- How can we go between the integer combination of the real-valued codewords and the linear combination of the finite field messages?
- How do we cope with the self-noise?
- Use (dithered) nested lattice codes from **Erez-Zamir '04**.

Nested Lattice Codes from q-ary Linear Codes

- Choose an $n \times k$ generator matrix $\mathbf{G} \in \mathbb{F}_q^{n \times k}$ for q-ary code.
- Integers serve as coarse lattice, $\Lambda = \mathbb{Z}^n$.
- Map elements $\{0, 1, 2, \ldots, q-1\}$ to equally spaced points between $-1/2$ and $1/2$.
- Place codewords $\mathbf{x} = \mathbf{G}\mathbf{w}$ into the fundamental Voronoi region $\mathcal{V} = [-1/2, 1/2)^n$
- One more step needed for shaping gain...



Effective Noise

- Only need channel state information at the receivers.

- Effective noise due to mismatch between channel coefficients $\mathbf{h}_m = [h_{m1} \cdots h_{mM}]^T$ and equation coefficients $\mathbf{a}_m = [a_{m1} \cdots a_{mM}]^T$.

$$N_{\mathsf{EFFEC},m} = 1 + P\|\mathbf{h}_m - \mathbf{a}_m\|^2$$

$$R = \min_m \frac{1}{2} \log^+ \left(\frac{P}{1 + P\|\mathbf{h}_m - \mathbf{a}_m\|^2} \right)$$

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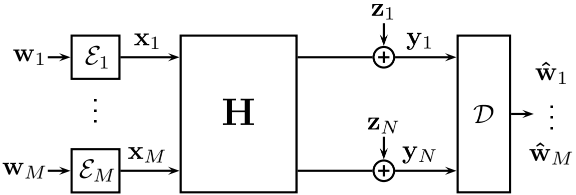
- Can do better with MMSE scaling.

$$N_{\mathsf{EFFEC},m} = \alpha_m^2 + P\|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2$$

$$R = \min_m \max_{\alpha_m} \frac{1}{2} \log \left(\frac{P}{\alpha_m^2 + P\|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right)$$

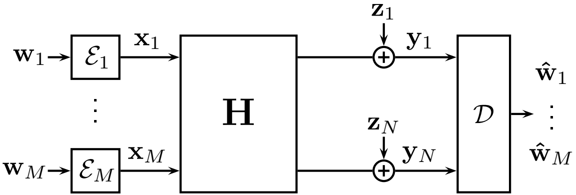
$$= \min_m \frac{1}{2} \log^+ \left(\frac{1 + P\|\mathbf{h}_m\|^2}{\|\mathbf{a}_m\|^2 + P(\|\mathbf{h}_m\|^2 \|\mathbf{a}_m\|^2 - (\mathbf{h}_m^T \mathbf{a}_m)^2)} \right)$$

Problem Statement



- M transmit and N receive antennas $(N \geq M)$.
- Channel model: $\mathbf{Y} = \mathbf{HX} + \mathbf{Z}$
- Each antenna encodes an independent message $\mathbf{w}_\ell$ (or data stream) of rate R/M to a codeword $\mathbf{x}_\ell \in \mathbb{R}^T$ with power at most SNR.
- i.i.d. Gaussian noise $\mathbf{z}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.
- Probability of error: $\mathbb{P}(\{\hat{\mathbf{w}}_1 \neq \mathbf{w}_1\} \cup \cdots \cup \{\hat{\mathbf{w}}_M \neq \mathbf{w}_M\}) < \epsilon$
- V-BLAST setting.

Joint ML Decoder

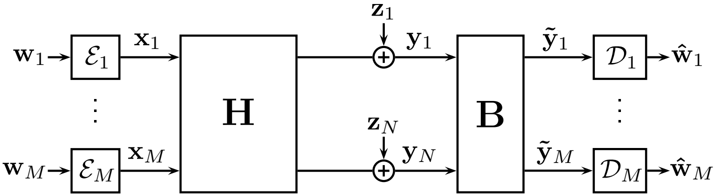


- Joint maximum likelihood decoder offers optimal performance,

$$R_{\mathsf{JOINT}}(\mathbf{H}) = \min_{\mathcal{S} \subseteq \{1,2,\ldots,M\}} \frac{M}{2|\mathcal{S}|} \log\det\left(\mathbf{I}_{\mathcal{S}} + \mathsf{SNR}\ \mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^T\right)$$

- Worst-case complexity is exponential in MT (where $M =$number of antennas and $T =$blocklength).
- We would like to approach this performance as closely as possible using single stream decoding.

Linear Receiver Architectures



- First project the received signal, $\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y}$.
- Each projected output stream $\tilde{\mathbf{y}}_m$ is fed into a decoder that attempts to recover a single message $\mathbf{w}_m$.
- Each decoder may face interference from other data streams,

$$R_{\text{LINEAR}}(\mathbf{H}) = \min_m \frac{M}{2} \log\left(1 + \frac{\text{SNR}\|\mathbf{b}_m^T\mathbf{h}_m\|^2}{\|\mathbf{b}_m\|^2 + \text{SNR}\sum_{i\neq m}\|\mathbf{b}_m^T\mathbf{h}_i\|^2}\right) .$$

- Worst-case complexity is exponential in T and linear in M.

- Zero-Forcing Receiver: Eliminate interference between data streams by setting projection matrix to

$$\mathbf{B} = \mathbf{H}^{\dagger} \ .$$

- Also known as a decorrelator.
- Effective channel: $\tilde{\mathbf{Y}} = \mathbf{X} + \mathbf{H}^{-1}\mathbf{Z}$ (if $\mathbf{H}$ is full rank). Performs poorly if $\mathbf{H}$ is ill-conditioned.

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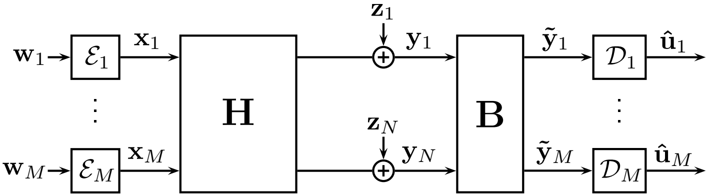
- MMSE Receiver: Can do slightly better with a regularized projection matrix

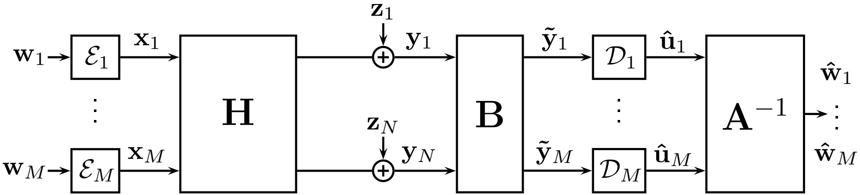
$$\mathbf{B} = \mathbf{H}^T \left(\mathbf{H}\mathbf{H}^T + \frac{1}{\mathsf{SNR}}\mathbf{I} \right)^{-1} \ .$$



$$\mathbf{u}_m = \bigoplus_{\ell=1}^{M} a_{m\ell}\mathbf{w}_\ell$$

Integer-Forcing Receiver:

- First project the received signal, $\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y}$.
- Each stream $\tilde{\mathbf{y}}_m$ is fed into a decoder that attempts to recover an equation $\mathbf{u}_m$.
- Equations can then be solved digitally for the original messages.
- Choose equation coefficients $\mathbf{A}$ to minimize effective noise.
- Worst-case complexity is exponential in T and exponential in M.



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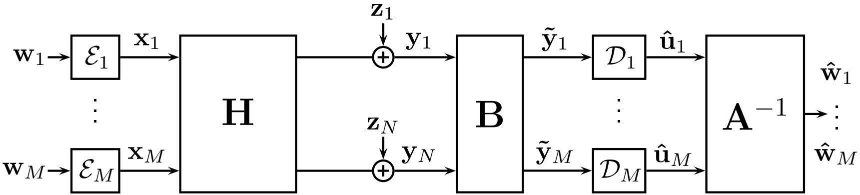

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- **Zhan-Nazer-Erez-Gastpar, submitted to IT <arXiv:1003.5966>**
  The achievable rate of integer-forcing is

$$R_{\mathsf{INTEGER}}(\mathbf{H}) = \max_{\substack{\mathbf{A} \in \mathbb{Z}^{M \times M} \\ \mathsf{rank}(\mathbf{A}) = M}} \max_{\mathbf{B} \in \mathbb{R}^{M \times N}} \min_{m} R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m)$$

$$R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m) = \frac{M}{2} \log^{+} \left(\frac{\mathsf{SNR}}{\|\mathbf{b}_m\|^2 + \mathsf{SNR} \|\mathbf{H}^T \mathbf{b}_m - \mathbf{a}_m\|^2} \right)$$

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- Only need to search over integer vectors $\mathbf{a}_m$ satisfying

$$\|\mathbf{a}_m\|^2 \leq 1 + \lambda_{\mathsf{MAX}}^2\mathsf{SNR}$$

  where λ_{MAX} is the maximum singular value of $\mathbf{H}$.

- Exact Integer-Forcing: Eliminate interference between data streams by setting projection matrix to

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$$R_{\mathsf{EXACT}}(\mathbf{H}) = \max_{\substack{\mathbf{A}\in\mathbb{Z}^{M\times M}\\ \mathsf{rank}(\mathbf{A})=M}} \min_{m} \frac{M}{2}\log^{+}\left(\frac{\mathsf{SNR}}{\|(\mathbf{H}^{T})^{\dagger}\mathbf{a}_m\|^2}\right) .$$

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Outage Formulation

- Channel matrix $\mathbf{H}$ is i.i.d. Gaussian, only known at the receiver.

- Assume there exists an architecture that encodes each data stream at the same rate and achieves sum rate $R(\mathbf{H})$. For a target rate R, then the outage probability is defined as

$$p_{\mathsf{OUT}}(R) = \Pr(R(\mathbf{H}) < R).$$

- For a fixed probability $p \in (0, 1]$, the *outage rate* is defined to be

$$R_{\mathsf{OUT}}(p) = \sup\{R : p_{\mathsf{OUT}}(R) \leq p\}.$$

- Linear receiver architectures are often augmented using successive interference cancellation (SIC).
- Basic idea: After decoding codeword $\mathbf{x}_\ell$, remove its effect from channel output to reduce the interference between data streams.

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- V-BLAST II: Select the decoding order for each channel realization to maximize the effective SNR for the data stream that sees the worst channel.
- V-BLAST III: Decodes and cancel the data streams in a predetermined order. The rate of each data stream is selected to maximize the sum rate. (Outside problem statement.)

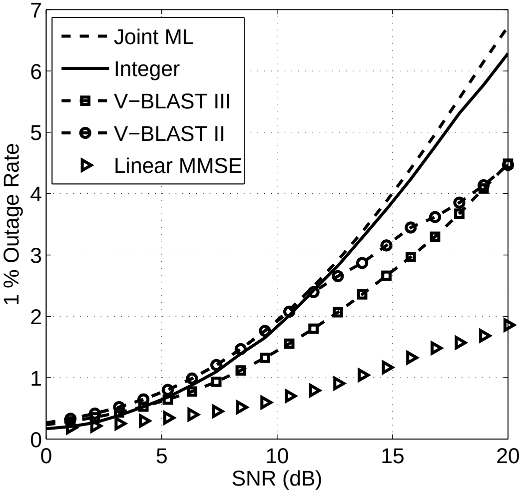

Figure: 1 percent outage rates for the 2×2 complex-valued MIMO channel with Rayleigh fading.

Simulation: Outage Probability

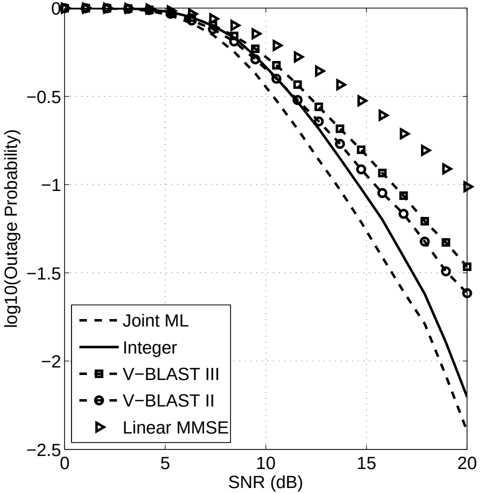

Figure: Outage probability for the 2×2 complex-valued MIMO channel with Rayleigh fading for a target sum rate of $R = 6$

Noise variance in received stream m:

$$\widetilde{\sigma}_m^2 = |\mathbf{v}_{\mathsf{min}}^T \mathbf{a}_m|^2 \frac{1}{\lambda_{\mathsf{min}}} + |\mathbf{v}_{\mathsf{max}}^T \mathbf{a}_m|^2 \frac{1}{\lambda_{\mathsf{max}}}$$

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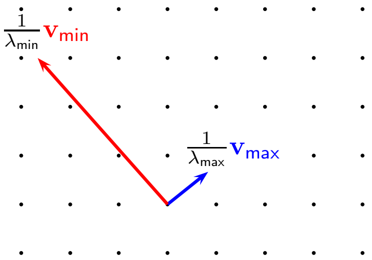

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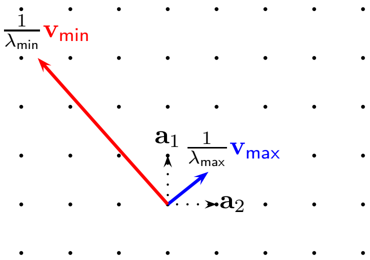
Decorrelator



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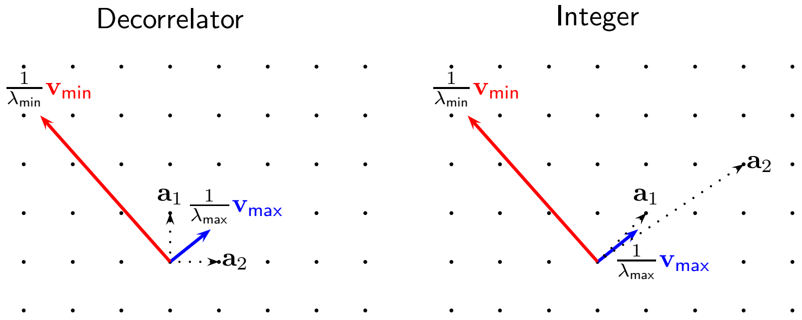

- **Zheng-Tse '03:** A family of codes is said to achieve spatial multiplexing gain r and diversity gain d if the total data rate and the average probability of error satisfy

$$\lim_{\mathsf{SNR}\to\infty} \frac{R(\mathsf{SNR})}{\log \mathsf{SNR}} \geq r$$
$$\lim_{\mathsf{SNR}\to\infty} \frac{\log P_e(\mathsf{SNR})}{\log \mathsf{SNR}} \leq -d.$$

- For our problem, the optimal diversity-multiplexing tradeoff (DMT) under Rayleigh fading is

$$d_{\mathsf{JOINT}}(r) = N\left(1 - \frac{r}{M}\right)$$

where $r \in [0, M]$ and can be achieved by joint ML decoding.

- **Zheng-Tse '03:** The DMTs achieved by the decorrelator and SIC architectures are

$$
\begin{aligned}
d_{\text{DECORR}}(r) &= \left(1 - \frac{r}{M}\right) \\
d_{\text{V-BLAST I}}(r) &= \left(1 - \frac{r}{M}\right) \\
d_{\text{V-BLAST II}}(r) &\leq (N-1)\left(1 - \frac{r}{M}\right) \\
d_{\text{V-BLAST III}}(r) &= \text{piecewise linear curve connecting points } \ (r_k, n-k) \\
&\qquad \text{where } r_0 = 0, r_k = \sum_{i=0}^{k-1} \frac{k-i}{n-i} \ \ 1 \leq k \leq n
\end{aligned}
$$

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$$\text{where } r_0 = 0, r_k = \sum_{i=0}^{k-1} \frac{k-i}{n-i} \quad 1 \leq k \leq n$$

- Integer-forcing recovers the optimal DMT:

$$d_{\text{INTEGER}}(r) = N\left(1 - \frac{r}{M}\right)$$

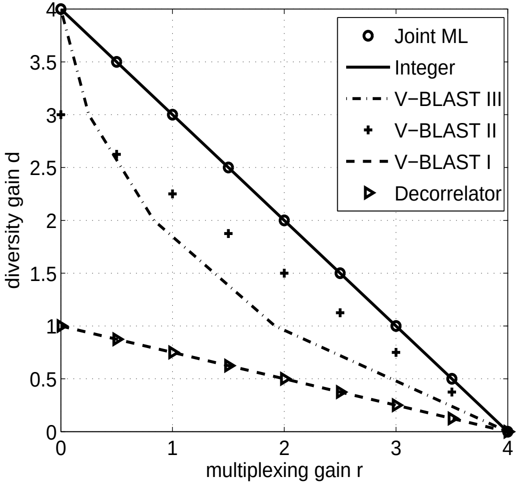
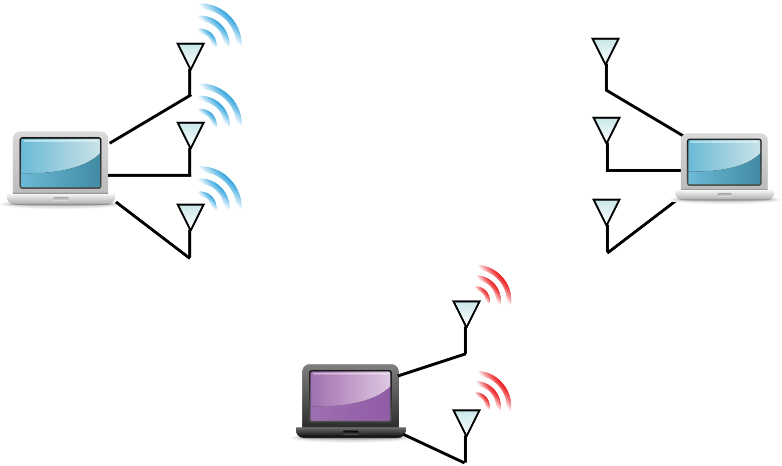
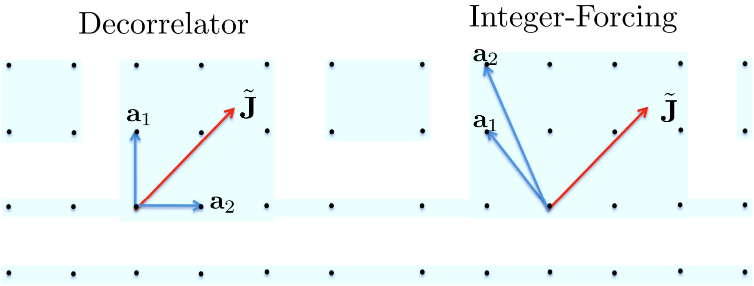


Figure: Diversity-multiplexing tradeoff for the 4×4 MIMO channel with Rayleigh fading.

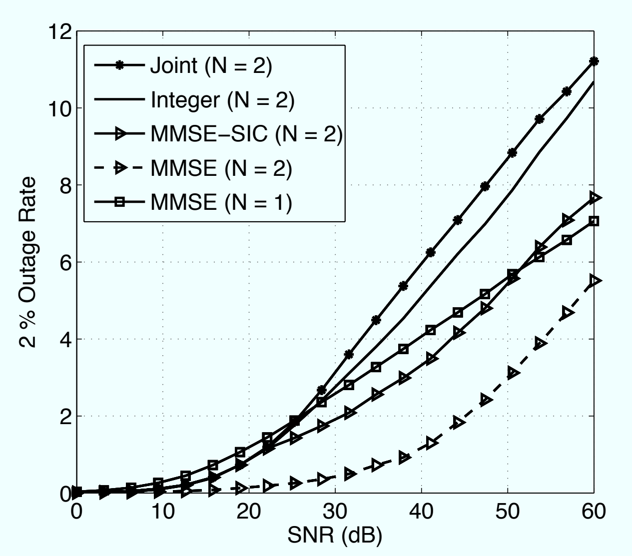


Choose integer vectors orthogonal to the interference space.



$$\mathbf{Y} = \mathbf{HX} + \mathbf{JS} + \mathbf{Z}$$

Integer-Forcing Advantage



2 by 2 MIMO, 1 dimensional interference, $\mathsf{INR} = \mathsf{SNR}^{0.2}$

Some Connections

- Lattice Reduction: **Yao-Wornell '02**, **Taherzadeh-Mobasher-Khandani '07**, **Jalden-Elia '09**
- Lattices for AWGN Capacity: **Erez-Zamir '04**
- Lattices for DMT: **El Gamal-Caire-Damen '04**
- Practical compute-and-forward: **Feng-Silva-Kschischang '10**, **Hern and Narayanan '11**, **Ordentlich and Erez '10**, **Osmane and Belfiore '11**

Concluding Remarks

- Algebraic structure enables us to do something in between treating interference as noise and decoding interference.
- See ISIT '11 Tutorial: Algebraic Structure in Network Information Theory for more information.
- Near-optimal receiver architecture with decoupled optimization problems.
- Question: Can this be generalized to include encoding across transmit antennas?

- V-BLAST I-V: Decodes and cancel the data streams in the best order, given the channel realization. The rate of each data stream is selected to maximize the sum rate. (Outside problem statement.)