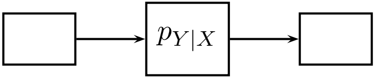
Towards an Algebraic Network Information Theory

Bobak Nazer (BU)

Tufts ECE Seminar
October 28, 2016

Motivation



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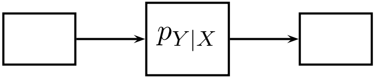

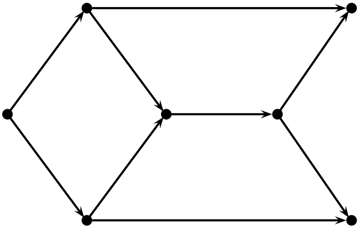

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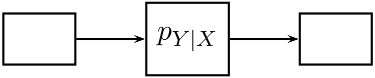

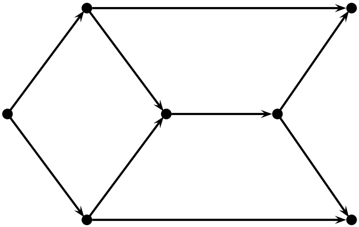

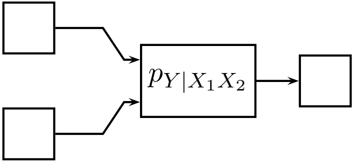

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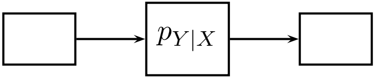

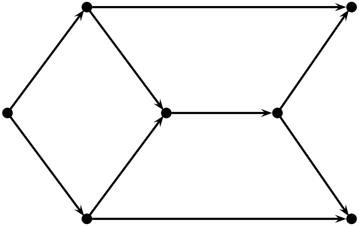

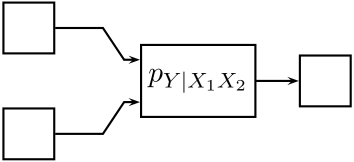

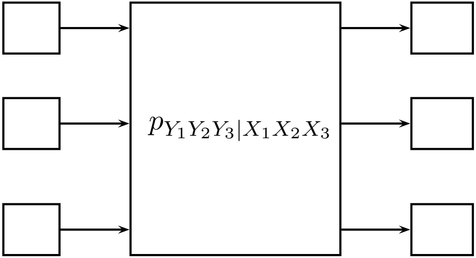

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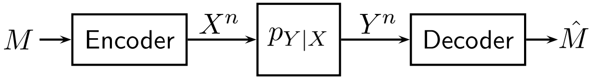
- Codes with algebraic structure are sought after to mimic the performance of random i.i.d. codes with low implementation complexity.

Point-to-Point Channels



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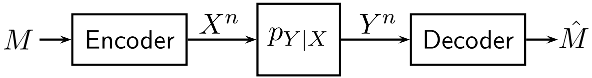

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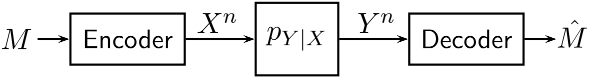

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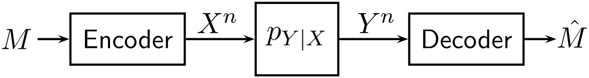

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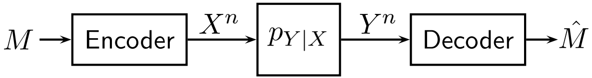

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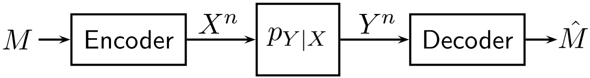

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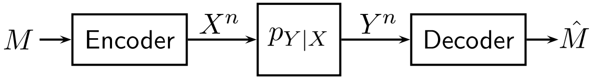

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- Proof relies on random i.i.d. codebooks combined with joint typicality decoding.

Typicality

- x^n is a length-n sequence with elements from finite alphabet $\mathcal{X}$
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- This motivates the typical set

$$\mathcal{T}_\epsilon^{(n)}(X) = \big\{x^n : |\pi(x|x^n) - p_X(x)| \le \epsilon p_X(x) \;\text{ for all } x \in \mathcal{X}\big\}$$

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- We can generalize this definition to pairs of sequences (X^n, Y^n) that are i.i.d. according to $p_{XY}(x, y)$ and so on...

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Intuition: Probability that i.i.d. $\tilde{X}^n$ looks jointly typical $\approx 2^{-nI(X;Y)}$

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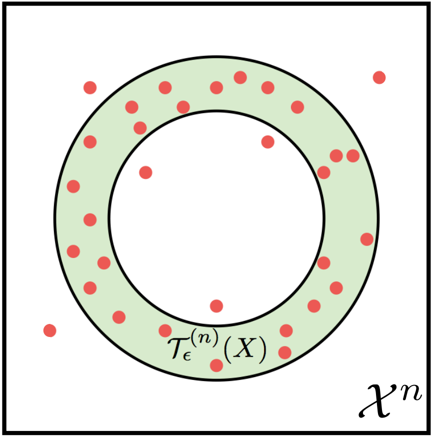
$$\begin{aligned}\mathsf{P}\Big\{ \bigcup_{\tilde{m} \neq m} \big\{(X^n(\tilde{m}), Y^n) \in \mathcal{T}_\epsilon^{(n)}\big\}\Big\} &\leq \sum_{\tilde{m} \neq m} \mathsf{P}\big\{(X^n(\tilde{m}), Y^n) \in \mathcal{T}_\epsilon^{(n)}\big\} \\ &\leq \sum_{\tilde{m} \neq m} 2^{-nI(X;Y) - \delta(\epsilon)} \\ &< 2^{nR}\, 2^{-nI(X;Y) - \delta(\epsilon)}\ .\end{aligned}$$

    Probability goes to zero if $R < I(X;Y) - \delta(\epsilon)$.



Random i.i.d. Codes

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This Talk: We build on previous work and propose a joint typicality approach to algebraic network information theory.

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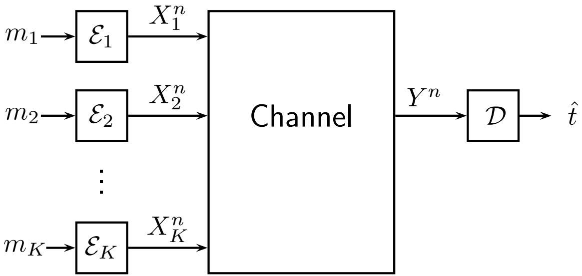

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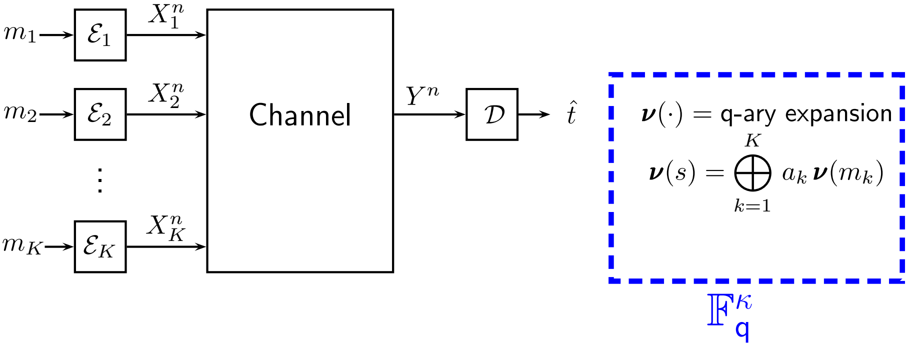

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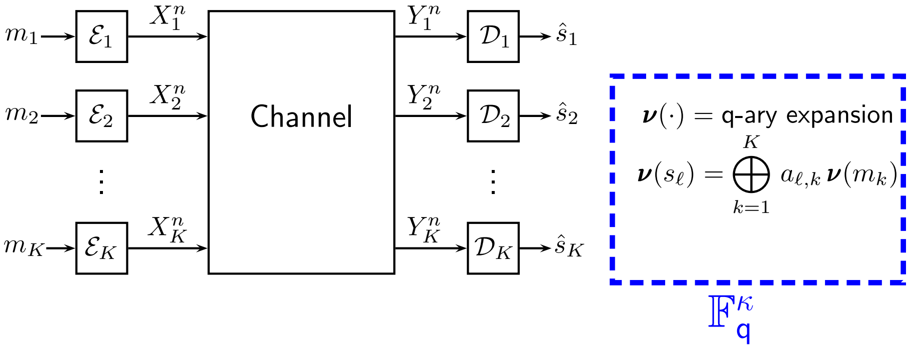

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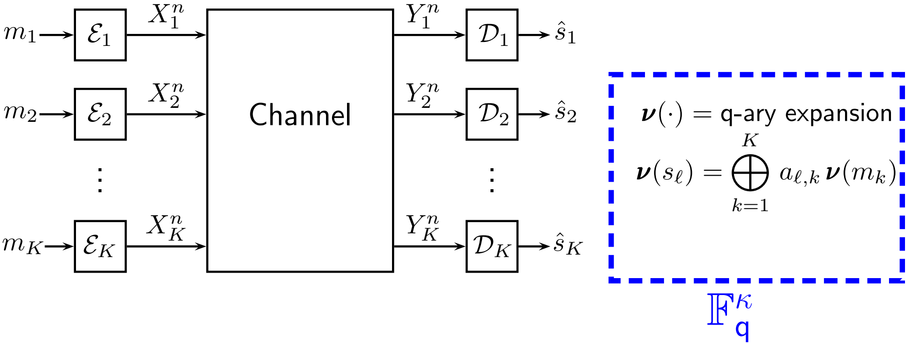

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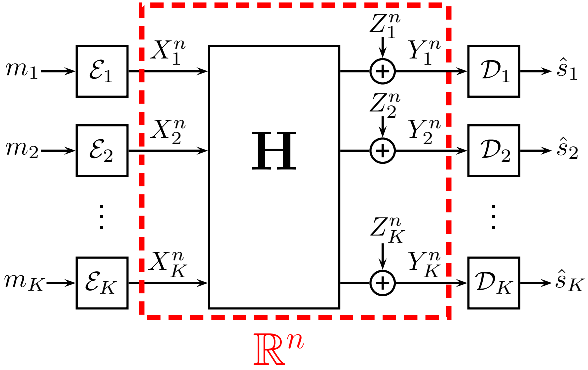


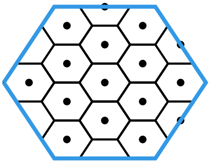
$\boldsymbol{\nu}(\cdot)$ = q-ary expansion

$$\boldsymbol{\nu}(s_\ell) = \bigoplus_{k=1}^{K} a_{\ell,k}\, \boldsymbol{\nu}(m_k)$$

$\mathbb{F}_{\mathsf{q}}^{\kappa}$

The Usual Approach

- Nested Lattice Code: Formed by taking all elements of Λ_{F} that lie in the fundamental Voronoi region of Λ_{C}.



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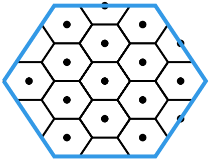

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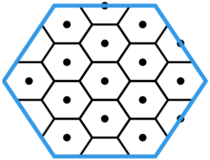

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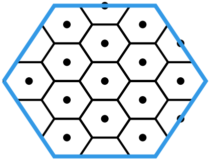

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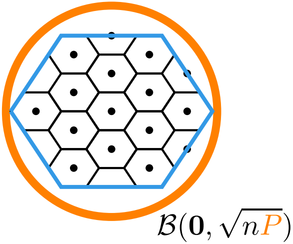

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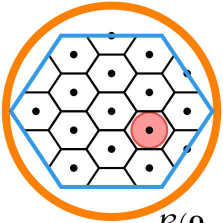

Nested Lattice Code Heuristics

- The Voronoi region $\mathcal{V}_\mathsf{C}$ of the coarse lattice Λ_C enforces the power constraint: If $\mathbf{x} \in \mathcal{V}_\mathsf{C}$, then $\frac{1}{n}\|\mathbf{x}\|^2 \leq P$.

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- The number of codewords in the nested lattice codebook $\Lambda_{\mathsf{F}} \cap \mathcal{V}_{\mathsf{C}}$ is

$$2^{nR} = \frac{\mathrm{Vol}(\mathcal{V}_{\mathsf{C}})}{\mathrm{Vol}(\mathcal{V}_{\mathsf{F}})} \approx \frac{\mathrm{Vol}\Big(\mathcal{B}\big(\mathbf{0}, \sqrt{nP}\big)\Big)}{\mathrm{Vol}\Big(\mathcal{B}\big(\mathbf{0}, \sqrt{n\sigma^2_{\mathsf{eff}}}\big)\Big)} = \left(\frac{P}{\sigma^2_{\mathsf{eff}}}\right)^{n/2}$$

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- Can show that the achievable rate satisfies $R > \frac{1}{2}\log\left(\frac{P}{\sigma^2_{\text{eff}}}\right) - \delta$.

Compute-and-Forward with Lattice Codes

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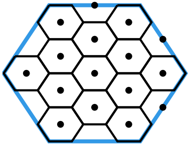
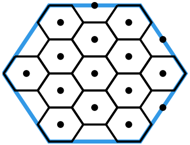
- We can decode $\mathbf{v}$ if $R < \frac{1}{2}\log\left(\frac{\text{SNR}}{\alpha^2 + \text{SNR}\|\alpha\mathbf{h} - \boldsymbol{a}\|^2}\right)$.
- Finding the best $\boldsymbol{a}$ corresponds to finding the shortest vector in the lattice $(\text{SNR}^{-1}\mathbf{I} + \mathbf{h}\mathbf{h}^{\mathsf{T}})^{-1/2}\mathbb{Z}^K$.

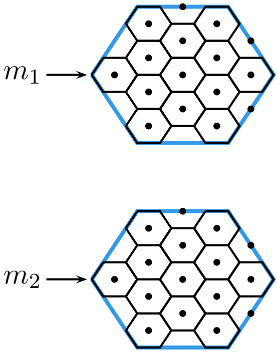
All users employ the same nested lattice code.





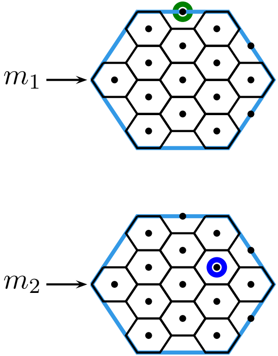
Compute-and-Forward: Illustration

Choose messages $m_k \in [2^{nR}]$.

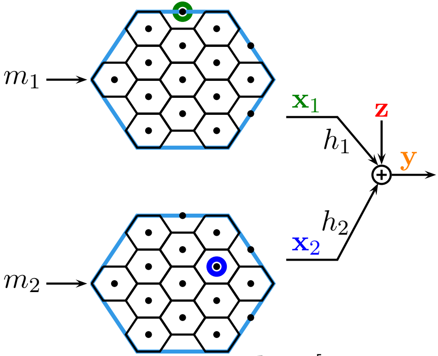


Compute-and-Forward: Illustration

Map m_k to lattice codeword $\mathbf{x}_k = \mathcal{E}_k(m_k)$.



Transmit lattice points over the channel.



$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\boldsymbol{a} = [\ 2 \quad 3\ ]$$

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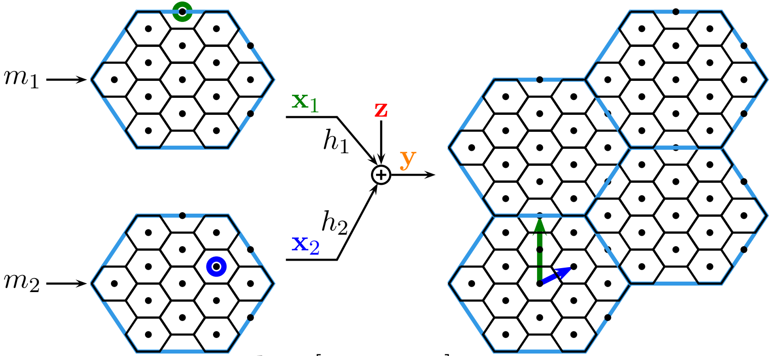

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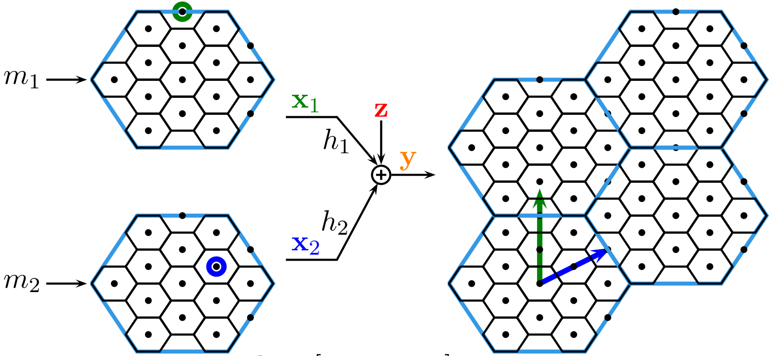
Compute-and-Forward: Illustration

Lattice codewords are scaled by channel coefficients.



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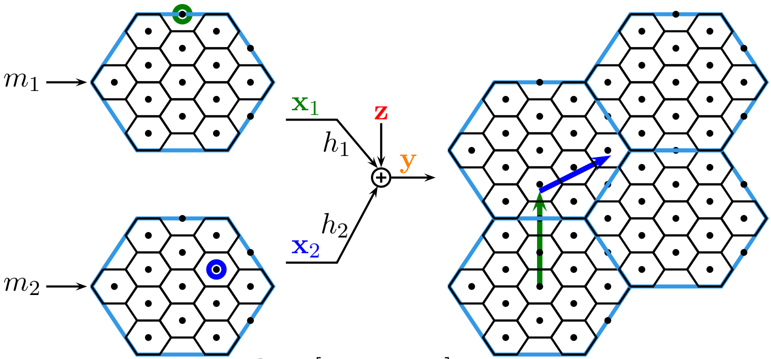
Compute-and-Forward: Illustration

Scaled codewords added together plus noise.



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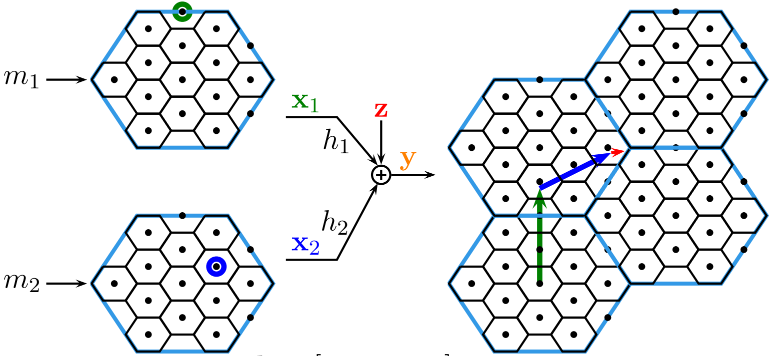

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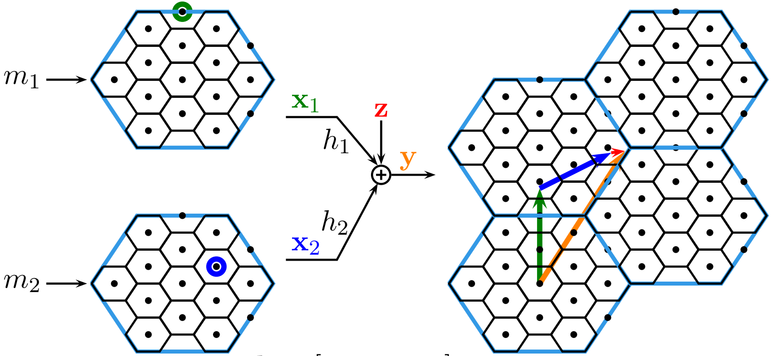
Compute-and-Forward: Illustration

Extra noise penalty for non-integer channel coefficients.



$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

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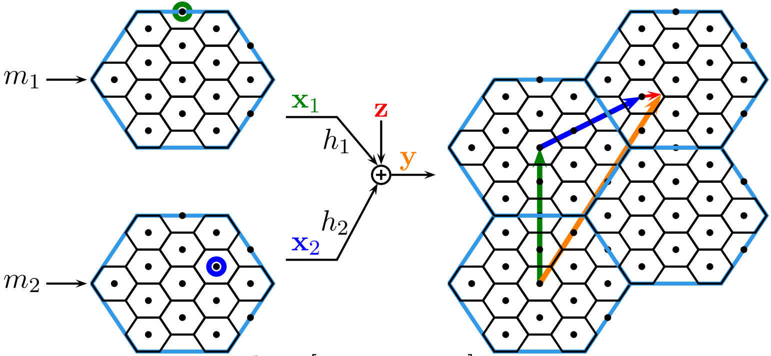
Effective noise: $1 + P\|\mathbf{h} - \mathbf{a}\|^2$

Compute-and-Forward: Illustration

Scale output by α to reduce non-integer noise penalty.



$$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$$

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Effective noise: $\alpha^2 + P\|\alpha\mathbf{h} - \boldsymbol{a}\|^2$

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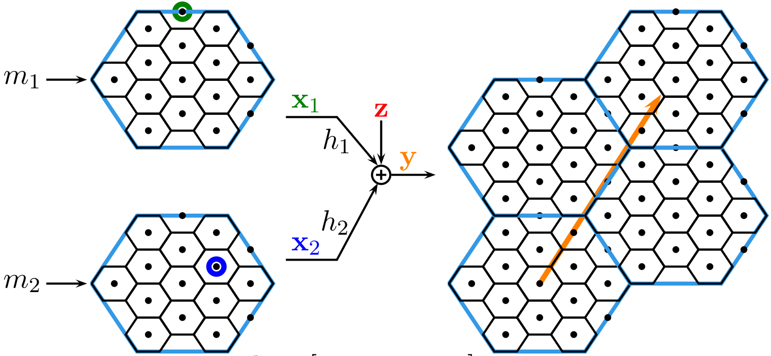

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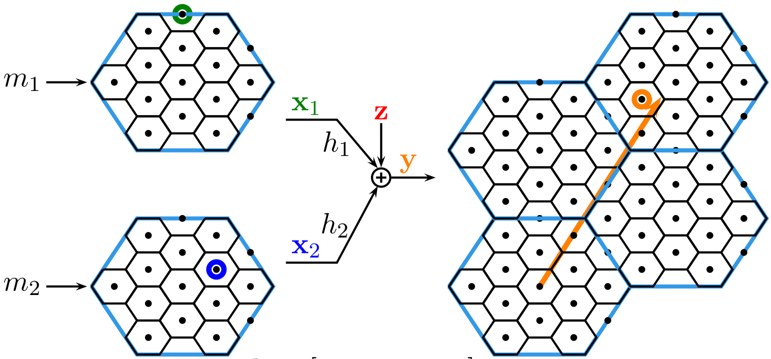
Compute-and-Forward: Illustration

Decode to the closest lattice point.



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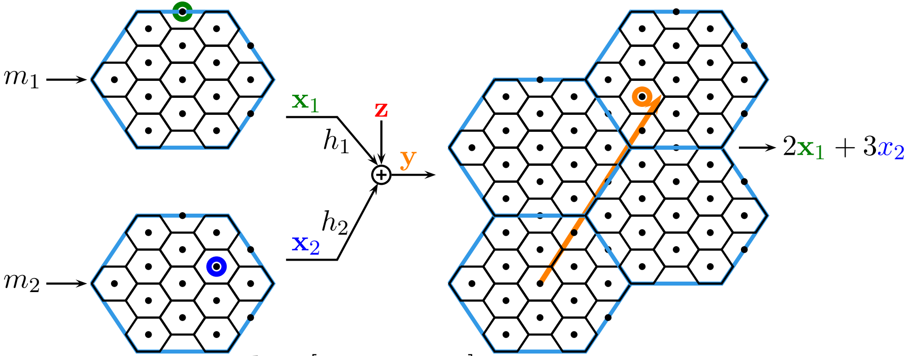
Compute-and-Forward: Illustration

Recover integer linear combination of the codewords.



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Compute-and-Forward: Achievable Rates

Theorem (Nazer-Gastpar '11)

A receiver can recover a linear combination with coefficient vector $\boldsymbol{a} \in \mathbb{Z}^K$ *over the channel vector* $\mathbf{h} \in \mathbb{R}^K$ *if* $R < R_{comp}(\mathbf{h}, \boldsymbol{a})$ *where*

$$R_{comp}(\mathbf{h}, \boldsymbol{a}) = \max_{\alpha \in \mathbb{R}} \frac{1}{2} \log^+ \left(\frac{P}{\alpha^2 + P \|\alpha \mathbf{h} - \boldsymbol{a}\|^2} \right).$$

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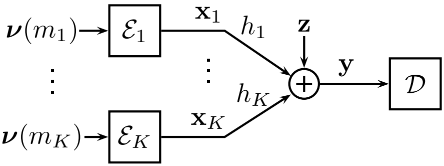

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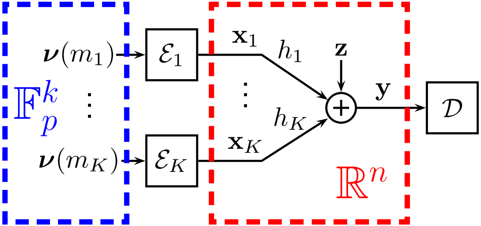

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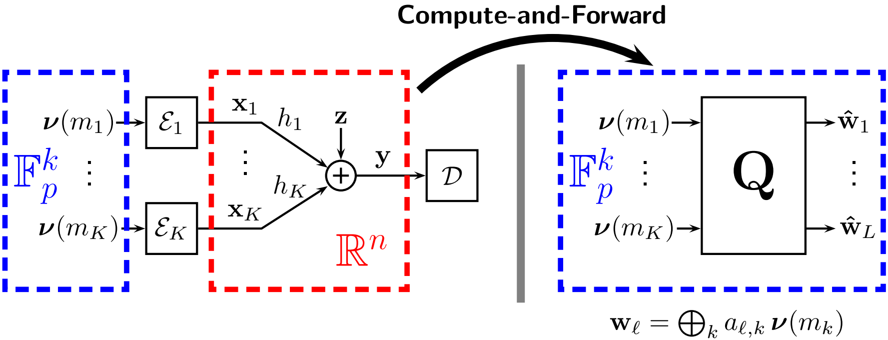

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Special Cases:

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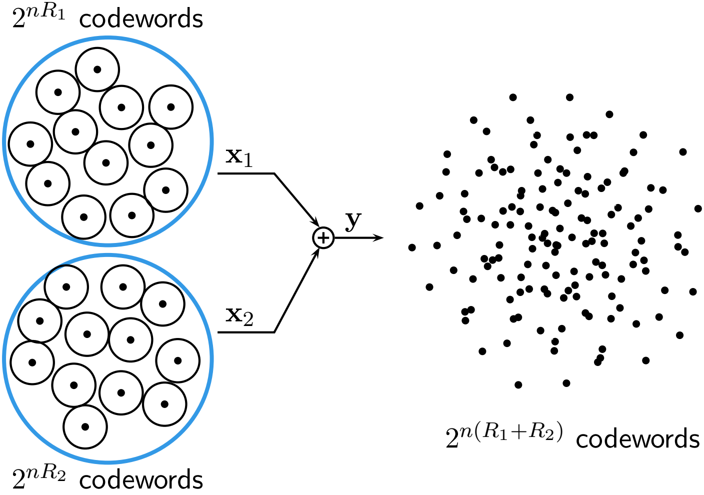
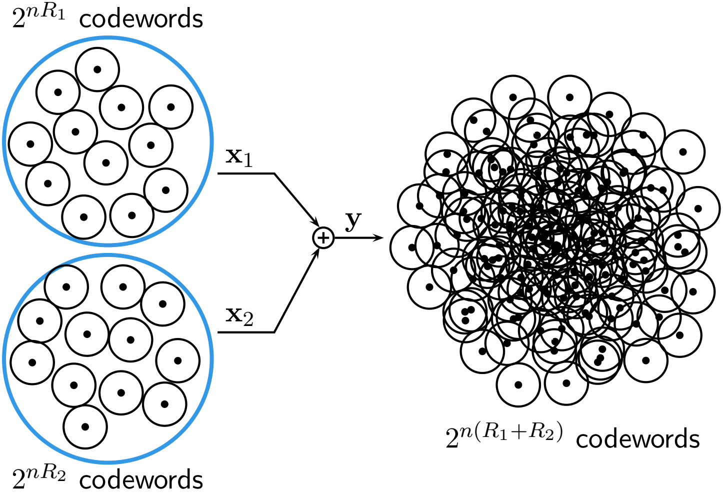
- Decode the k^{th} Message:
$$R_{\text{comp}}\Big(\mathbf{h}, [\underbrace{0 \ \cdots \ 0}_{k-1 \text{ zeros}} \ 1 \ 0 \ \cdots \ 0]^\mathsf{T}\Big) = \frac{1}{2} \log \left(1 + \frac{h_k^2 P}{1 + P \sum_{\ell \neq k} h_\ell^2} \right)$$

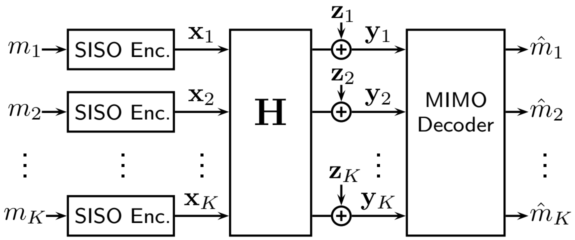
Random i.i.d. codes are not good for computation



Random i.i.d. codes are not good for computation



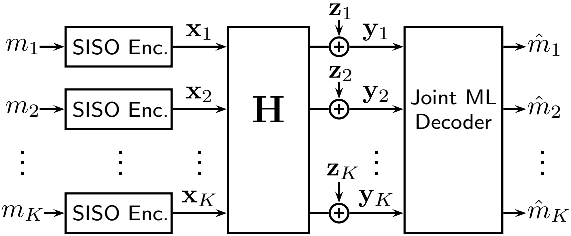
Application: MIMO Uplink Channel



Usual Assumptions:

- Each antenna carries an independent data stream $\mathbf{x}_\ell \in \mathbb{C}^n$ of rate R (e.g., V-BLAST setting, cellular uplink). $\mathbf{X} = [\mathbf{x}_1 \ \cdots \ \mathbf{x}_K]^\mathsf{T}$.
- Usual power constraint: $\|\mathbf{x}_\ell\|^2 \leq nP$.
- Channel model: $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z}$
- $\mathbf{Z}$ is elementwise i.i.d. $\mathcal{CN}(0,1)$.
- CSIR: Only the receiver knows channel realization $\mathbf{H} \in \mathbb{C}^{K \times K}$.

MIMO Uplink Channel: Joint ML Decoding

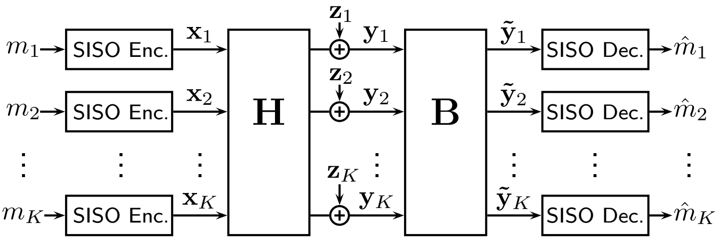


Joint Maximum Likelihood Decoding:

$$R_{\text{joint}}(\mathbf{H}) = \min_{\mathcal{S} \subseteq \{1,\ldots,K\}} \frac{1}{|\mathcal{S}|} \log\det\left(\mathbf{I} + P\ \mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^*\right)$$

- Corresponds to the (symmetric) outage capacity.
- Naive implementation has prohibitively high complexity.
- Of course, there are many clever ways to reduce the complexity!

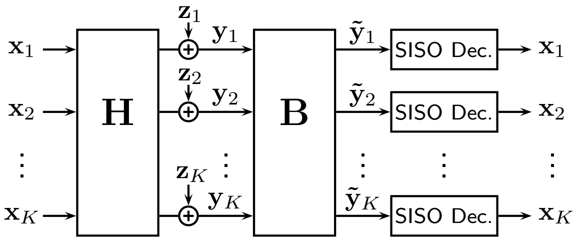
MIMO Uplink Channel: Zero-Forcing and Linear MMSE



Zero-Forcing and Linear MMSE Receivers:

- Project the received signal, $\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y}$ to eliminate interference between data streams.
- After projection, single-user decoders attempt to recover the individual data streams.
- Optimal $\mathbf{B}$ is the MMSE projection.

MIMO Uplink Channel: Zero-Forcing and Linear MMSE



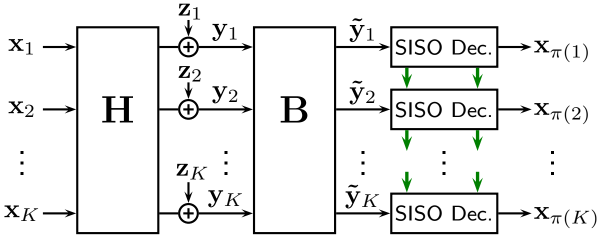
Zero-Forcing and Linear MMSE Receivers:

- The k^{th} SISO decoder tries to recover $\mathbf{x}_k$ from $\mathbf{b}_k^{\mathsf{T}}\mathbf{Y}$:

$$\mathsf{SINR}_{\mathsf{LMMSE},k}(\mathbf{H}) = \max_{\mathbf{b}_k} \frac{P\ \|\mathbf{b}_k^{\mathsf{T}}\mathbf{h}_k\|^2}{1 + P\ \sum_{\ell \neq k} \|\mathbf{b}_k^{\mathsf{T}}\mathbf{h}_\ell\|^2}$$

- Rate per user:

$$R_{\mathsf{LMMSE}}(\mathbf{H}) = \min_{k=1,\ldots,K} \log\left(1 + \mathsf{SINR}_{\mathsf{LMMSE},k}(\mathbf{H})\right)$$



Successive Interference Cancellation Receivers:

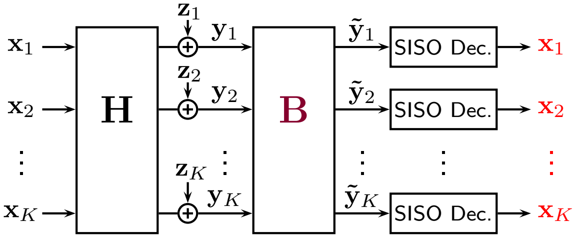
- Decode in order π. Cancel $\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(k-1)}$ from $\tilde{\mathbf{y}}_k$:

$$\mathsf{SINR}_{\mathsf{SIC},\pi(m)}(\mathbf{H}) = \max_{\mathbf{b}_m} \frac{P \, \|\mathbf{b}_k^{\mathsf{T}} \mathbf{h}_{\pi(k)}\|^2}{1 + \mathsf{SNR} \, \sum_{\ell=k+1}^{K} \|\mathbf{b}_k^{\mathsf{T}} \mathbf{h}_{\pi(\ell)}\|^2}$$

- Rate per user:

$$R_{\mathsf{V\text{-}BLAST\ II}}(\mathbf{H}) = \max_{\pi} \min_{k=1,\ldots,K} \log\left(1 + \mathsf{SINR}_{\mathsf{SIC},\pi(k)}(\mathbf{H})\right)$$

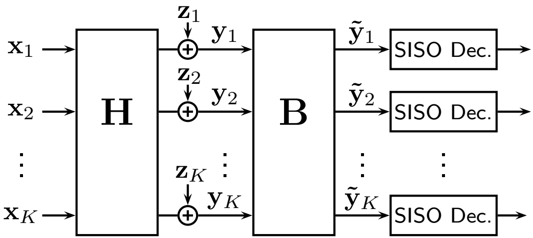
MIMO Uplink Channel: Integer-Forcing



What if we could decode something else?

- Zero-Forcing / LMMSE: First, eliminate interference.
  Then, decode individual data streams.

MIMO Uplink Channel: Integer-Forcing



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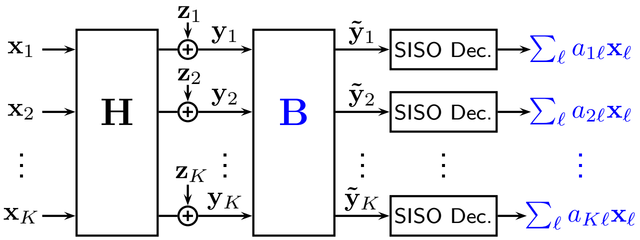
First, decode

MIMO Uplink Channel: Integer-Forcing



What if we could decode something else?

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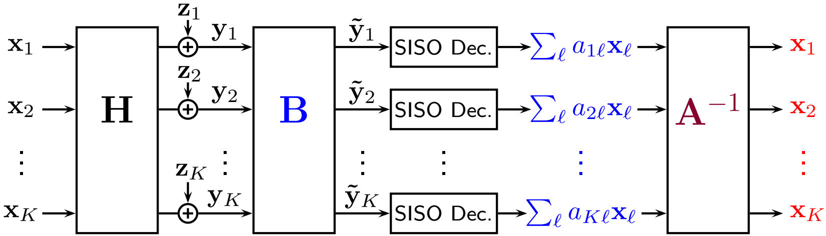
- Integer-Forcing: First, decode integer-linear combinations.

MIMO Uplink Channel: Integer-Forcing



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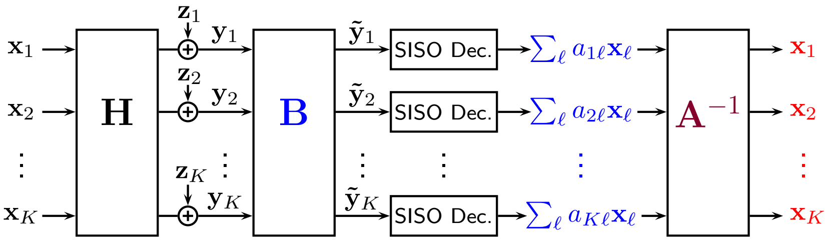

What if we could decode something else?

- Zero-Forcing / LMMSE: First, eliminate interference.
  Then, decode individual data streams.

- Integer-Forcing: First, decode integer-linear combinations.
  Then, eliminate interference.

- If the integer matrix $\mathbf{A}$ is full rank, we can successfully recover the individual data streams.

Integer-Forcing Linear Receivers:

- The k^{th} effective channel after projection is

$$\mathbf{b}_k^{\mathsf{T}}\mathbf{Y} = \mathbf{b}_k^{\mathsf{T}}\mathbf{H}\mathbf{X} + \mathbf{b}_k^{\mathsf{T}}\mathbf{Z}$$

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&= \underbrace{\sum_{\ell=1}^{K} a_{k\ell}\mathbf{x}_\ell^{\mathsf{T}}}_{\text{Codeword}} + \underbrace{(\mathbf{b}_k^{\mathsf{T}}\mathbf{H} - \mathbf{a}_k^{\mathsf{T}})\mathbf{X} + \mathbf{b}_k^{\mathsf{T}}\mathbf{Z}}_{\text{Effective Noise}}
\end{aligned}
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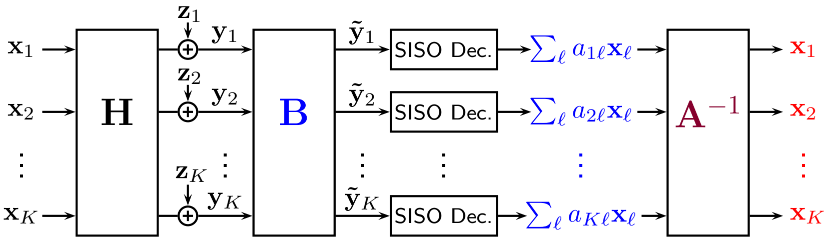
- The $a_{k\ell} \in \mathbb{Z}[j]$ are Gaussian integers and the codebook should be closed under integer-linear combinations.

- We are free to choose any full-rank integer-valued matrix $\mathbf{A}$.

MIMO Uplink Channel: Integer-Forcing



Integer-Forcing Linear Receivers: (Zhan-Nazer-Erez-Gastpar '14)

- The k^{th} SISO decoder tries to recover $\sum_\ell a_{k\ell}\mathbf{x}_\ell$ from $\mathbf{b}_k^\mathsf{T}\mathbf{Y}$:

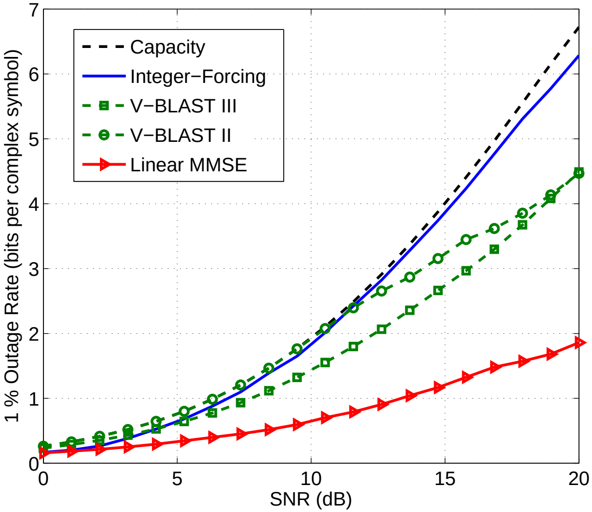
$$\mathsf{SINR}_{\mathsf{IF},k}(\mathbf{H},\mathbf{A}) = \max_{\mathbf{b}_k} \frac{P}{\|\mathbf{b}_k\|^2 + P\|\mathbf{b}_k^\mathsf{T}\mathbf{H} - \mathbf{a}_k^\mathsf{T}\|^2}$$

- Rate per user:

$$R_{\mathsf{IF}}(\mathbf{H}) = \max_{\mathbf{A}} \min_{k=1,\dots,K} \log^+\left(\mathsf{SINR}_{\mathsf{IF},k}(\mathbf{H},\mathbf{A})\right)$$

- Includes linear MMSE as a special case by setting $\mathbf{A} = \mathbf{I}$.

Comparison: Outage Rates



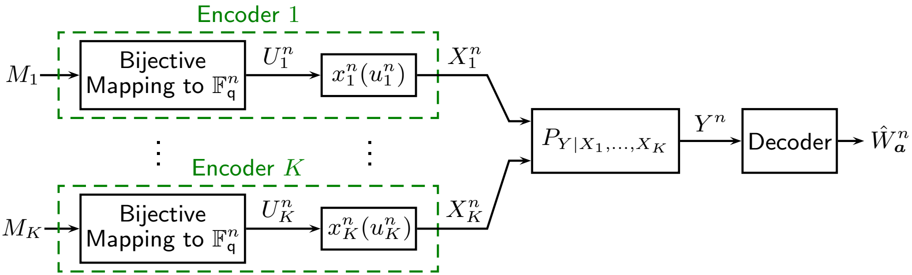
2 users, 2 receive antennas, Rayleigh fading, 1% outage.

Many Other Applications

- Distributed Source Coding: **Körner-Marton '79**, **Krithivasan-Pradhan '09,'11**, **Wagner '11**, **Tse-Maddah-Ali '10**
- Relaying: **Wilson-Narayanan-Pfister-Sprintson '10**, **Nam-Chung-Lee '10, '11**, **Goseling-Gastpar-Weber '11**, **Song-Devroye '13**, **Nokleby-Aazhang '12**
- Cellular Networks: **Sanderovich-Peleg-Shamai '11**, **Nazer-Sanderovich-Gastpar-Shamai '09**, **Hong-Caire '13**
- Distributed Dirty-Paper Coding: **Philosof-Zamir '09**, **Philosof-Zamir-Erez-Khisti '11**, **Wang '12**
- Joint Source-Channel Coding: **Kochman-Zamir '09**, **Nazer-Gastpar '07, '08**, **Soundararajan-Vishwanath '12**
- Physical-Layer Secrecy: **He-Yener '11, '14**, **Kashyap-Shashank-Thangaraj '12**

- For the rest of the talk, I will discuss our recent efforts to bring these lattice coding ideas into the joint typicality framework.
- This is joint work with Sung Hoon Lim, Chen Feng, Adriano Pastore, and Michael Gastpar.
- See arXiv for our June 2016 pre-print.

Compute-and-Forward: Beyond Gaussian Channels



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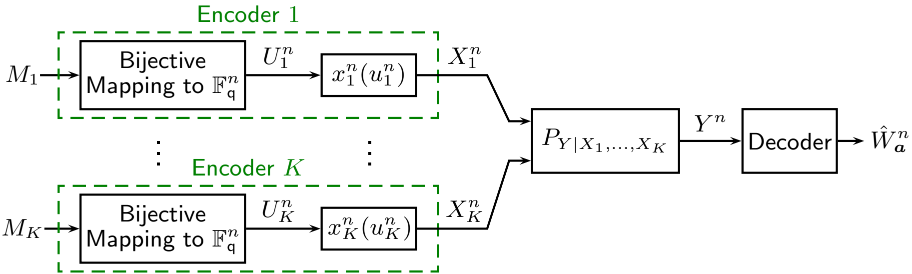
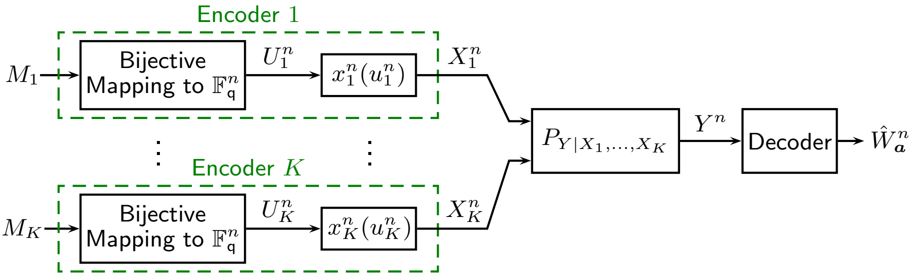
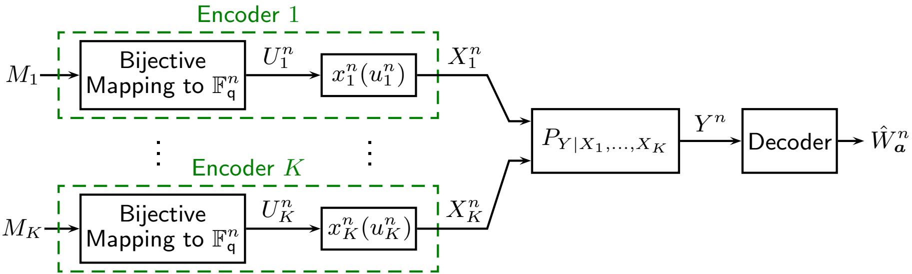


- Messages: $m_k \in [2^{nR_k}] \triangleq \{0, \ldots, 2^{nR_k} - 1\}$, $k = 1, \ldots, K$.



- Messages: $m_k \in [2^{nR_k}] \triangleq \{0, \ldots, 2^{nR_k} - 1\}$, $k = 1, \ldots, K$.
- Encoders: mappings $(u_k^n, x_k^n)(m_k) \in \mathbb{F}_q^n \times \mathcal{X}_k^n$, $k = 1, \ldots, K$ such that $u_k^n(m_k)$ is *bijective.*



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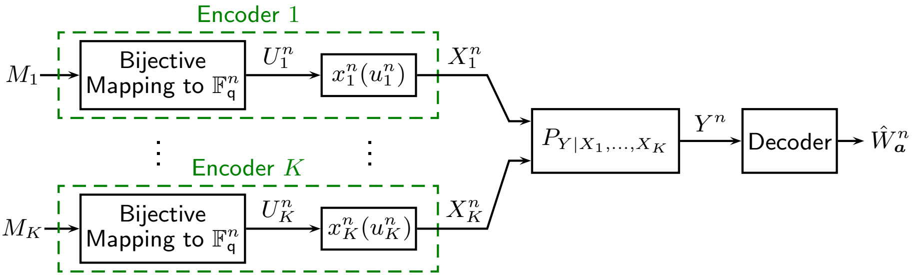
- Linear Combination: $w_{\boldsymbol{a}}^n \triangleq \bigoplus_k a_k u_k^n(m_k)$, $\boldsymbol{a} = [a_1 \ \cdots \ a_K] \in \mathbb{F}_\mathsf{q}^K$

Compute-and-Forward: Beyond Gaussian Channels



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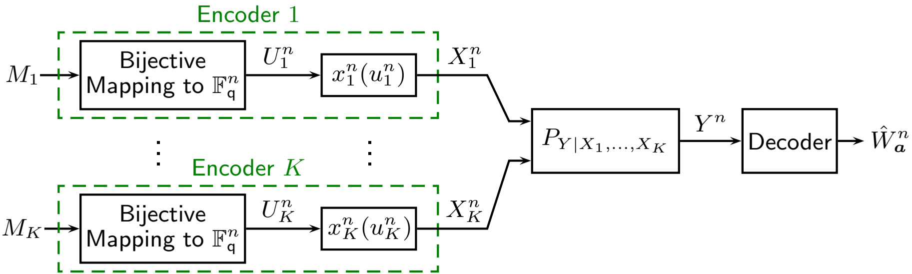
Compute-and-Forward: Beyond Gaussian Channels



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- Decoder: assigns an estimate $\hat{w}_{\boldsymbol{a}}^n \in \mathbb{F}_\mathsf{q}^n$ to each $y^n \in \mathcal{Y}^n$.
- Probability of Error: For uniformly distributed messages $M_1, \ldots, M_K$, want vanishing probability of error $\mathsf{P}\{\hat{W}_{\boldsymbol{a}}^n \neq W_{\boldsymbol{a}}^n\}$.

High-Level Intuition:

- Input Distribution: Want U_k^n to look typical with respect to pmf $p_{U_k}(u_k)$. There are $\approx 2^{nH(U_k)}$ typical sequences.

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- Decoder searches for sequences $\tilde{w}_{\boldsymbol{a}}^n$ that are jointly typical with Y^n. There are $\approx 2^{nH(W_{\boldsymbol{a}}|Y)}$ possible sequences. If only one such sequence is jointly typical, declare it as the estimate $\hat{W}_{\boldsymbol{a}}^n$ of the linear combination $W_{\boldsymbol{a}}^n = a_1 U_1^n \oplus \cdots \oplus a_K U_K^n$.

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- We can show that, for this decoding strategy, we can achieve any rate tuple $(R_1, \ldots, R_K)$ satisfying

$$R_k < H(U_k) - H(W_{\boldsymbol{a}}|Y).$$

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(Proof is actually a bit involved.)
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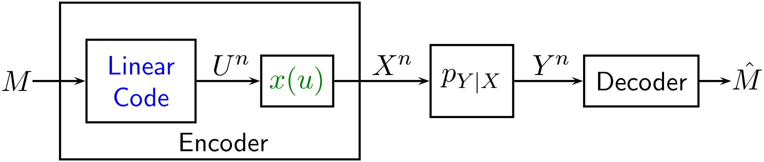
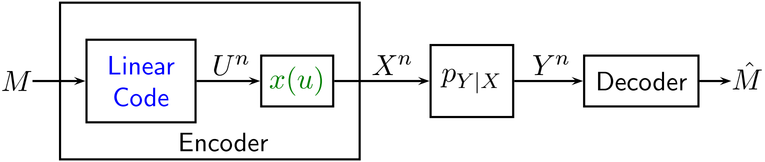
(Not a mutual information and can be negative.)



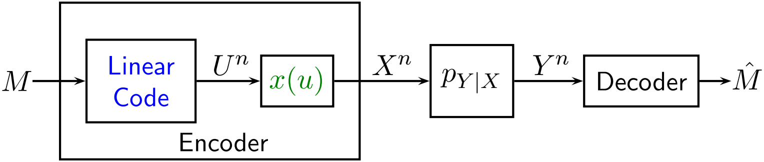
Code Construction:

Point-to-Point Channels: Linear Codes



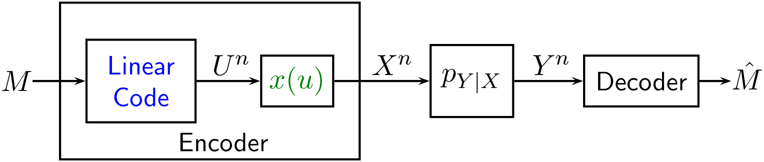
Code Construction:

- Pick a finite field $\mathbb{F}_{\mathsf{q}}$ and a symbol mapping $x : \mathbb{F}_{\mathsf{q}} \to \mathcal{X}$.



Code Construction:

- Pick a finite field $\mathbb{F}_{\mathsf{q}}$ and a symbol mapping $x : \mathbb{F}_{\mathsf{q}} \to \mathcal{X}$.
- Set $\kappa = nR/\log(\mathsf{q})$.



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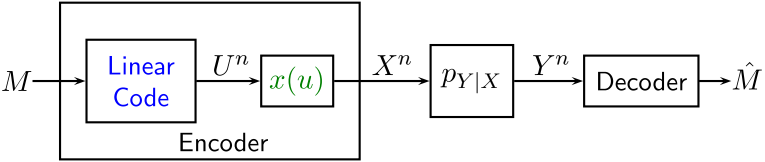

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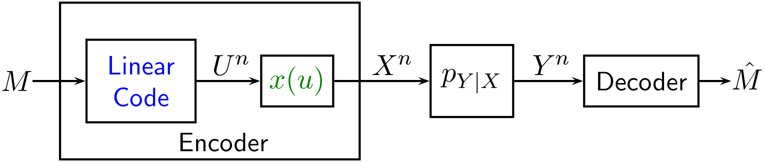

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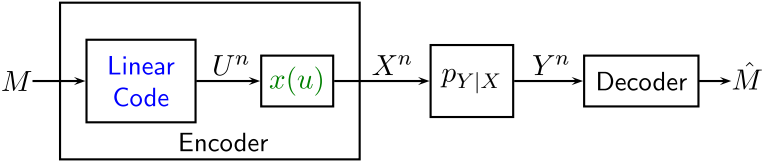

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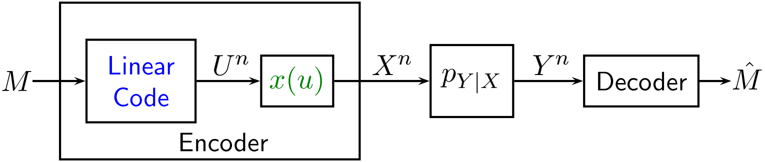

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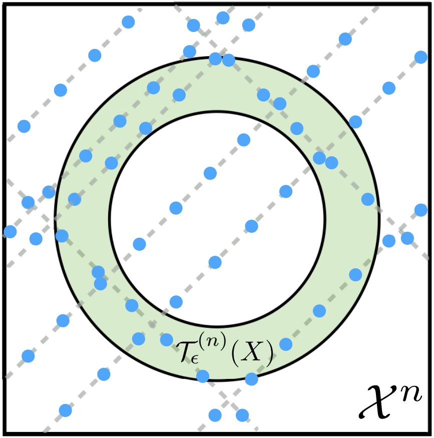
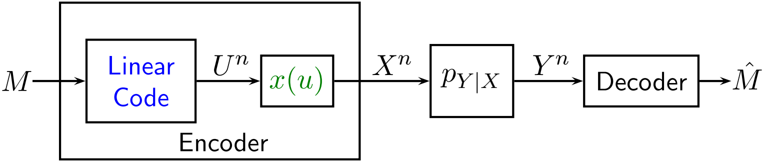
- Channel input at time i is $x_i(m) = x(u_i(m))$.



Random Linear Codes

- Codewords are pairwise independent of one another.
- Codewords are uniformly distributed over $\mathbb{F}_q^n$.

Point-to-Point Channels: Linear Codes



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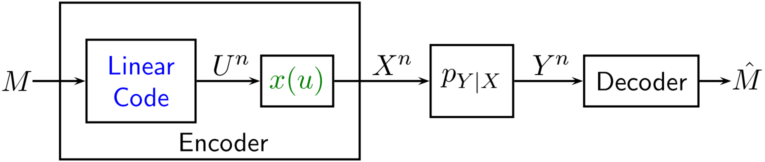

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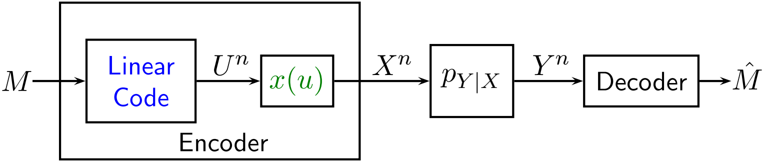

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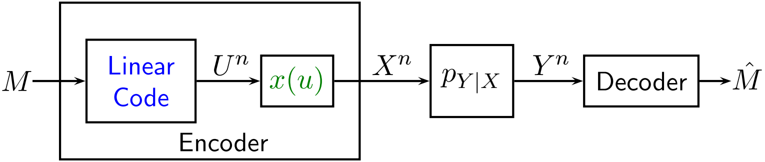

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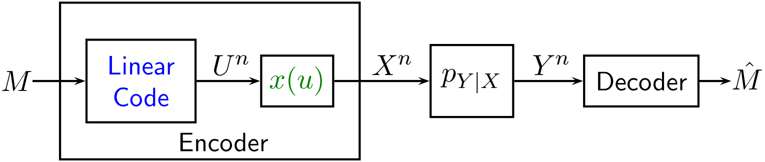

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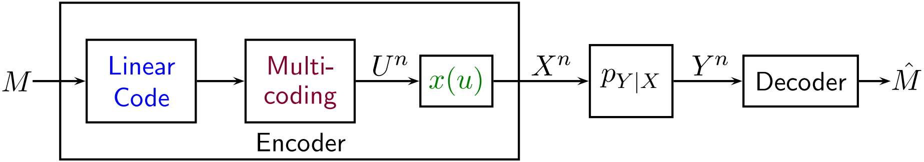
- **Padakandla-Pradhan '13:** It is possible to shape the input distribution using nested linear codes.
- Basic idea: Generate many codewords to represent one message. Search in this "bin" to find a codeword with the desired type, i.e., multicoding.

Point-to-Point Channels: Linear Codes + Multicoding



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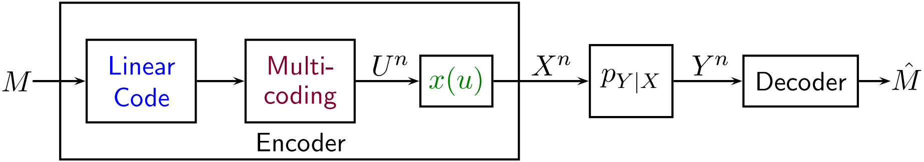

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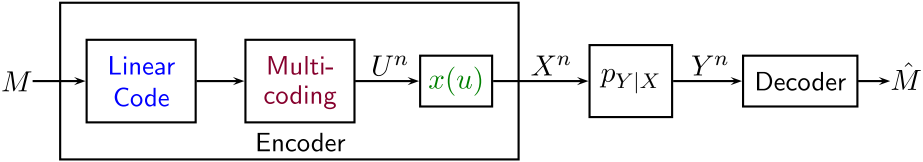

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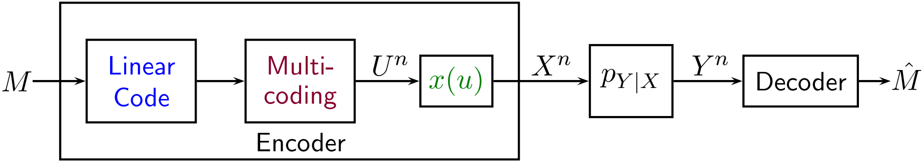

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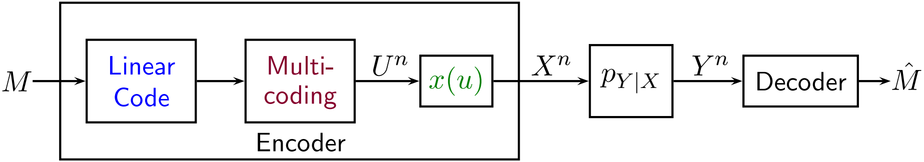

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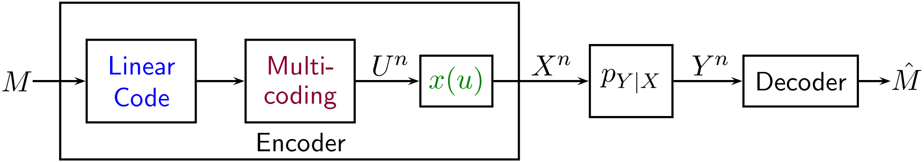

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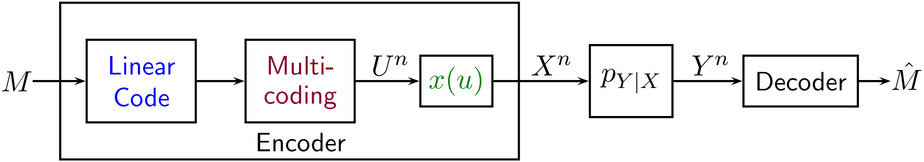
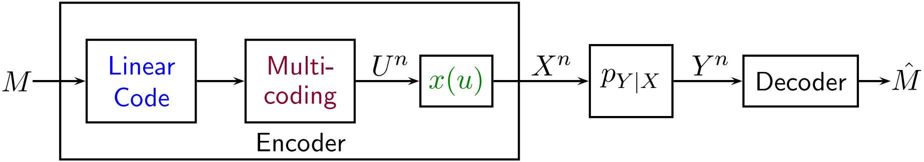

M
Linear Code
Multi-coding
U^n
$x(u)$
Encoder
X^n
$p_{Y|X}$
Y^n
Decoder
$\hat{M}$

Encoding:

Point-to-Point Channels: Linear Codes + Multicoding



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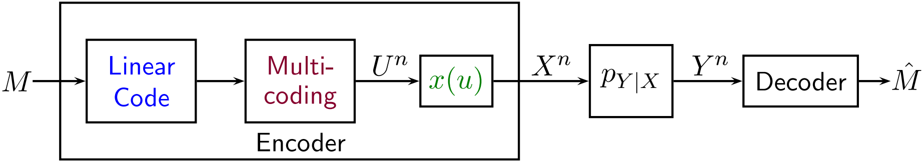

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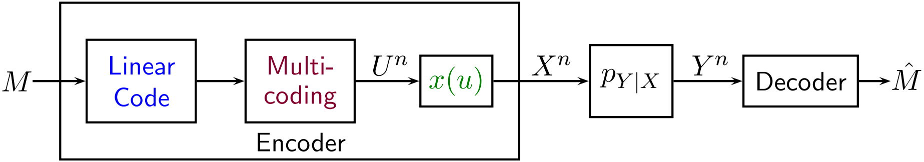

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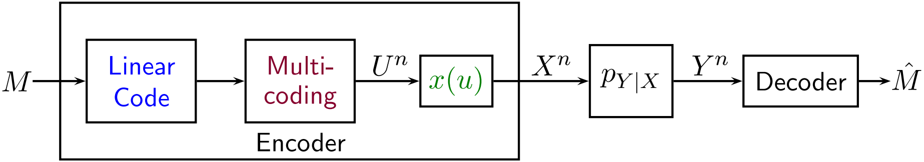

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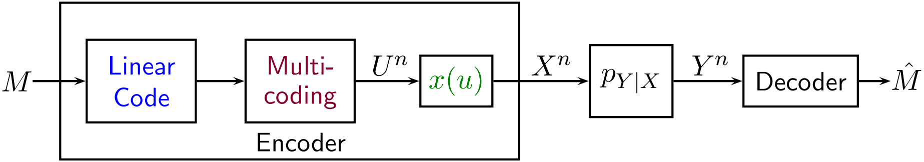

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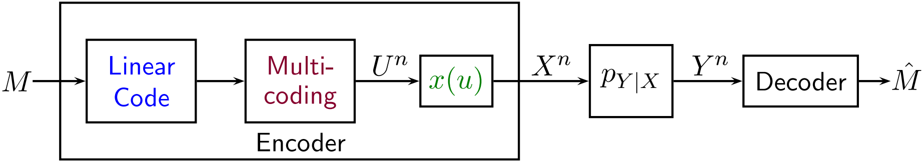

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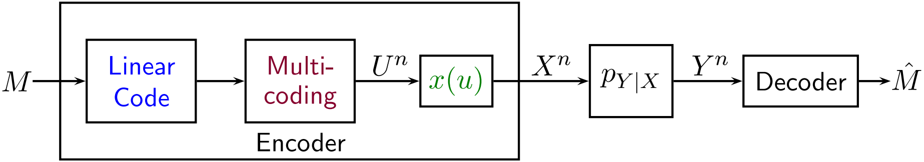

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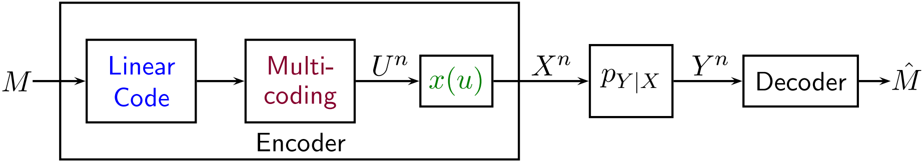
Point-to-Point Channels: Linear Codes + Multicoding



Theorem (Padakandla-Pradhan '13)

Any rate R satisfying

$$R < \max_{p(u),\, x(u)} I(U;Y)$$

is achievable. This is equal to the capacity if $\mathsf{q} \geq |\mathcal{X}|$.

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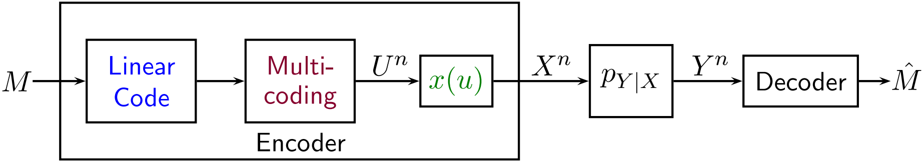

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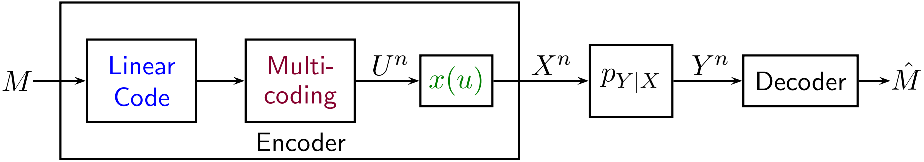

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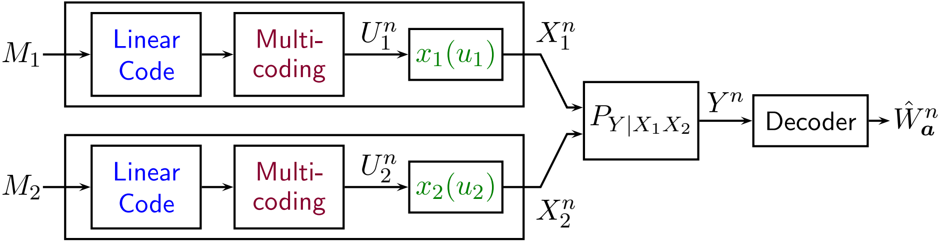
- Next, let's examine a two-transmitter, one-receiver "compute-and-forward" network.



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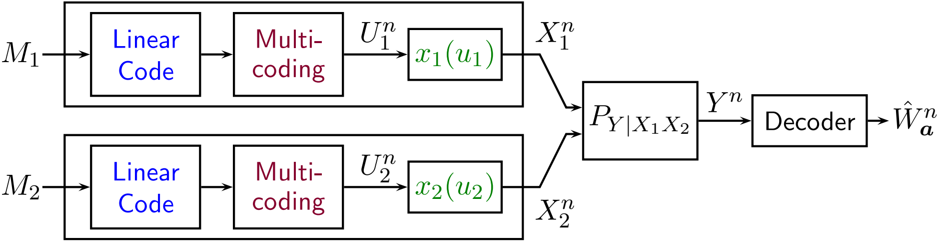
Nested Linear Coding Architecture



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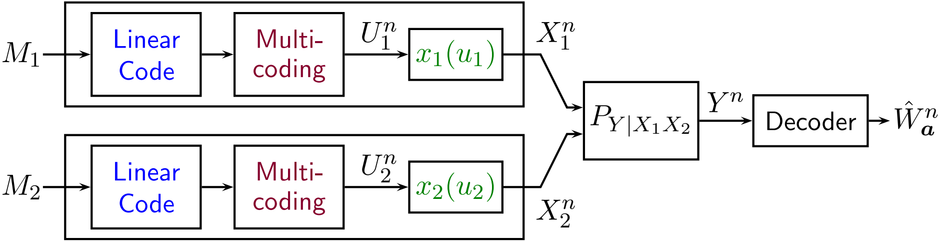

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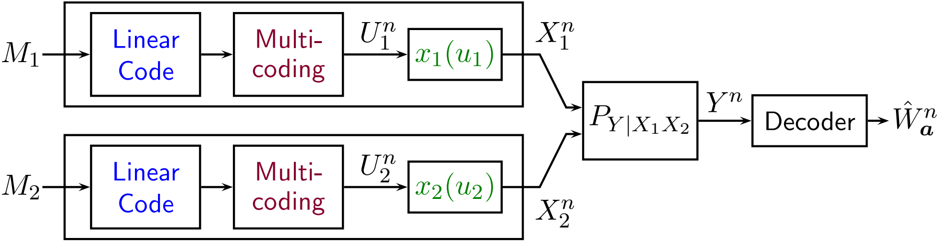

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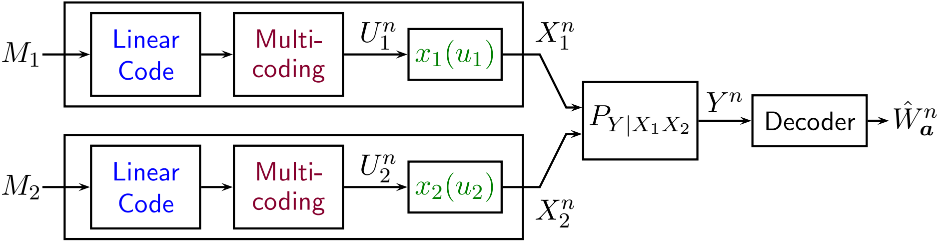
- Take q-ary expansions $\left[\boldsymbol{\nu}(m_1) \quad \boldsymbol{\nu}(l_1)\right] \in \mathbb{F}_{\mathsf{q}}^{\kappa}$

  $\left[\boldsymbol{\nu}(m_2)\ \boldsymbol{\nu}(l_2)\ \mathbf{0}\right] \in \mathbb{F}_{\mathsf{q}}^{\kappa}$ Zero-padding

Nested Linear Coding Architecture



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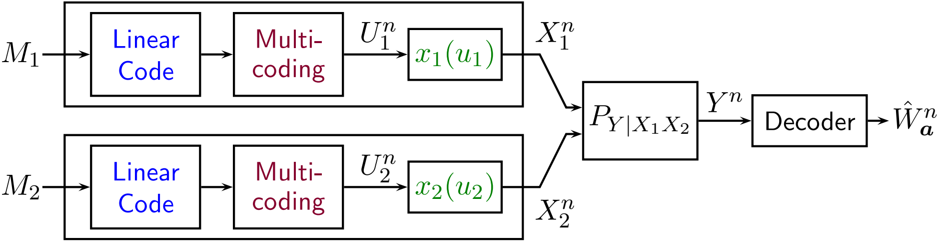

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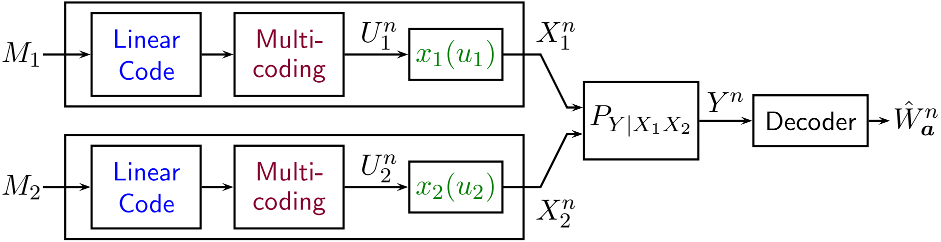
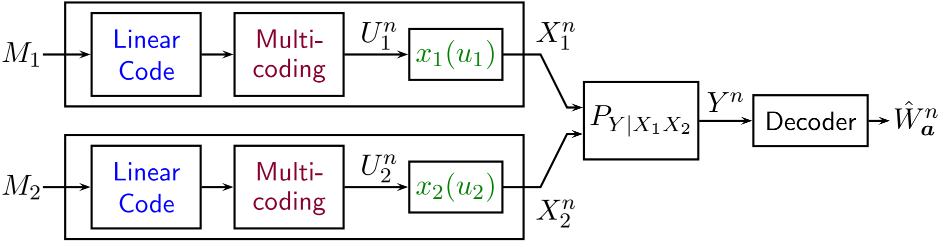
Nested Linear Coding Architecture



Encoding:

Nested Linear Coding Architecture



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- Fix pmfs $p(u_1)$, $p(u_2)$ and mappings $x_1(u_1)$, and $x_2(u_2)$.

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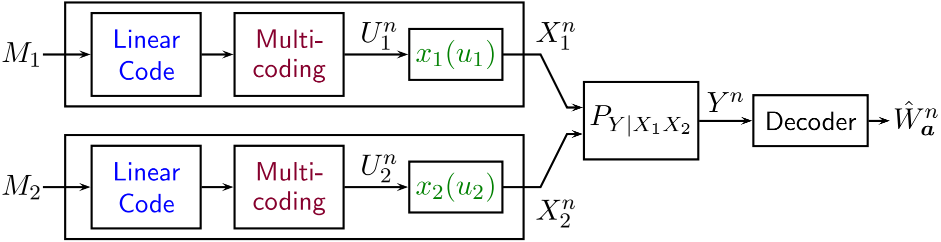

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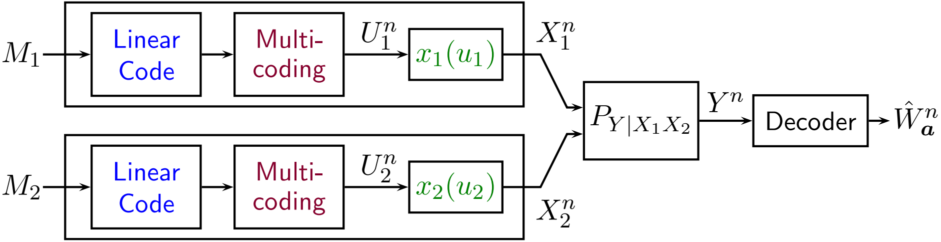

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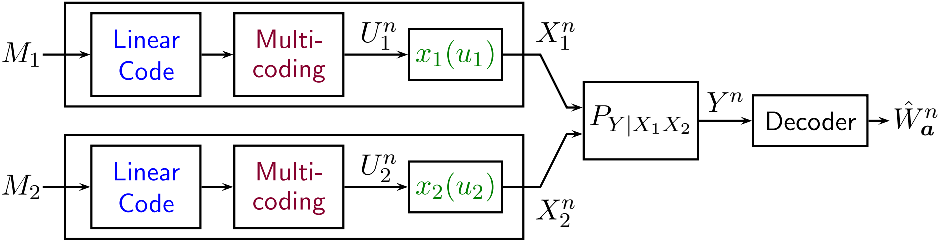

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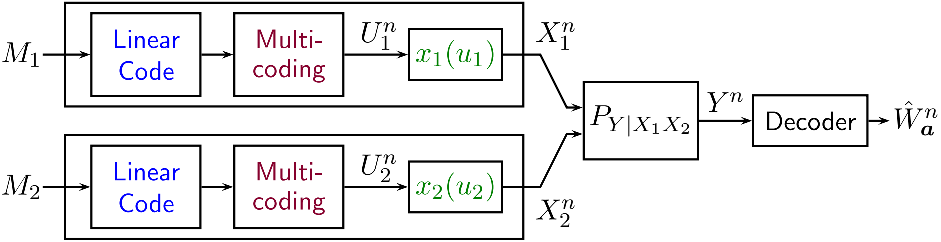
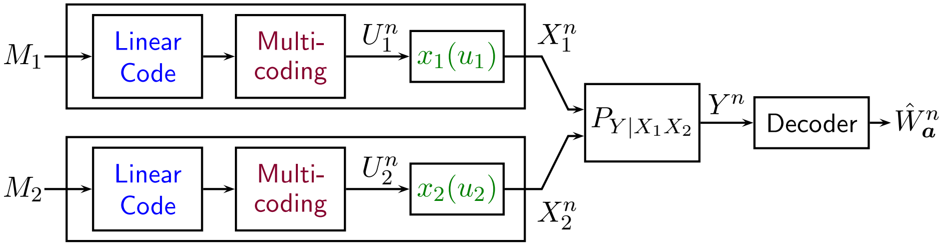


Computation Problem:

Nested Linear Coding Architecture



Computation Problem:

- For $m_k \in [2^{nR_k}]$, $l_k \in [2^{n\hat{R}_k}]$, we can express the linear combination of codewords as

$$\begin{aligned} w_{\boldsymbol{a}}^n &= a_1 u_1^n(m_1, l_1) \oplus a_2 u_2^n(m_2, l_2) \\ &= \big[a_1 \boldsymbol{\eta}(m_1, l_1) \oplus a_2 \boldsymbol{\eta}(m_2, l_2)\big] \mathsf{G} \oplus a_1 d_1^n \oplus a_2 d_2^n \\ &= \boldsymbol{\nu}(s_{\boldsymbol{a}}) \mathsf{G} \oplus a_1 d_1^n \oplus a_2 d_2^n \end{aligned}$$

where $s_{\boldsymbol{a}} \in [2^{n \max\{R_1+\hat{R}_1, R_2+\hat{R}_2\}}]$.

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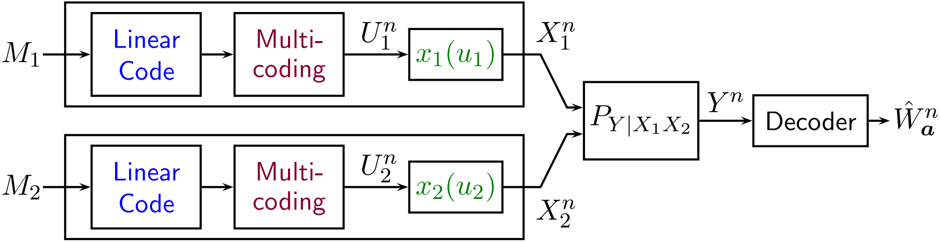

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- Let $\epsilon' < \epsilon$.

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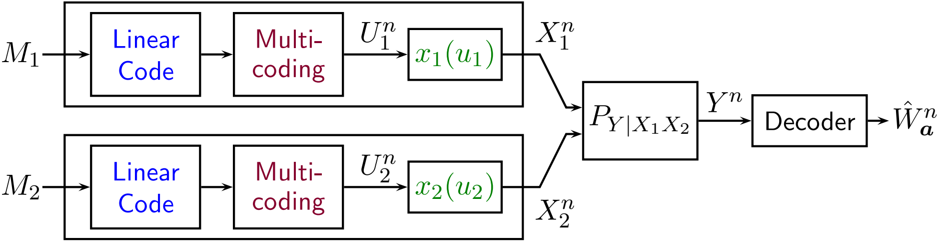

Decoding:

- Let $\epsilon' < \epsilon$.
- Search for a unique index $s_{\boldsymbol{a}} \in [2^{n\max\{R_1+\hat{R}_1, R_2+\hat{R}_2\}}]$ such that

$$(u_1^n(m_1, l_1), u_2^n(m_2, l_2), y^n) \in \mathcal{T}_\epsilon^{(n)}(U_1, U_2, Y),$$

for some $(m_1, l_1, m_2, l_2) \in [2^{nR_1}] \times [2^{n\hat{R}_1}] \times [2^{nR_2}] \times [2^{n\hat{R}_2}]$ such that

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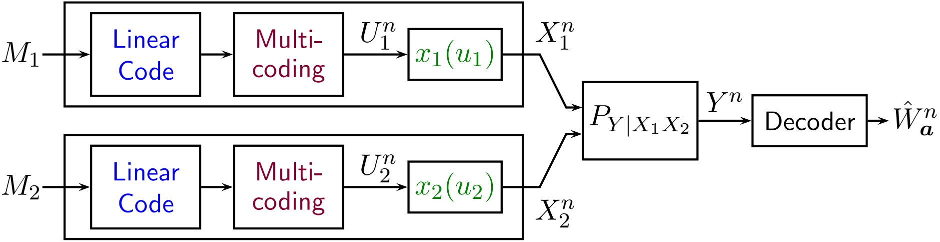

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- If there is no such index, or more than one, the decoder declares an error.

An error occurs only if one or more of the following events occur,

- For some message, we cannot find a typical linear codeword:

$$\mathcal{E}_1 = \{U_k^n(m_k, l_k) \notin \mathcal{T}_{\epsilon'}^{(n)} \text{ for all } l_k, \text{ for some } m_k, k = 1, 2\}.$$

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Then, by the union of events bound,

$$\mathsf{P}\{\hat{W}_{\boldsymbol{a}}^n \neq W_{\boldsymbol{a}}^n\} \leq \mathsf{P}\{\mathcal{E}_1\} + \mathsf{P}\{\mathcal{E}_2 \cap \mathcal{E}_1^c\} + \mathsf{P}\{\mathcal{E}_3 \cap \mathcal{E}_1^c\}.$$

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- Proof just requires second moment method.

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- Our proof handles these statistical dependencies by breaking up the possible error events according to the underlying rank of the selected linear codewords. (Markov Lemma for Nested Linear Codes.)
- Prior work by **Padakandla-Pradhan '13** developed a bound that also requires $\hat{R}_k < D(p_{U_k} \| p_{\mathsf{q}}) + 3\delta(\epsilon)$.

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$$\mathsf{P}\{\mathcal{E}_3 \cap \mathcal{E}_1^c\} \leq \mathsf{P}\Big\{(W_{\boldsymbol{a}}^n(s_{\boldsymbol{a}}), Y^n) \in \mathcal{T}_\epsilon^{(n)},\ \mathcal{E}_1^c,\ s_{\boldsymbol{a}} \neq S_{\boldsymbol{a}}\Big\}$$

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  2. "Multiple-Access Decoding" Bound

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- Can show that $\lim_{n\to\infty} \mathsf{P}\{\mathcal{E}_3 \cap \mathcal{E}_1^c\} = 0$ if

$$R_1 < I_{\mathsf{CF},1}(\boldsymbol{a}) \triangleq H(U_1) - H(W_{\boldsymbol{a}}|Y)$$
$$R_2 < I_{\mathsf{CF},2}(\boldsymbol{a}) \triangleq H(U_2) - H(W_{\boldsymbol{a}}|Y),$$

which matches our intuition from earlier.

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OR

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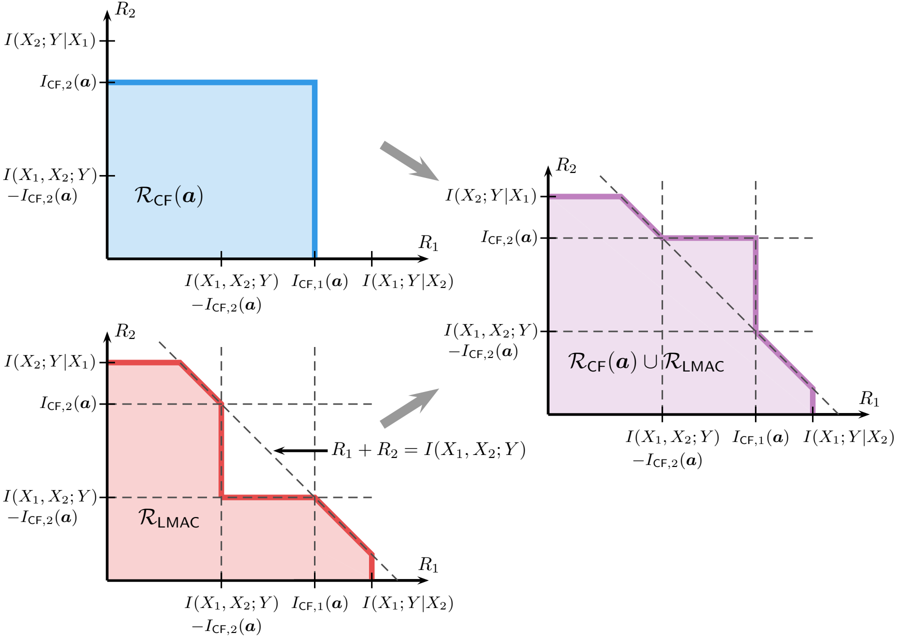
- The $I_{\mathsf{CF},2}(\boldsymbol{b})$ term plays a key role in handling the dependencies between competing pairs of linear codewords.

Rate Region



R_2
$I(X_2;Y|X_1)$
$I_{\mathsf{CF},2}(\boldsymbol{a})$
$I(X_1,X_2;Y)$
$-I_{\mathsf{CF},2}(\boldsymbol{a})$
$\mathcal{R}_{\mathsf{CF}}(\boldsymbol{a})$
R_1
$I(X_1,X_2;Y)$
$-I_{\mathsf{CF},2}(\boldsymbol{a})$
$I_{\mathsf{CF},1}(\boldsymbol{a})$
$I(X_1;Y|X_2)$
R_2
$I(X_2;Y|X_1)$
$I_{\mathsf{CF},2}(\boldsymbol{a})$
$I(X_1,X_2;Y)$
$-I_{\mathsf{CF},2}(\boldsymbol{a})$
$\mathcal{R}_{\mathsf{LMAC}}$
$R_1 + R_2 = I(X_1,X_2;Y)$
R_1
$I(X_1,X_2;Y)$
$-I_{\mathsf{CF},2}(\boldsymbol{a})$
$I_{\mathsf{CF},1}(\boldsymbol{a})$
$I(X_1;Y|X_2)$
R_2
$I(X_2;Y|X_1)$
$I_{\mathsf{CF},2}(\boldsymbol{a})$
$I(X_1,X_2;Y)$
$-I_{\mathsf{CF},2}(\boldsymbol{a})$
$\mathcal{R}_{\mathsf{CF}}(\boldsymbol{a}) \cup \mathcal{R}_{\mathsf{LMAC}}$
R_1
$I(X_1,X_2;Y)$
$-I_{\mathsf{CF},2}(\boldsymbol{a})$
$I_{\mathsf{CF},1}(\boldsymbol{a})$
$I(X_1;Y|X_2)$

Concluding Remarks

- First steps towards bringing algebraic network information theory back into the realm of joint typicality.
- Joint decoding rate region for compute-and-forward that outperforms parallel and successive decoding.