EK381: Exam 1 Review

College of Engineering
Boston University

Sets:

- A set is a collection of elements.
- The universal set Ω is the set of all elements (for the specific context).
- A subset A of a set B is a set consisting of some (or none) of the elements of B. Usually written as $A \subset B$.
- The empty set (or null set) ϕ is the set with no elements.

Set Operations:

- Complement: $A^{\mathsf{c}} = \{x : x \notin A\}$.
- Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- Difference: $A - B = \{x : x \in A \text{ and } x \notin B\}$.

Other Set Concepts:

- A collection of sets $A_1, \ldots, A_n$ is mutually exclusive if $A_i \cap A_j = \phi$ for $i \neq j$.
- A collection of sets $A_1, \ldots, A_n$ is collectively exhaustive if $A_1 \cup \cdots \cup A_n = \Omega$.
- A collection of sets $A_1, \ldots, A_n$ is a partition if it is both mutually exclusive and collectively exhaustive.

De Morgan's Laws:

$$\big(A \cup B\big)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$$

$$\left(\bigcup_{i=1}^{n} A_i\right)^{\mathsf{c}} = \bigcap_{i=1}^{n} A_i^{\mathsf{c}}$$

$$\big(A \cap B\big)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$$

$$\left(\bigcap_{i=1}^{n} A_i\right)^{\mathsf{c}} = \bigcup_{i=1}^{n} A_i^{\mathsf{c}}$$

Basic Probability Model:

- An experiment is a procedure that generates observable outcomes.
- An outcome is a possible observation of an experiment.
- The sample space Ω is the set of all possible outcomes.
- An event is a set of outcomes of an experiment.

Probability Axioms:

- Non-negativity: For any event A, $\mathbb{P}[A] \geq 0$.
- Normalization: $\mathbb{P}[\Omega] = 1$.
- Additivity: For any countable collective $A_1, A_2, \ldots$ of mutually exclusive events, $\mathbb{P}[A_1 \cup A_2 \cup \cdots] = \mathbb{P}[A_1] + \mathbb{P}[A_2] + \cdots$.

Other Useful Properties:

- Complement: $\mathbb{P}[A^{\mathsf{c}}] = 1 - \mathbb{P}[A]$.
- Inclusion-Exclusion: $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$.

Conditional Probability:

- The conditional probability of event A given that B occurs is

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \ .$$

- Conditional probability satisfies the probability axioms:
  - Non-negativity: For any event A, $\mathbb{P}[A|B] \geq 0$.
  - Normalization: $\mathbb{P}[\Omega|B] = 1$.
  - Additivity: For any countable collective $A_1, A_2, \ldots$ of mutually exclusive events, $\mathbb{P}[A_1 \cup A_2 \cup \cdots |B] = \mathbb{P}[A_1|B] + \mathbb{P}[A_2|B] + \cdots$.

Multiplication Rule:

- For any events $A_1, A_2, \ldots, A_n$,

$$\mathbb{P}\left[\bigcap_{i=1}^{n} A_i\right] = \mathbb{P}[A_1]\,\mathbb{P}[A_2|A_1]\,\mathbb{P}[A_3|A_1 \cap A_2] \cdots \mathbb{P}[A_n|A_1 \cap \cdots \cap A_{n-1}] \, .$$

- Two events A and B: $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B|A] = \mathbb{P}[B]\,\mathbb{P}[A|B]$.

Law of Total Probability:

- For a partition $B_1, \dots, B_n$ satisfying $\mathbb{P}[B_i] > 0$ for all i,

$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A|B_i]\mathbb{P}[B_i] \ .$$

Bayes' Rule:

- This is a method to "flip" conditioning:

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A|B]\,\mathbb{P}[B]}{\mathbb{P}[A]} \ .$$

- Sometimes, it is useful to solve for the denominator using the law of total probability. For a partition $B_1, \dots, B_n$ satisfying $\mathbb{P}[B_i] > 0$ for all i,

$$\mathbb{P}[B_j|A] = \frac{\mathbb{P}[A|B_j]\,\mathbb{P}[B_j]}{\mathbb{P}[A]} = \frac{\mathbb{P}[A|B_j]\,\mathbb{P}[B_j]}{\sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i]}$$

Independence:

- Two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$.
- Events $A_1, \ldots, A_n$ are independent if
  - All collections of $n-1$ events chosen from $A_1, \ldots, A_n$ are independent.
  - $\mathbb{P}[A_1 \cap \cdots \cap A_n] = \mathbb{P}[A_1] \cdots \mathbb{P}[A_n]$
- Independence means that no subset of the events can be used to improve the prediction of any other subset of events.
- If $A_1, \ldots, A_n$ only satisfy $\mathbb{P}[A_i \cap A_j] = \mathbb{P}[A_i]\mathbb{P}[A_j]$ for all $i \neq j$, then we say they are pairwise independent (but not independent).

Conditional Independence:

- The events A and B are conditionally independent given C if

$$\mathbb{P}[A \cap B | C] = \mathbb{P}[A|C]\,\mathbb{P}[B|C] \ .$$

- Independence does not imply conditional independence.
- Conditional independence does not imply independence.

Counting:

- If an experiment is composed of m subexperiments and the i^{th} subexperiment consists of n_i outcomes (that can be freely chosen), then the total number of outcomes $n_1 \, n_2 \, \cdots \, n_m$.

Sampling:

- Number of ways to make k selections out of n distinguishable elements

	Order	
	Dependent	Independent
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Discrete Random Variables:

- A random variable is a mapping that assigns (real) numbers to outcomes in the sample space.
- Random variables are denoted by capital letters (such as X) and their specific values are denoted by lowercase letters (such as x).
- The range of a random variable X is denoted by R_X.

Probability Mass Function (PMF):

- Probability that a discrete random variable X takes the value x:

$$P_X(x) = \mathbb{P}[X = x] \ .$$

Basic PMF Properties

- Non-negativity: $P_X(x) \geq 0$ for all x.
- Normalization: $\displaystyle\sum_{x \in R_X} P_X(x) = 1$.
- Additivity: For any event $B \subset R_X$, the probability that X falls in B is $\mathbb{P}[X \in B] = \displaystyle\sum_{x \in B} P_X(x)$.

Expectation:

- The expected value of a discrete random variable X is

$$\mathbb{E}[X] = \sum_{x \in R_X} x \, P_X(x) \ .$$

Functions of a Random Variable:

- A function of $Y = g(X)$ of a discrete random variable X is itself a discrete random variable.
- Range: $R_Y = \{g(x) : x \in R_X\}$.
- PMF: $P_Y(y) = \sum_{x : g(x) = y} P_X(x)$.
- Expected Value: $\mathbb{E}[Y] = \sum_{y \in R_Y} y \, P_Y(y) = \sum_{x \in R_X} g(x) \, P_X(x) \ .$
- Linearity of Expectation: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$.

Variance:

- The variance measures how spread out a random variable is around its mean, $\text{Var}[X] = \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)^2\Big]$.
- Alternate formula: $\text{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$.
- Variance of a linear function: $\text{Var}[aX + b] = a^2\text{Var}[X]$.
- Standard Deviation: $\sigma_X = \sqrt{\text{Var}[X]}$.

Important Families of Discrete Random Variables:

<u>Bernoulli Random Variables:</u>

- X is a Bernoulli(p) random variable if it has PMF

$$P_X(x) = \begin{cases} 1-p & x = 0, \\ p & x = 1 \end{cases}.$$

- Range: $R_X = \{0, 1\}$.
- Expected Value: $\mathbb{E}[X] = p$.
- Variance: $\text{Var}[X] = p(1-p)$.
- Interpretation: Single trial with success probability p.

Important Families of Discrete Random Variables:

Geometric Random Variables:

- X is a Geometric(p) random variable if it has PMF

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $R_X = \{1, 2, \ldots\}$.
- Expected Value: $\mathbb{E}[X] = \frac{1}{p}$.
- Variance: $\text{Var}[X] = \frac{1-p}{p^2}$.
- Interpretation: $\#$ of independent Bernoulli(p) trials until first success.

Important Families of Discrete Random Variables:

<u>Binomial Random Variables:</u>

- X is a Binomial(n, p) random variable if it has PMF

$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $R_X = \{0, 1, \ldots, n\}$.
- Expected Value: $\mathbb{E}[X] = np$.
- Variance: $\text{Var}[X] = np(1-p)$.
- Interpretation: # of successes in n independent Bernoulli(p) trials.

Important Families of Discrete Random Variables:

Discrete Uniform Random Variables:

- X is a Discrete Uniform(a, b) random variable if it has PMF

$$P_X(x) = \begin{cases} \dfrac{1}{b-a+1} & x = a, a+1, \ldots, b, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $R_X = \{a, a+1, \ldots, b\}$.
- Expected Value: $\mathbb{E}[X] = \dfrac{a+b}{2}$.
- Variance: $\text{Var}[X] = \dfrac{(b-a)(b-a+2)}{12} = \dfrac{(b-a+1)^2 - 1}{12}$.
- Interpretation: equally likely to take any integer value from a to b.

Important Families of Discrete Random Variables:

Poisson Random Variables:

- X is a Poisson(λ) random variable if it has PMF

$$P_X(x) = \begin{cases} \frac{\lambda^x}{x!}e^{-\lambda} & x = 0, 1, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $R_X = \{0, 1, 2, \ldots\}$.
- Expected Value: $\mathbb{E}[X] = \lambda$.
- Variance: $\text{Var}[X] = \lambda$.
- Interpretation: $\#$ of arrivals in a fixed time window.

Cumulative Distribution Function (CDF):

- The CDF returns the probability that a random variable X is less than or equal to a value x:

$$F_X(x) = \mathbb{P}[X \leq x] \ .$$

Basic CDF Properties:

- Non-negativity: $F_X(x)$ is a non-decreasing function of x.
- Normalization: $\lim_{x \to \infty} F_X(\infty) = 1$.
- Probability of an Interval: $\mathbb{P}[a < X \leq b] = F_X(b) - F_X(a)$.

Conditioning for Discrete Random Variables:

- The conditional PMF of X given an event $\{X \in B\}$ is

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} \quad \text{where} \quad \mathbb{P}[X \in B] = \sum_{x \in B} P_X(x)$$

  - Non-negativity: $P_{X|B}(x) \geq 0$ for all x.
  - Normalization: $\sum_{x \in B} P_{X|B}(x) = 1$.
  - Additivity: For any event $A \subset R_X$, the probability that X falls in A given that X falls in B is $\mathbb{P}\big[\{X \in A\}\big|\{X \in B\}\big] = \sum_{x \in A} P_{X|B}(x)$.

- The conditional expected value of X given an event B is

$$\mathbb{E}[X|B] = \sum_{x \in B} x P_{X|B}(x) \ .$$

- The conditional expected value of a function $g(X)$ given event B is

$$\mathbb{E}[g(X)|B] = \sum_{x \in B} g(x) P_{X|B}(x)$$

Foundations of Probability: True/False Practice Questions

For each of the following questions, indicate whether the statement is always true or it can be false by clearly writing “True” or “False.” Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

Foundations of Probability: True/False #1

If $\mathbb{P}[A|B] = \mathbb{P}[A]$, then A and B are independent.

True.

$\mathbb{P}[A|B] = \dfrac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \mathbb{P}[A]$, which implies independence, $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$.

Foundations of Probability: True/False #2

If A, B, and C are independent, then $\mathbb{P}[A \cap B|C] = \mathbb{P}[A]\mathbb{P}[B]$.

True.

$$\mathbb{P}[A \cap B|C] = \frac{\mathbb{P}[A \cap B \cap C]}{\mathbb{P}[C]} = \frac{\mathbb{P}[A]\mathbb{P}[B]\mathbb{P}[C]}{\mathbb{P}[C]} = \mathbb{P}[A]\mathbb{P}[B].$$

Foundations of Probability: True/False #3

If A and B are independent, then they are also conditionally independent given C.

False.

Independence does not imply conditional independence.

Foundations of Probability: True/False #4

$\mathbb{P}[A \cap B \cap C] + \mathbb{P}[A \cap B^c \cap C^c] = \mathbb{P}[A]$.

False.

The events $B \cap C$ and $B^c \cap C^c$ are not a partition so we cannot use the Total Probability Theorem. (We could instead use $B \cap C$ and $(B \cap C)^c$ as a partition).

Foundations of Probability: True/False #5

If A and B are independent, $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]\mathbb{P}[A^c]$.

True.

$$\begin{aligned}\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] &= \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A]\mathbb{P}[B] \\ &= \mathbb{P}[A] + \mathbb{P}[B]\big(1 - \mathbb{P}[A]\big) \\ &= \mathbb{P}[A] + \mathbb{P}[B]\mathbb{P}[A^c]\end{aligned}$$

Foundations of Probability: True/False #6

If A contains more outcomes than B, then $\mathbb{P}[A] > \mathbb{P}[B]$.

False.

Even if A contains more outcomes, they could have lower combined probability than the outcomes in B.

Foundations of Probability: True/False #7

$\mathbb{P}[A \cap B] \leq \mathbb{P}[A] + \mathbb{P}[B] - 1$.

False.

Combining $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$ with the fact that $\mathbb{P}[A \cup B] \leq 1$, we have that $1 \geq \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$. Rearranging terms, we get

$\mathbb{P}[A \cap B] \geq \mathbb{P}[A] + \mathbb{P}[B] - 1$, which has the inequality in the opposite direction.

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

Discrete Random Variables: True/False #1

$P_X(a) \geq P_X(b)$ for all $a \geq b$.

False.

For example, if X is Bernoulli$(1/4)$, then $P_X(1) < P_X(0)$ even though $1 > 0$.

Discrete Random Variables: True/False #2

For $a \geq b$, $\mathbb{P}[X \geq a | X \geq b] = \dfrac{\mathbb{P}[X \geq a]}{\mathbb{P}[X \geq b]}$.

True.

$$\mathbb{P}[X \geq a | X \geq b] = \frac{\mathbb{P}\big[\{X \geq a\} \cap \{X \geq b\}\big]}{\mathbb{P}[X \geq b]} = \frac{\mathbb{P}[X \geq a]}{\mathbb{P}[X \geq b]} \text{ since } a \geq b.$$

Discrete Random Variables: True/False #3

For any real number a, $\mathbb{E}\big[(X+a)^2\big] = \mathbb{E}[X^2] + 2a\mathbb{E}[X] + a^2$.

True.

$\mathbb{E}\big[(X+a)^2\big] = \mathbb{E}\big[X^2 + 2aX + a^2\big] = \mathbb{E}[X^2] + 2a\mathbb{E}[X] + a^2$ using the linearity of expectation.

Discrete Random Variables: True/False #4

If, for all values $a > 0$, $P_X(a) = P_X(-a)$, then $\mathbb{E}[X] = 0$.

True.

This means that the PMF is symmetric and centered on 0, so it has mean 0.

Discrete Random Variables: True/False #5

$\mathrm{Var}[X] \geq 0$

True.

$\mathrm{Var}[X] = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big]$ so it is the average of non-negative terms and must be non-negative.

Discrete Random Variables: True/False #6

$\mathbb{E}\big[(X^2 - \mathbb{E}[X^2])^2\big] = \mathbb{E}[X^4] - \mathbb{E}[X^2]$.

False.

Define $Y = X^2$. We know that $\mathrm{Var}[Y] = \mathbb{E}\big[(Y - \mathbb{E}[Y])^2\big] = \mathbb{E}[Y^2] - \big(\mathbb{E}[Y]\big)^2$. Plugging in $Y = X^2$, we get $\mathbb{E}\big[(X^2 - \mathbb{E}[X^2])^2\big] = \mathbb{E}[X^4] - \big(\mathbb{E}[X^2]\big)^2$. The proposed equation is missing the square in the second term.

Practice Question #1

Consider the following game. There is a hat containing 5 blue balls and 3 red balls. Without looking, you reach into the hat and pull out 4 balls. You win the game if you pull out exactly 2 blue and 2 red balls. After each game, the balls are returned to the hat and mixed up again for the next game. (All outcomes within a single game are equally likely and that each game is independent of the others.)

a. What is the probability that you win a single game?

b. Let's say you play a total of 7 games. Let W denote the total number of games that you win out of 7. What kind of random variable is W?

c. What is the expected value of the number of games you will win?

d. Say that you lost the first 2 out of 7 games. What is the probability that you will win at least 3 games in total?

Practice Question #1 Solution

a) $\mathbb{P}[\text{win}] = \dfrac{\#\text{ ways to win}}{\#\text{ total ways to select}}$ (since equally likely).

- There are $\binom{8}{4} = \frac{8!}{4!4!} = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1} = 70$ ways to select 4 balls out of $5+3=8$ total ways to select.
- $$\begin{aligned}\#\text{ ways to win} &= \#\text{ ways to select 2 out of 3 red balls}\\ &\quad\times \#\text{ ways to select 2 out of 5 blue balls}\\ &= \binom{3}{2} \times \binom{5}{2} = \frac{5!}{3!2!}\cdot\frac{3!}{1!2!} = 30\end{aligned}$$
- $\mathbb{P}[\text{win}] = \dfrac{30}{70} = \dfrac{3}{7}$

b) $W \sim \text{Binomial}(7, 3/7)$

c) $\mathbb{E}[W] = np = 7\cdot\dfrac{3}{7} = 3$

Practice Question #1 Solution

(d) Since the games are independent, we can just ignore the first 2 losses and focus on the probability of winning 3 out of 5 games. We can express this as a new random variable X that is Binomial$(5, 3/7)$. The probability of winning at least 3 games is

$$\begin{aligned}\mathbb{P}[X \geq 3] &= P_X(3) + P_X(4) + P_X(5) \\ &= \binom{5}{3}\left(\frac{3}{7}\right)^3\left(\frac{4}{7}\right)^2 + \binom{5}{4}\left(\frac{3}{7}\right)^4\left(\frac{4}{7}\right)^1 + \binom{5}{5}\left(\frac{3}{7}\right)^5\left(\frac{4}{7}\right)^0 \\ &= \frac{6183}{16807}\end{aligned}$$

Practice Question # 2

You walk up to the Green Line subway platform and wait for the train. You know from past experience that the number of minutes M (rounded up to the nearest minute) one has to wait for a train is a Geometric$(1/10)$ random variable.

a) What is the probability that M is greater than or equal to 2 minutes?

b) Calculate the average number of minutes you need to wait.

c) Your frustration F is equal to the square of the number of minutes you wait. Calculate your average frustration.

d) Assume that you have already waited one minute and would like to predict how much longer you will wait. Specifically, assume the event $B = \{M \geq 2\}$ has occurred. Determine the conditional PMF $P_{M|B}(m)$.

e) As in (d), assume that you have already waited one minute. Let $Y = M - 1$ be the random variable corresponding to the number of additional minutes you will wait. Using your answer from part (d), calculate $\mathbb{E}[Y|B]$, the average number of additional minutes.

Practice Question #2 Solution

a. Since M is Geometric($\frac{1}{10}$) its PMF is

$$P_M(m) = \begin{cases} \frac{1}{10}\left(\frac{9}{10}\right)^{m-1} & m = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{P}[M \geq 2] = 1 - \mathbb{P}[M \leq 1] = 1 - P_M(1) = 1 - \frac{1}{10} = \frac{9}{10}$$

b. $\mathbb{E}[M] = \frac{1}{p} = \frac{1}{\frac{1}{10}} = 10$

c. First, recall that $\text{Var}[M] = \mathbb{E}[M^2] - \left(\mathbb{E}[M]\right)^2$. Since M is a Geometric($\frac{1}{10}$) random variable, $\text{Var}[M] = \frac{9/10}{(1/10)^2} = 90$ and $\mathbb{E}[M] = 10$. Therefore,

$$\mathbb{E}[F] = \mathbb{E}[M^2] = \text{Var}[M] + \left(\mathbb{E}[M]\right)^2 = 90 + (10)^2 = 190.$$

d. $$P_{M|B}(m) = \begin{cases} \frac{P_M(m)}{\mathbb{P}[B]} & m \in B \\ 0 & m \notin B \end{cases} = \begin{cases} \frac{1}{10}\left(\frac{9}{10}\right)^{m-2} & m = 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

(e) Since $Y = M - 1$, we can substitute in $m = y + 1$ into the conditional PMF of M given B to get

$$P_{Y|B}(y) = P_{M|B}(y+1) = \begin{cases} \frac{1}{10}\left(\frac{9}{10}\right)^{y+1-2} & y+1 = 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{10}\left(\frac{9}{10}\right)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

This is just a Geometric$(1/10)$ random variable, so $\mathbb{E}[Y|B] = \frac{1}{\frac{1}{10}} = 10$.
You could also get this by realizing that the past independent trials have no effect on the future trials, so M conditioned on $\{M > k\}$ is just a shifted Geometric distribution and $Y - k$ conditioned on $\{M > k\}$ is Geometric.

Practice Question # 3

You are in charge of monitoring an online message board. You believe that the number of messages M posted in an hour is well-modeled as a Poisson random variable. After careful analysis, you have determined that the probability that *at least* one message is posted in an hour is $1 - e^{-3}$.

- a What is the average number of messages posted in an hour?
- b Given that between 1 and 3 messages (inclusive) are posted in an hour, what is the probability of seeing exactly 2 messages?
- c Calculate $\mathbb{E}[10M^2 + 50M]$.
- d You consider it a busy hour if 4 or more messages are posted. If you know that at least 2 messages have been posted, what is the probability that it is a busy hour?
- e Assume that the activity for each hour in a day is independent. Let T be the total number of busy hours in a day. What kind of random variable is T? (Don't forget the parameters.)

Practice Question #3 Solution

- **a** We are given that $\mathbb{P}[M \geq 1] = 1 - e^{-3}$, which implies that $\mathbb{P}[M \leq 0] = 1 - \mathbb{P}[M \geq 1] = e^{-3}$. Since Poisson random variables are non-negative, this tells us that $P_M(0) = e^{-3}$. We know from the Poisson PMF that $P_M(0) = \frac{\lambda^0}{0!}e^{-\lambda} = e^{-\lambda}$ so $\lambda = 3$ in this case. Finally, the average of a Poisson random variable is $\mathbb{E}[M] = \lambda = 3$.

- **b** Define the events $A = \{M = 2\}$ and $B = \{M \in \{1, 2, 3\}\}$. The conditional probability of A given B is

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{P_M(2)}{P_M(1) + P_M(2) + P_M(3)}$$

$$= \frac{e^{-3}\frac{3^2}{2!}}{e^{-3}\left(\frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right)} = \frac{\frac{9}{2}}{3 + \frac{9}{2} + \frac{27}{6}} = \frac{3}{8}\ .$$

Practice Question #3 Solution

(c) First, we use the variance formula $\mathrm{Var}[M] = \mathbb{E}[M^2] - \big(\mathbb{E}[M]\big)^2$ to solve for the second moment, $\mathbb{E}[M^2] = \mathrm{Var}[M] + \big(\mathbb{E}[M]\big)^2 = \lambda + \lambda^2 = 3 + 3^2 = 12$. Now, using the linearity of expectation, $\mathbb{E}[10M^2 + 50M] = 10\mathbb{E}[M^2] + 50\mathbb{E}[M] = 10 \cdot 12 + 50 \cdot 3 = 270$.

(d) Define the events $C = \{M \geq 4\}$ and $D = \{M \geq 2\}$. The conditional probability of C given D is

$$\begin{aligned}\mathbb{P}[C|D] = \frac{\mathbb{P}[C \cap D]}{\mathbb{P}[D]} &= \frac{\mathbb{P}[M \geq 4]}{\mathbb{P}[M \geq 2]} \\ &= \frac{1 - \big(P_M(0) + P_M(1) + P_M(2) + P_M(3)\big)}{1 - \big(P_M(0) + P_M(1)\big)} \\ &= \frac{1 - e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right)}{1 - e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!}\right)} = \frac{1 - 13\,e^{-3}}{1 - 4\,e^{-3}}\end{aligned}$$

Practice Question #3 Solution

(e) We can think of a busy hour as a success and the number of busy hours in a day as the sum of 24 independent trials with success probability $p = \mathbb{P}[M \geq 4] = 1 - 13\,e^{-3}$. Therefore, T is a Binomial$(24, 1 - 13\,e^{-3})$ random variable.