Erasure codes for distributed storage and related problems

Alexander Barg

University of Maryland, College Park

NASIT, July 2019

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A Data Center

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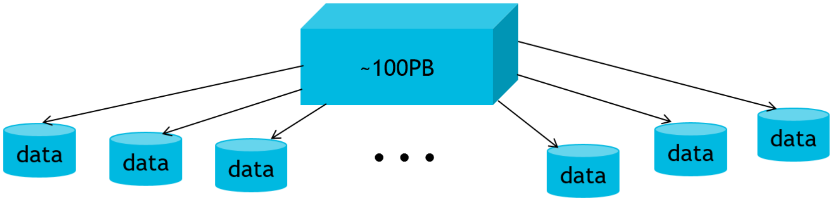
Data Center Equipment

- Power backup provided by 1.75 generator capacity being fed from 4000 gallon diesel fuel cells
- N+1 redundant HVAC system using multiple units with backup units standing by
  Redundant cable routing system
- Anti-static environment
- Triple power feeds, UPS
- N+1 cooling units
- N+1 UPS Systems
- Rated to withstand Class 3 – 4 hurricane strength
- NOC (Network Operations Center) staffed with senior system technicians 24 x 7 x 365

Network

- 100 Gigabit Ethernet Core
- All switch based internal network
- 100% network uptime guarantee
- Current generation Terrathon class routers with terrabit routing capacity.
- Backbone connections from Level 3, MCI, and Time Warner
- Over 300 direct private peering relationships to ensure the best delivery of your traffic
- Over 77 Gbps in total bandwidth capacity
- Servers are placed in segmented network
- Up to10000 Mbps port, fully burstable

Motivation: Distributed Storage Systems (DSS)



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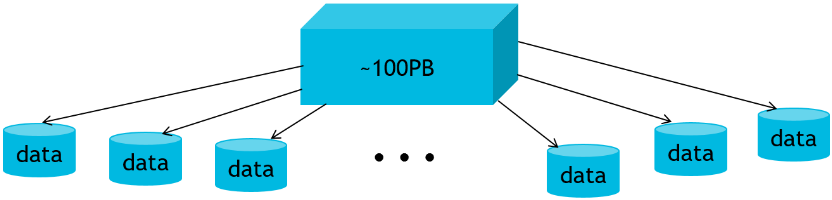

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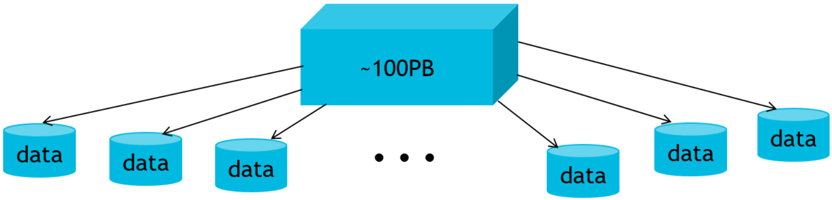
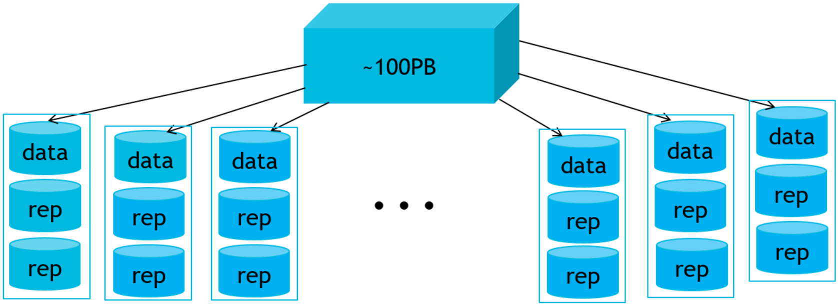


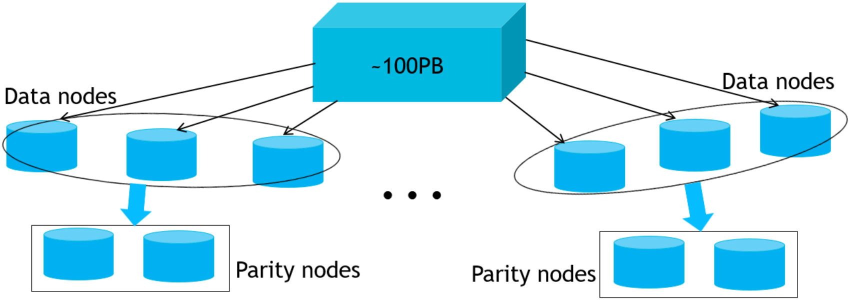
- DSS spread data across thousands of storage nodes
- Individual storage nodes fail frequently
- To protect the data we rely on erasure codes

Replication: large storage overhead



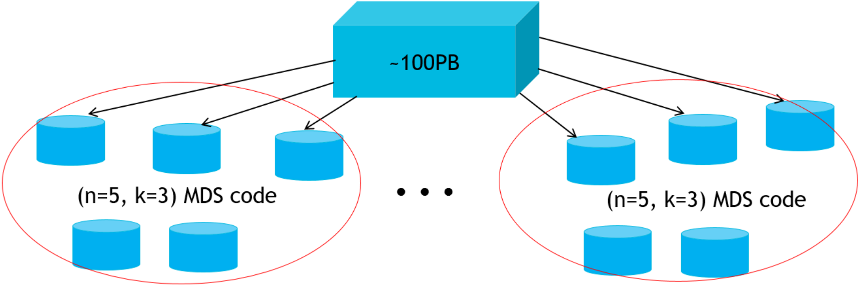
Can tolerate any 2 node failures
Storage overhead $= \quad 3\times$

MDS codes: Optimal storage efficiency



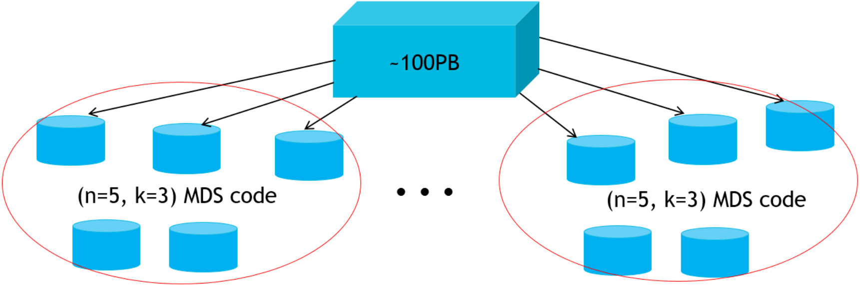
Add 2 parity nodes to every 3 data nodes
Form an $(n = 5, k = 3)$ MDS code

MDS codes: Optimal storage efficiency



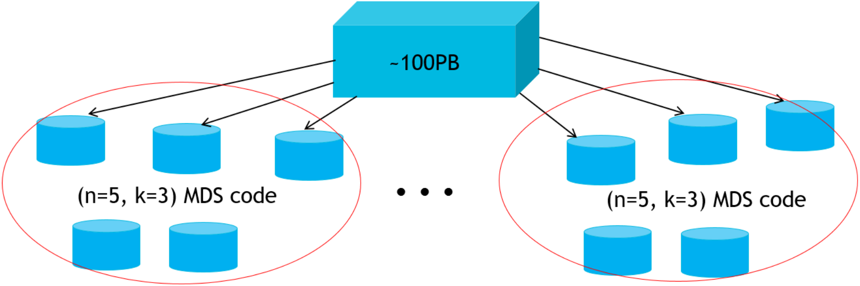
Can tolerate any 2 node failures
Storage overhead $= \quad 1.67\times$

MDS codes: Optimal storage efficiency



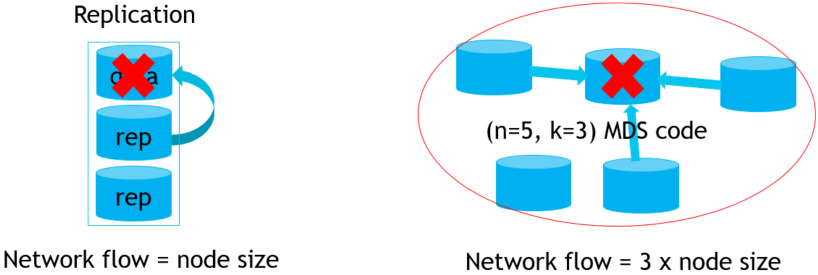
Block-based system model: Data blocks are encoded independently

MDS codes: Optimal storage efficiency



Each node stores l symbols of the block

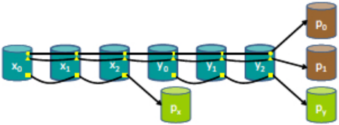
Repair Bandwidth



MDS code uses much more network bandwidth during data regeneration

Repair degree

A combination of local and global parity checks for single and multiple nodes failures



(**C. Huang** at al., Erasure coding in Windows Azure Storage, USENIX Conf. 2012)

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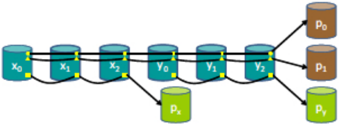

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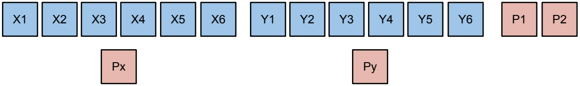
Other similar constructions (Windows Azure code)



Pyramid codes (**C. Huang** et al., 2007)

Main ideas

REPAIR DEGREE

- P. GOPALAN, C. HUANG, H. SIMITCI, AND S. YEKHANIN, *On the locality of codeword symbols*, T-IT, 2012
  - C. HUANG, M. CHEN, AND J. LI, Pyramid codes, Proc. 6th IEEE Int. Symp. NCA, 2007

REPAIR BANDWIDTH

- A.G. DIMAKIS, P.B. GODDFREY, Y. WU, M.J. WAINWRIGHT, AND K. RAMCHANDRAN, *Network coding for distributed storage*, T-IT, 2010

REPAIR OF RS CODES

- V. GURUSWAMI AND M. WOOTTERS, *Repairing Reed-Solomon codes*, T-IT, 2017

Different versions of the repair problem

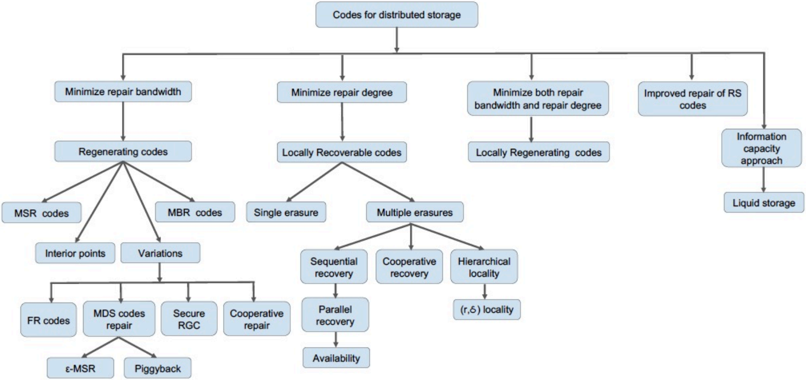


Figure 1 from "Erasure coding for distributed storage: An overview"

S.B. Balaji et al., arXiv:1806.04437

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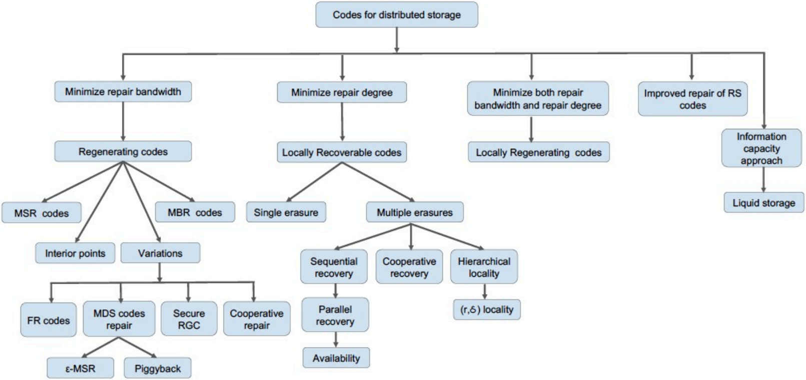

Figure 1 from "Erasure coding for distributed storage: An overview"

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Current literature count is in the hundreds

Outline of the tutorial

1. Repair degree
    1.1 Locally recoverable codes and their parameters
    1.2 Constructions for low repair degree
    1.3 Related problems:
        1.3.1 availability,
        1.3.2 repair of several nodes,
        1.3.3 hierarchical locality,
        1.3.4 sequential recovery
    1.4 Open questions: MR codes, maximum length of optimal LRC codes
2. Repair bandwidth
    2.1 Information flow graphs and the cutset bound
        2.1.1 Regenerating codes
        2.1.2 The MSR case
        2.1.3 Construction of MSR codes
        2.1.4 Multiple erasures: Centralized and cooperative repair
        2.1.5 Optimal access and subpacketization
        2.1.6 Repair of Reed-Solomon codes
    2.2 Heterogeneous storage
3. Looking forward: Storage networks, random failures

Correcting one erasure

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- Consider a binary code that encodes 3-bit messages into 7-bit code blocks:

$$(110) \times \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$
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Any 4 bits identify the codeword uniquely (in other words, the code can correct up to 3 erasures)

- At the same time, a single erasure can be recovered from 2 bits: For instance

$$(0 \quad \cancel{1} \quad 1 \quad 1 \quad 1 \quad 0 \quad 0)$$

$$c_2 = c_4 \oplus c_6$$

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- Coordinates of the codeword $C = (C_1, \dots, C_n) \in \mathcal{C}$ are called nodes; If a node C_i is erased (failed), we look at the other (surviving) coordinates, called helper nodes, and download their values, or functions of these values.

Reminder: Finite fields

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- Consider the set of integers modulo 13

$$\mathbb{F}_{13} = \{0, 1, 2, \dots, 12\}$$

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- Polynomials and linear algebra work over $\mathbb{F}_q$ in many ways as over $\mathbb{R}$

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(a_0, a_1, a_2, a_3)

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$$(a_0, a_1, a_2, a_3) \rightarrow f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \rightarrow f(x) = (f(P_1), f(P_2), ..., f(P_n))$$

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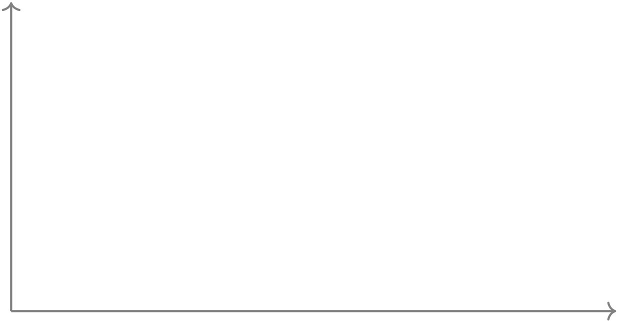

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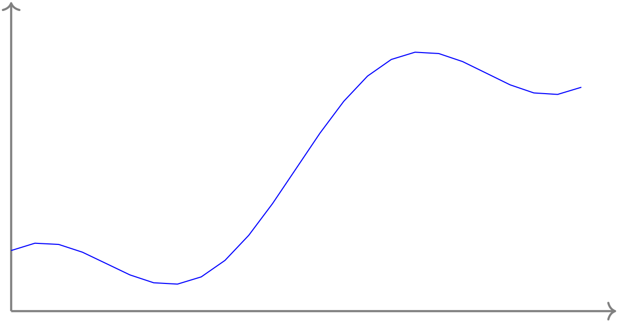

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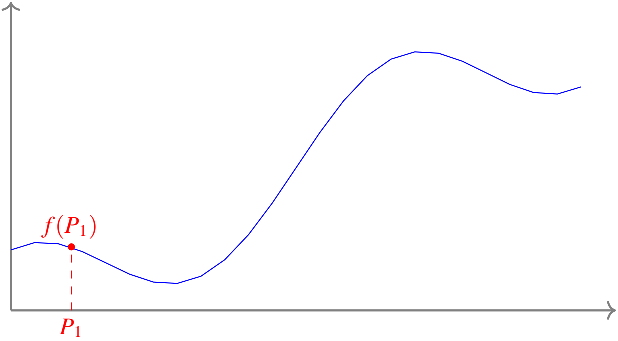

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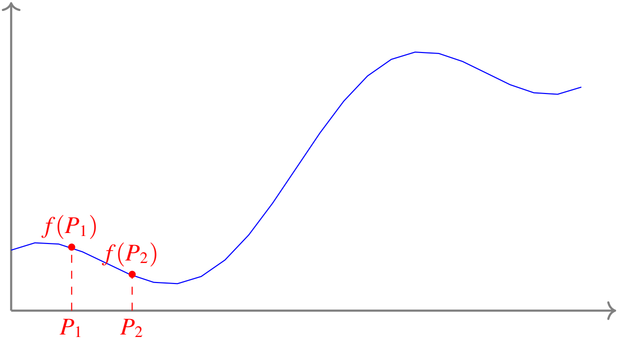

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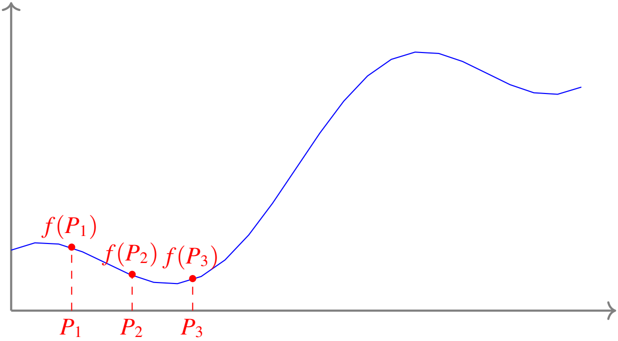

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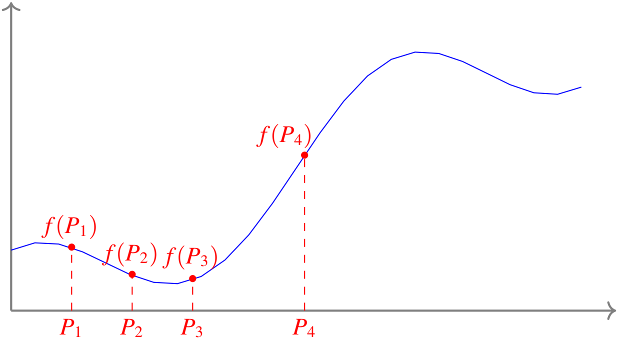

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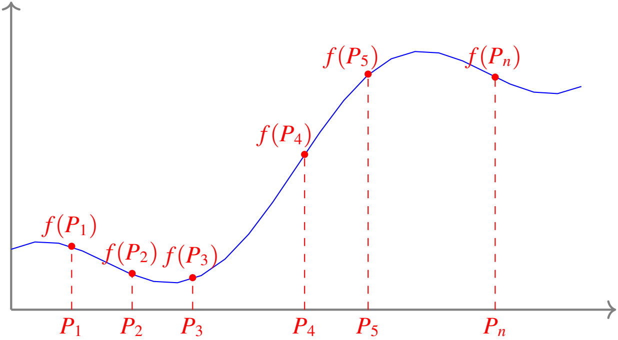

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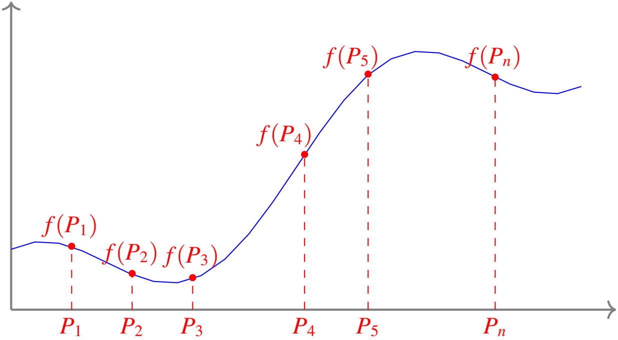

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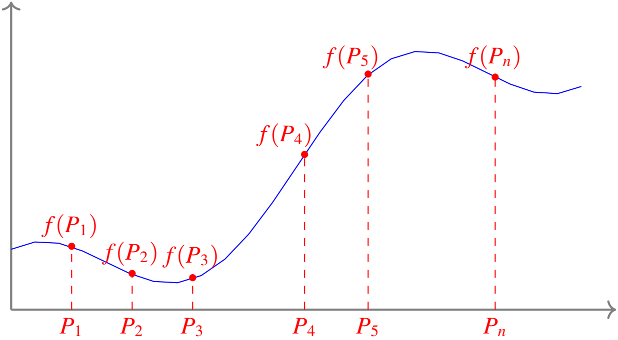

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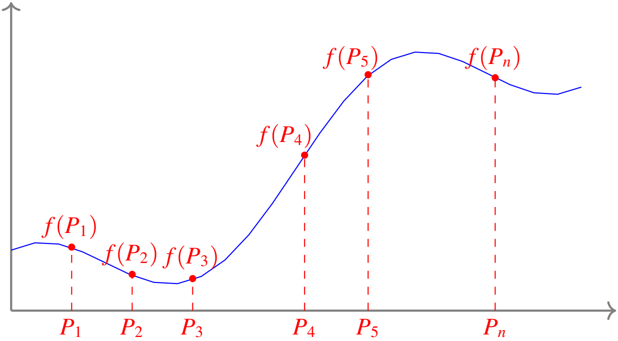

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An RS code is a set of vectors obtained by evaluating all polynomials of degree up to $k-1$.
The ***minimum distance*** of the RS code is $n-(k-1)$; and this is the largest possible value according to the Singleton bound (MDS code).

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Given a polynomial $f \in \mathbb{F}_q[x]$ and a set $A = \{P_1, \dots, P_n\} \subset \mathbb{F}_q$ define the map

$$ev_A : f \mapsto (f(P_i), i = 1, \dots, n)$$

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RS code $\mathcal{C}$ encodes messages of k symbols.

Let $V_k(q) = \{f \in \mathbb{F}_q[x] : \deg(f) \leqslant k-1\}$

$$\begin{aligned} \mathcal{C} : V_k(q) &\to \mathbb{F}_q^n \\ f &\mapsto ev_A(f) = (f(P_i), i = 1, \dots, n) \end{aligned}$$

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Example: Let $q = 8, f(x) = 1 + \alpha x + \alpha x^2$

$$f(x) \mapsto (1, \alpha^4, \alpha^6, \alpha^4, \alpha, \alpha, \alpha^6)$$

Limitations of Reed-Solomon codes

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Example: $[14, 10]$ RS code

Limitations of Reed-Solomon codes

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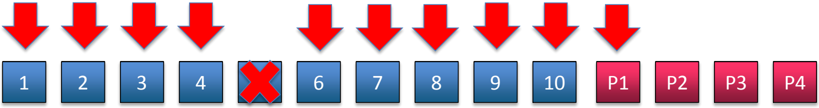

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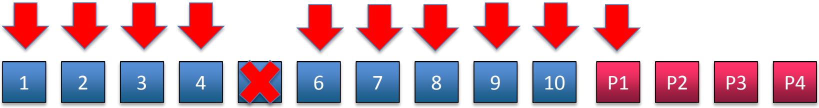

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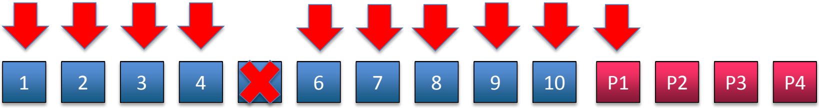

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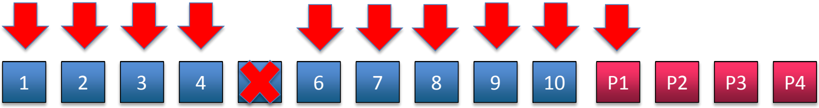

Locally Recoverable Codes - Definition

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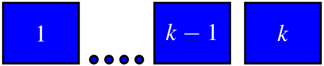

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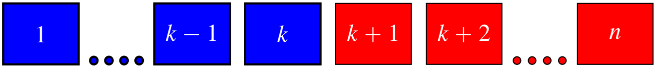

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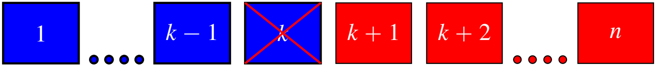

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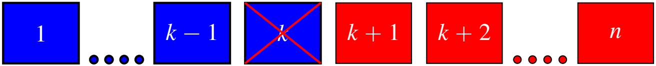

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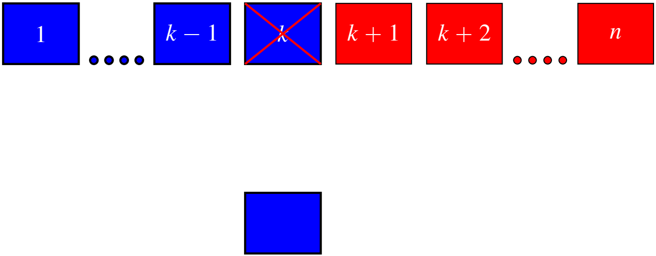

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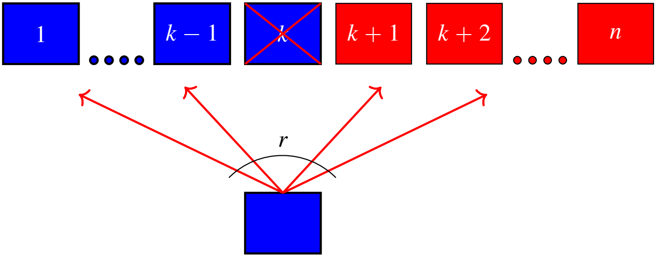

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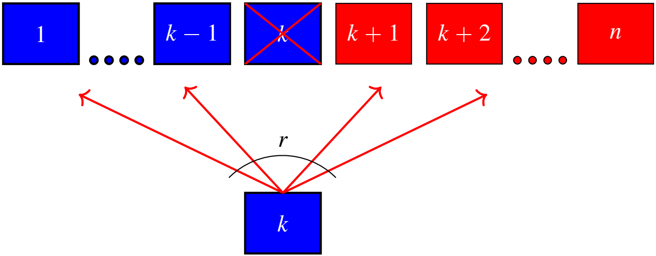

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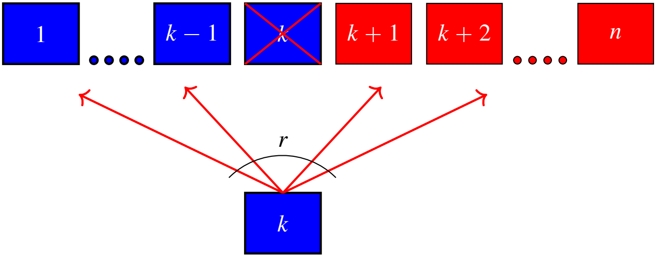

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3. Upon exposing at most $\frac{nr}{r+1}$ coordinates, we recover the entire codeword

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- Optimal (n, k, r) LRC code achieves the bound with equality

The distance bound

Main idea.

Let $\mathcal{C}$ be a q-ary code of length n, size q^k. The distance $d(\mathcal{C})$ satisfies

$$d(\mathcal{C}) \leqslant n - \{|S| : |\mathcal{C}_S| < q^k\}$$

Details:

- $\frac{k}{n} \leqslant \frac{r}{r+1} \implies \exists$ a set I of $\lfloor \frac{k-1}{n} \rfloor$ redundant coordinates
- Set $\mathcal{R} = \cup_{i\in I}\mathcal{R}_i$. Clearly $|\mathcal{R}| \leqslant k-1$
- If $|\mathcal{R}| < k-1$ add to it

The Singleton bound (with locality):

Let $I_i \subset [n], |I_i| \leqslant r$ be the recovery set for the symbol $c_i, i = 1, \dots, n$.

Let $J_m = \cup_{i=1}^m I_i$, where $m = \lfloor (k-1)/r \rfloor$. Clearly $|J_m| \leqslant k-1$.

Consider the subset $J_m' = J_m \cup \{1, \dots, m\}$. We have $\mathcal{C}_{J_m'} \leqslant q^{k-1}$.

If $|J_m'| < k-1$, add to J_m' any $k-1-|J_m|$ other coordinates to form the set $L_m \subset [n]$.

We have

$$|\mathcal{C}_{L_m}| < q^k$$
$$|L_m| = k-1+m = k-1+\left\lfloor \frac{k-1}{r} \right\rfloor = k-2+\left\lceil \frac{k}{r} \right\rceil$$

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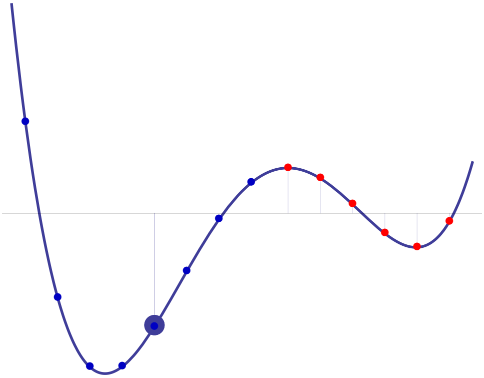
- Q: Generalize the optimal codes for $r = 1, k$ to codes with arbitrary r?

Reed-Solomon codes



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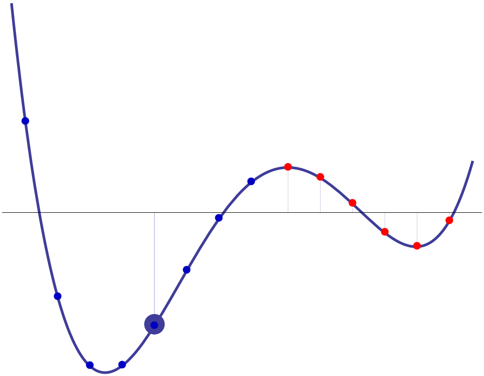

To recover one erased value we need to read k other values

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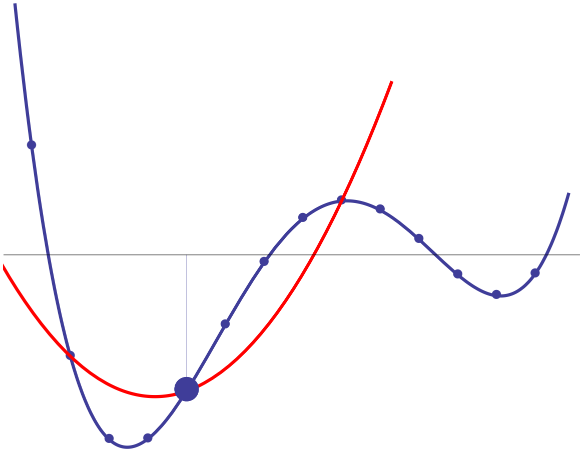

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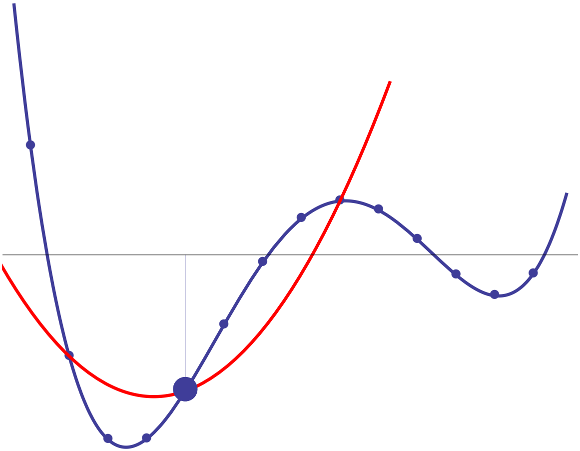

It is possible to construct such codes by carefully choosing subcodes of the RS codes

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Set of points: A={1,2,3,4,5,6,9,10,12}

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Say $c_1 = f_a(1)$ is erased. We access the recovering set A_1 to construct a line $\delta(x) = 2x + 2$ such that $\delta(3) = 8, \delta(9) = 7$.

Construction of (n, k, r) LRC codes: Example

Parameters: $n = 9, k = 4, r = 2, q = 13;$

Set of points: A={1,2,3,4,5,6,9,10,12}

$$\mathcal{A} = \{A_1 = \{1, 3, 9\}, A_2 = \{2, 6, 5\}, A_3 = \{4, 12, 10\}\}$$

Message: $a = (a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}) \in \mathbb{F}_q^k$

Polynomial space:

$$V_k(q) := \{a_{0,0} + a_{1,0}x + a_{0,1}x^3 + a_{1,1}x^4\}$$

E.g., $a = (1, 1, 1, 1), f_a(x) = 1 + x + x^3 + x^4; ev_A(f) = (4, 8, 7, 1, 11, 2, 0, 0, 0)$

Say $c_1 = f_a(1)$ is erased. We access the recovering set A_1 to construct a line $\delta(x) = 2x + 2$ such that $\delta(3) = 8, \delta(9) = 7$.

Compute c_1 as $\delta(1) = 4$

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Suppose there exists a polynomial $g(x) \in \mathbb{F}[x]$ such that

1. $\deg g = r + 1,$
2. There exists a partition $\mathcal{A} = \{A_1, ..., A_{\frac{n}{r+1}}\}$ of A into sets of size $r+1$, such that g is constant on each set A_i in the partition. For all $i = 1, \dots, n/(r+1)$, and any $\alpha, \beta \in A_i$,

$$g(\alpha) = g(\beta).$$

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$$g(\alpha) = g(\beta).$$

E.g., $n = 9, r = 2, q = 13$;

$$\mathcal{A} = \{A_1 = \{1, 3, 9\}, A_2 = \{2, 6, 5\}, A_3 = \{4, 12, 10\}\},$$

Then $g(x) = x^3$ is constant on each of the A_i's

Construction of (n, k, r) LRC codes

Given $A \subset \mathbb{F}$, partition $\mathcal{A}$ into $(r+1)$-subsets.

To encode the message $a \in \mathbb{F}^k$, write $a = (a_{ij}, i = 0, \ldots, r-1; j = 0, ..., \frac{k}{r} - 1)$

Define the encoding polynomial

$$f_a(x) = \sum_{i=0}^{r-1} x^i \sum_{j=0}^{\frac{k}{r}-1} a_{ij} g(x)^j$$

A linear code $\mathcal{C}$ is constructed as follows:

$$\begin{aligned} Ev : & \mathbb{F}^k \to \mathbb{F}^n \\ & a \mapsto (f_a(\beta), \beta \in A) \end{aligned}$$

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It is easy to show that the parameters of the constructed codes meet the Gopalan et al. bound with equality

I. Tamo and A.B., *A family of optimal locally recoverable codes*, T-IT August 2014

Constructing $g(x)$

Proposition

Let H be a subgroup of $\mathbb{F}_q^*$ or $\mathbb{F}_q^+$. The annihilator polynomial of H

$$g(x) = \prod_{h \in H} (x - h)$$

is constant on each coset of H.

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Further constructions:

J. Liu, S. Mesnager and L. Chen, *New constructions of optimal locally recoverable codes via good polynomials*, T-IT 2018

Summary of the construction

The optimal RS-like LRC codes are constructed as follows:

1. Take an RS code over $\mathbb{F}_q$ of length n and dimension $\frac{r+1}{r}k-2$
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These codes are studied outside the storage context:

L. HOLZBAUR AND A. WACHTER-ZEH, *List decoding of locally repairable codes*, arXiv:1801.04229

A. MAZUMDAR, *Caoacity of locally repairable codes*, arXiv:1801.04229

S. KADHE AND R. CALDERBANK, *LRC codes with small availability*, arXiv:1701.02456

Generalization of the main construction

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- E.g., message $(1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5)$

$$F(x, y) = 1 + \alpha y + \alpha^2 y^2 + \alpha^3 x + \alpha^4 xy + \alpha^5 xy^2$$

$$F(0, 0) = 1 \text{ etc.}$$

A.B., I. Tamo, and S. Vlăduţ, *LRC codes on algebraic curves*, T-IT, Aug. 2017

Random LRC codes and a Gilbert-Varshamov type bound

Let $M(n, r, \delta n)$ be the max size of a code of length n, distance d, locality r

$$R(r, \delta) := \limsup_{n\to\infty} \frac{1}{n} \log M(n, r, \delta n)$$

GV-type bound:

$$R(r, \delta) \geqslant 1 - \min_{0<s\leqslant 1} \left\{ \frac{1}{r+1} \log_2((1+s)^{r+1} + (1-s)^{r+1}) - \delta \log_2 s \right\}.$$

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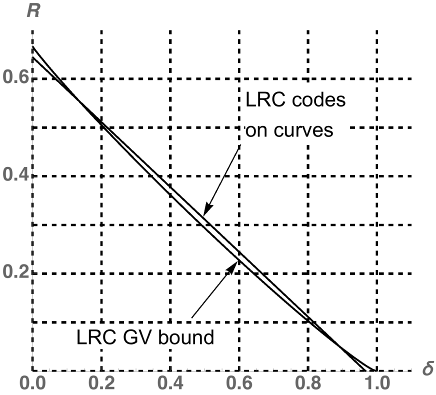
$$\Pr(\{d(\mathcal{C}) < \delta n\}) \leqslant \delta n q^{-n(\frac{r}{r+1} - R)} \min_{0<s\leqslant 1} \frac{b(s)^{\frac{n}{r+1}}}{s^{\delta n}}$$

H_L is a matrix with independent uniformly chosen elements of $\mathbb{F}_q$

V. Cadambe and A. Mazumdar, T-IT 2015; I. Tamo, A.B., and A. Frolov, T-IT, June 2016

Improving GV bound using LRC codes on curves



Locality $r = 2$, $q = 2^{10}$

A.B., I. Tamo, and S. Vlăduţ, *LRC codes on algebraic curves*, T-IT, Aug. 2017

More on bounds:

A. Agarwal et al., *Combinatorial alphabet-dependent bounds for locally recoverable codes*, T-IT 2018

Extensions

- Codes with availability
- Correcting $2, 3, \ldots$ erasures locally
- Hierarchical locality
- Maximally recoverable codes
- Maximum length of optimal LRC codes
- Cyclic LRC codes

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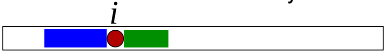

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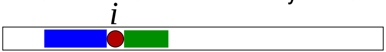
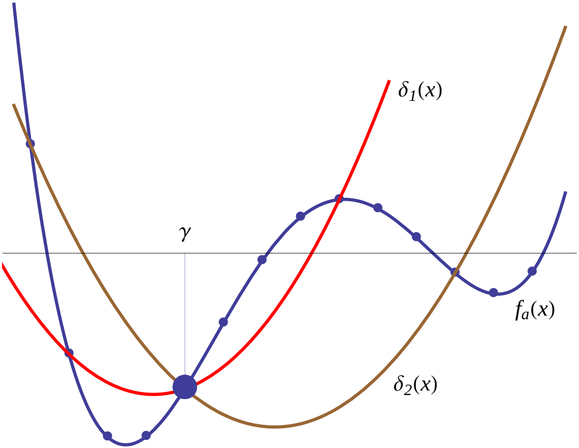


- A code $\mathcal{C}$ is called an $LRC(2)$ code if every coordinate i has 2 disjoint recovering sets $R_{1,i}, |R_{1,i}| \leqslant r_1; R_{2,i}, |R_{2,i}| \leqslant r_2$

Multiple recovery sets: Idea of construction



$f_a(\gamma)$ can be found by interpolating $\delta_1(x)$ as well as $\delta_2(x)$

Multiple recovery sets: Example

Take $\mathbb{F} = \mathbb{F}_{13}$; $G, H \leqslant \mathbb{F}^*$; $G = \langle 5 \rangle, H = \langle 3 \rangle$

$$\mathcal{A}_G = \{\{1, 5, 12, 8\}, \{2, 10, 11, 3\}, \{4, 7, 9, 6\}\}$$

$$\mathcal{A}_H = \{\{1, 3, 9\}, \{2, 6, 5\}, \{4, 12, 10\}, \{7, 8, 11\}\}$$

Let

$$\mathbb{F}_{\mathcal{A}_G}[x] = \{f \in \mathbb{F}[x] : f \text{ is constant on } A_i, i = 1, 2, 3;\ \deg f < |\mathbb{F}^*|\}$$

$$\mathbb{F}_{\mathcal{A}_G}[x] = \langle 1, x^4, x^8 \rangle, \quad \mathbb{F}_{\mathcal{A}_H}[x] = \langle 1, x^3, x^6, x^9 \rangle$$

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We construct an LRC $(12, 4, \{2,3\})$, distance $\geqslant 6$, code $\mathcal{C} : \mathbb{F}^4 \to \mathbb{F}^{12}$

$$a = (a_0, a_1, a_2, a_3) \mapsto f_a(x) = a_0 + a_1 x + a_2 x^4 + a_3 x^6$$

$$f_a(x) = \sum_{i=0}^{2} f_i(x) x^i, \text{ where } f_0(x) = a_0 + a_2 x^4,\ f_1(x) = a_1,\ f_2(x) = a_3 x^4; f_i \in \mathbb{F}_{\mathcal{A}}[x]$$

$$f_a(x) = \sum_{j=0}^{1} g_j(x) x^j \text{ where } g_0(x) = a_0 + a_3 x^6, g_1(x) = a_1 + a_2 x^3; g_j \in \mathbb{F}_{\mathcal{A}_H}[x]$$

E.g., $f_a(1)$ can be recovered by computing $\delta_1(x), x \in \{5, 12, 8\}$ OR $\delta_2(x), x \in \{3, 9\}$

Other constructions

- Product codes
- Codes on bipartite graphs
- Direct-sum codes

Open problem: Bounds on codes with availability

- Known bounds:

  Let $\mathcal{C}$ be an (n,k,r,t) LRC code with t disjoint recovering sets of size r. Then the rate of $\mathcal{C}$ satisfies

$$\frac{k}{n} \leqslant \frac{1}{\prod_{j=1}^{t}(1+\frac{1}{jr})}$$

  The minimum distance of $\mathcal{C}$ is bounded above as follows:

$$d \leqslant n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor.$$

I. Tamo, A.B., and A Frolov, *Bounds on the Parameters of Locally Recoverable Codes,* T-IT 2016

$$d \leqslant n-k+2-\left\lceil \frac{t(k-1)+1}{t(r-1)+1} \right\rceil$$

A. Wang and Z. Zhang, *Repair locality with multiple erasure tolerance,* T-IT 2014

More on bounds:

N. Prakash, V. Lalitha, and P. V. Kumar, *Codes with locality for two erasures*, ISIT 2014
S. B. Balaji and P. V. Kumar, *Bounds on ... codes with availability*, ISIT 2017 (improved results for linear codes)

- The RS-like construction can be extended to $t \geqslant 2$ recovering sets, but the resulting codes are not known to be optimal

Remarks on the bounds, and Graph-theoretic connections

- The bound on the rate of codes with availability t can be simplified:

$$\frac{k}{n} \leqslant \frac{1}{\sqrt[r]{t+1}}$$

- Tighter bounds are available in some cases (BALAJI-KUMAR, arXiv1611.00159)

$$R(r,2) \leqslant \frac{r}{r+2}, \; R(r,3) \leqslant \frac{r^2}{(r+1)^2}$$

- Problems related to multiplicities, e.g., availability or sequential repair, often can be interpreted in terms of graph theory or matroid theory.
- To derive the bounds, we note that multiple repair groups create dependence relations on the set of coordinates; we analyze the "expansion" of dependencies in the recovering graph as we add vertices successively.

M. GRETZEL AND C. HOLLANTI, *The complete hierarchical locality of the punctured simplex code*, arXiv:1901.03149

R. FREIJ-HOLLANTI, C. HOLLANTI, AND T. WESTERBCK, Matroid theory and storage codes, arXiv:1704.04007

Correcting $\geqslant 2$ erasures locally

In the event that more than one node in the encoding have failed, we need to correct more than one erasure locally

A code $\mathcal{C}$ is said to have the (ρ, r) locality property if each coordinate i is contained in a subset $A_i \subset [n], |A_i| \leqslant r + \rho - 1$ such that the restriction C_{A_i} forms a code of distance $\geqslant \rho$.

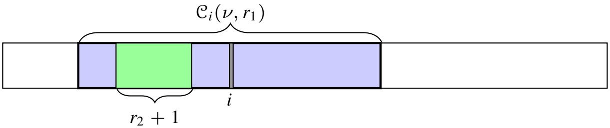
The distance of the code $\mathcal{C}$ satisfies the bound

$$d \leqslant n - k + 1 - \left(\left\lceil \frac{k}{r} \right\rceil - 1 \right) (\rho - 1)$$

G.M. Kamath et al., *Codes with local regeneration and erasure correction*, T-IT. Aug. 2014

The RS-like construction can be extended to this case, the parameters of the resulting codes meet this bound

Hierarchical locality



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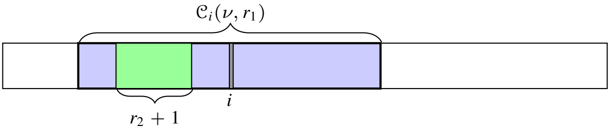

Every coordinate i is in a code $\mathcal{C}_i$ that

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B. Sasidharan et al., *Codes with hierarchical locality*, ISIT 2015

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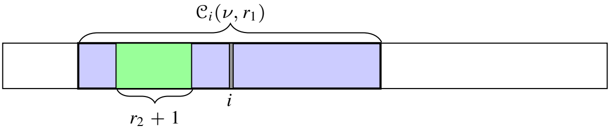

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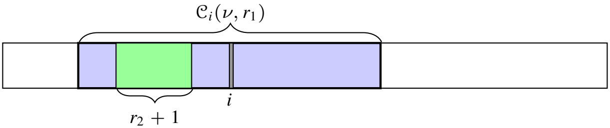

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B. Sasidharan et al., *Codes with hierarchical locality*, ISIT 2015

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Constructions of optimal RS-type codes and of codes on algebraic curves

S. Ballentine et al., *Codes with hierarchical locality from covering maps of curves*, T-IT 2019

Open problem: Maximally recoverable codes

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- Partial MDS codes - array configuration

M. Blaum et al.,*Partial MDS codes and their application to RAID type of architectures*, T-IT 2013

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- By the MR property, S suffices for decoding

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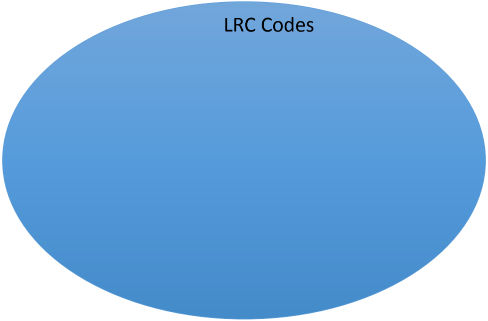

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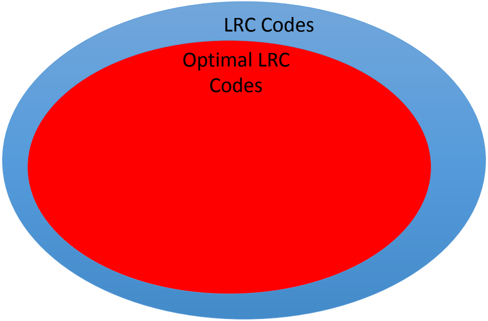

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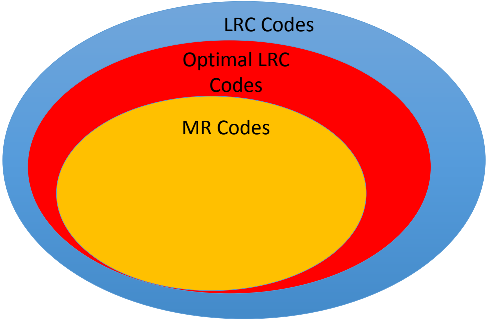

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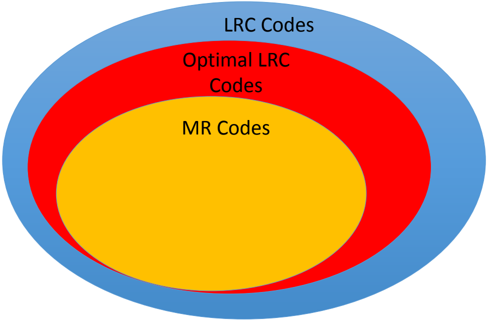

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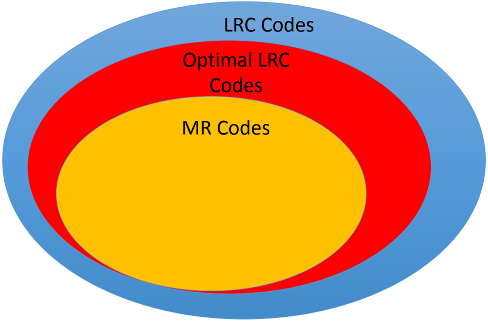

Q: MR codes = Optimal LRC codes ? Ans: No

MR codes through linearized polynomials

A. S. RAWAT ET AL.,,*Optimal locally repairable and secure codes for distributed storage systems*, T-IT 2014

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- Can we do better?

U. MARTINEZ-PEÑAS AND F. KSCHISCHANG, *Universal and dynamic locally repairable codes with maximal recoverability via sum-rank codes*, arXiv:1809.11158

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- L.JIN, T-IT 2019: Opt-LRC codes of length q^2 and distance $5, 6$

Further variants of the repair problem

- Sequential repair: For a subset $c_{i_1}, \ldots, c_{i_t}$ of t erased nodes, it is possible to find a repair group of size $\leqslant r$ to recover c_{i_1}, then a repair group of size $\leqslant r$ (possibly including c_{i_1}) that recovers c_{i_2}, then another repair group of size $\leqslant r$ for c_{i_3}, etc.

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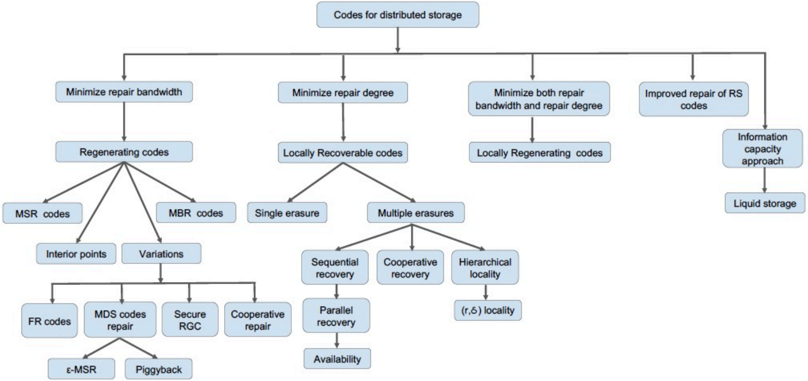
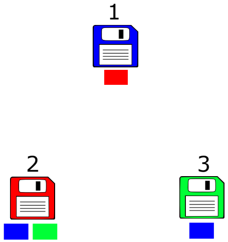
  S. KADHE ET AL., *On an Equivalence Between Single-Server PIR with Side Information and Locally Recoverable Codes*, arXiv:1907.00598

S.B. BALAJI et al., *Erasure coding for distributed storage: An overview*, arXiv:1806.04437

Different versions of the repair problem



LRC codes on graphs [Mazumdar 2014, Shanmugam and Dimakis 2014]



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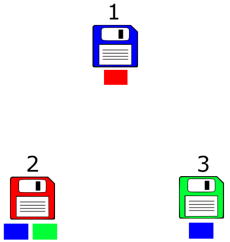

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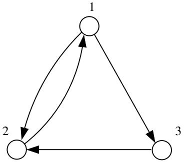

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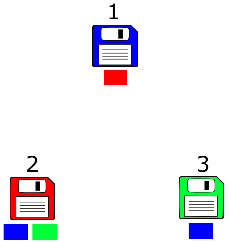

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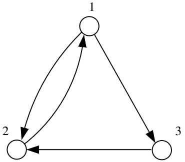

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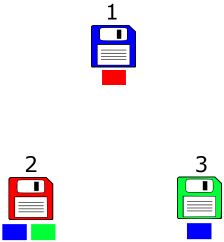

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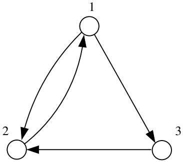

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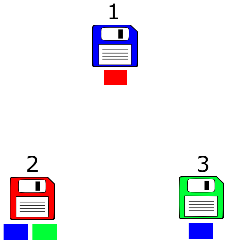

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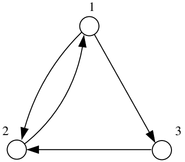

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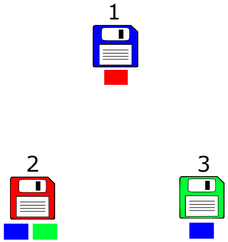

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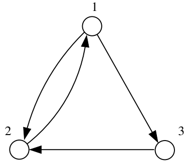

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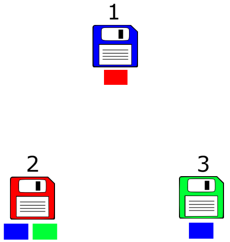

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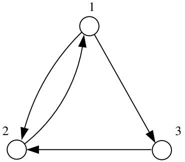

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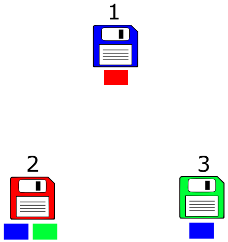

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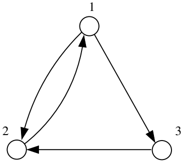

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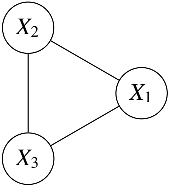

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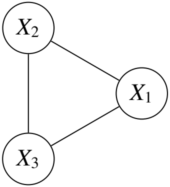

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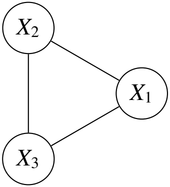

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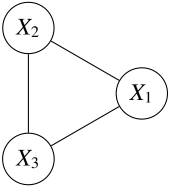

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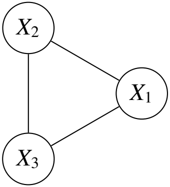
LRC codes on graphs - Example



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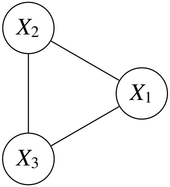

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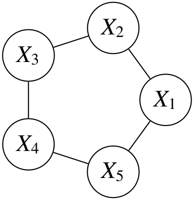

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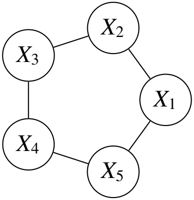

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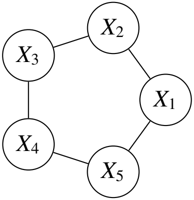

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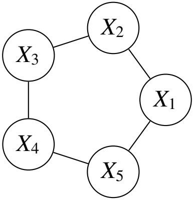

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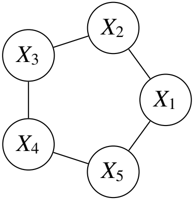

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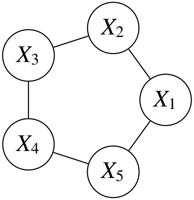

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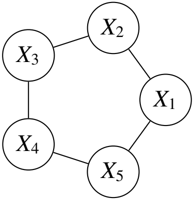

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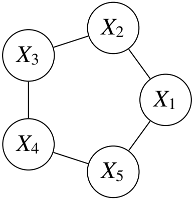

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Storage Capacity

Generally

$$|\text{Maximum matching}| \leqslant \text{Cap}(G) \leqslant |\text{Vertex cover}|$$

The bounds are separated by a factor of 2.

For planar graphs there is a 1.5 approximation

• A. MAZUMDAR ET AL.*Storage capacity as an information-theoretic vertex cover and the index coding rate*, T-IT 2019

Results for planar graphs and cycles with chords

Duality between Storage Capacity and Index Coding

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Many open problems