# EK381: Exam 2 Review

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**Probability Density Function (PDF):**

- The probability density function (PDF) is the derivative of the CDF:

$$f_X(x) = \frac{d}{dx} F_X(x)$$

- This does not directly tell us the probability of $X = x$, which is always $0$ for continuous random variables.
- It does tell us the density of probability around $x$.

**Basic PDF Properties:**

- Normalization: $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.
- Non-negativity: $f_X(x) \geq 0$.
- Probability of an interval: $\mathbb{P}[a < X \leq b] = \int_a^b f_X(x)\, dx$.
- PDF $\to$ CDF: $\int_{-\infty}^{x} f_X(u)\, du = F_X(x)$.

**Expected Value:**

- The expected value of a continuous random variable $X$ is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- Also known as the mean, the average, or the expectation.
- The expected value of a function of a continuous random variable $X$ is

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Linearity of Expectation:

- For any constants $a$ and $b$,

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

**Variance:**

- The variance of a random variable $X$ is

$$\mathsf{Var}[X] = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right]$$

- Captures how "spread out" a random variable is around its mean.
- Another useful formula is $\mathsf{Var}[X] = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2$
- The standard deviation is the square root of the variance:

$$\sigma_X = \sqrt{\mathsf{Var}[X]}\ .$$

Variance of a Linear Function:

- For any constants $a$ and $b$,

$$\mathsf{Var}[aX + b] = a^2\mathsf{Var}[X]$$

**Important Families of Continuous Random Variables:**

Uniform Random Variables:

- $X$ is a Uniform$(a, b)$ random variable if it has PDF

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x < b \\ 0 & \text{otherwise.} \end{cases}$$

- CDF: $F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x \end{cases}$
- Expected Value: $\mathbb{E}[X] = \dfrac{a+b}{2}$.
- Variance: $\text{Var}[X] = \dfrac{(b-a)^2}{12}$.

**Important Families of Continuous Random Variables:**

Exponential Random Variables:

- $X$ is an Exponential($\lambda$) random variable if it has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0\ . \end{cases}$$

- CDF: $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0\ . \end{cases}$
- Expected Value: $\mathbb{E}[X] = \dfrac{1}{\lambda}$.
- Variance: $\text{Var}[X] = \dfrac{1}{\lambda^2}$.

**Important Families of Continuous Random Variables:**

Gaussian Random Variables:

- $X$ is a Gaussian$(\mu, \sigma^2)$ random variable if it has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( - \frac{(x-\mu)^2}{2\sigma^2} \right)$$

- CDF: $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ for $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left( - \frac{w^2}{2} \right) dw$
- $\Phi(z)$ is the standard normal CDF and $Q(z) = 1 - \Phi(z)$ is the standard normal complementary CDF.
- $\Phi(-z) = 1 - \Phi(z) = Q(z)$
- Expected Value: $E[X] = \mu$.
- Variance: $\mathrm{Var}[X] = \sigma^2$.
- Linear function of a Gaussian is Gaussian:
  If $X$ is a Gaussian$(\mu, \sigma^2)$ random variable,
  then $Y = aX + b$ is a Gaussian$(a\mu + b, a^2\sigma^2)$ random variable.

**Conditioning for Continuous Random Variables:**

- The conditional PDF of $X$ given an event $B$ is

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} \quad \text{where } \mathbb{P}[X \in B] = \int_B f_X(x)\,dx$$

- The conditional expected value of $X$ given an event $B$ is

$$\mathbb{E}[X|B] = \int_{-\infty}^{\infty} x\, f_{X|B}(x)\,dx \ .$$

- The conditional expected value of a function $g(X)$ given an event $B$ is

$$\mathbb{E}\big[g(X)|B\big] = \int_{-\infty}^{\infty} g(x)\, f_{X|B}(x)\,dx \ .$$

- The conditional variance of $X$ given an event $B$ is

$$\mathrm{Var}\big[X|B\big] = \mathbb{E}\Big[\big(X - \mathbb{E}[X|B]\big)^2 \Big| B\Big] = \mathbb{E}\big[X^2|B\big] - \big(\mathbb{E}[X|B]\big)^2$$

**Joint Cumulative Distribution Function (CDF):**

- This is the probability that $X$ and $Y$ are less than or equal to the values $x$ and $y$, respectively:

$$F_{X,Y}(x, y) = \mathbb{P}[X \leq x,\ Y \leq y]$$

- Unifies discrete and continuous random variables.
- A pair of random variables is continuous if they have a continuous joint CDF that is differentiable almost everywhere.
- See lecture notes for basic properties.

**Pairs of Discrete Random Variables:**

- Joint PMF: $P_{X,Y}(x,y) = \mathbb{P}[X = x,\ Y = y]$.
- Range $R_{X,Y} = \{(x,y) : P_{X,Y}(x,y) > 0\}$.
- Marginal PMFs $P_X(x)$ and $P_Y(y)$ are just the PMFs of the individual random variables $X$ and $Y$, respectively.

$$P_X(x) = \sum_{y \in R_Y} P_{X,Y}(x,y) \qquad P_Y(y) = \sum_{x \in R_X} P_{X,Y}(x,y)$$

- Conditional PMFs give the probability of one random variable when the other is fixed to a certain value:

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} \qquad P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$$

for $(x,y) \in R_{X,Y}$, otherwise the conditional PMF is $0$.

**Pairs of Continuous Random Variables:**

- Joint PDF: $f_{X,Y}(x, y) = \dfrac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}$
- Range $R_{X,Y} = \{(x, y) : f_{X,Y}(x, y) > 0\}$.
- Marginal PDFs $f_X(x)$ and $f_Y(y)$ are just the PMFs of the individual random variables $X$ and $Y$, respectively.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$

- Conditional PDFs give the probability density of one random variable when the other is fixed to a certain value:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \qquad f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

for $(x, y) \in R_{X,Y}$, otherwise the conditional PDF is $0$.

**Joint PMF/PDF Properties:**

- Non-negativity: $P_{X,Y}(x,y) \geq 0$ (discrete)

  $f_{X,Y}(x,y) \geq 0$ (continuous)

- Normalization: $$\sum_{x \in R_X} \sum_{y \in R_Y} P_{X,Y}(x,y) = 1 \quad \text{(discrete)}$$

  $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \quad \text{(continuous)}$$

- Probability of an Event $B \subset R_{X,Y}$:

$$\mathbb{P}[B] = \mathbb{P}[(X,Y) \in B] = \sum_{(x,y) \in B} P_{X,Y}(x,y) \quad \text{(discrete)}$$

$$\mathbb{P}[B] = \mathbb{P}[(X,Y) \in B] = \iint_B f_{X,Y}(x,y)\,dx\,dy \quad \text{(continuous)}$$

**Conditional PMF/PDF Properties:**

- Non-negativity: $P_{X|Y}(x|y) \geq 0 \quad P_{Y|X}(y|x) \geq 0$

  $f_{X|Y}(x|y) \geq 0 \quad f_{Y|X}(y|x) \geq 0$

- Normalization: $\sum_{x \in R_X} P_{X|Y}(x|y) = \sum_{y \in R_Y} P_{Y|X}(y|x) = 1$

  $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\, dx = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, dy = 1$

- Additivity: For any event $B \subset R_X$, the probability that $X$ falls in $B$ given $Y = y$ is

$$\mathbb{P}[X \in B | Y = y] = \sum_{x \in B} P_{X|Y}(x|y) \quad \text{(discrete)}$$

$$\mathbb{P}[X \in B | Y = y] = \int_B f_{X|Y}(x|y)\, dx \quad \text{(continuous)}$$

- Multiplication Rule:

$$P_{X,Y}(x,y) = P_{X|Y}(x|y) P_Y(y) = P_{Y|X}(y|x) P_X(x)$$

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y) = f_{Y|X}(y|x) f_X(x)$$

**Independence of Random Variables:**

- $X$ and $Y$ are independent if $F_{X,Y}(x, y) = F_X(x)F_Y(y)$.
- Equivalently, we can just check if

$$P_{X,Y}(x, y) = P_X(x)P_Y(y) \quad \text{(discrete)}$$

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \quad \text{(continuous)}$$

- Special cases where $X$ and $Y$ are **not** independent:
    - Discrete: If there is a zero entry in the joint PMF table for which neither the entire column or entire row is zero.
    - Continuous: If the range is not a collection of rectangles parallel to the axes

**Expected Values of a Function of Random Variables:**

- The expected value of a function $W = g(X, Y)$ is

$$\mathbb{E}[W] = \sum_{x \in R_X} \sum_{y \in R_Y} g(x, y) P_{X,Y}(x, y) \quad \text{(discrete)}$$

$$\mathbb{E}[W] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy \quad \text{(continuous)}$$

Linearity of Expectation:

- For any constants $a$,$b$, and $c$,

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

even if $X$ and $Y$ are dependent.

Expectation of Products:

- If $X$ and $Y$ are independent, then $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$.

**Conditional Expectation:**

- The conditional expected value of $X$ given $Y = y$ is

$$\mathbb{E}[X|Y = y] = \sum_{x \in R_X} x\, P_{X|Y}(x|y) \quad \text{(discrete)}$$

$$\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y) dx \quad \text{(continuous)}$$

- $\mathbb{E}[X|Y = y]$ is a deterministic function of $y$.

- $\mathbb{E}[X|Y]$ is a random variable. It is in fact a function of the random variable $Y$. It can be obtained by substituting in $Y$ for $y$ in $\mathbb{E}[X|Y = y]$.

- Law of Total Expectation: $\mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}[X]$.

**Covariance:**

- The covariance of random variables $X$ and $Y$ is

$$\mathsf{Cov}[X,Y] = \mathbb{E}\big[\big(X - \mathbb{E}[X]\big)\big(Y - \mathbb{E}[Y]\big)\big]$$

- Another useful formula is $\mathsf{Cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- The covariance satisfies the following basic properties:
  - $\mathsf{Cov}[X,Y] = \mathsf{Cov}[Y,X]$
  - $\mathsf{Cov}[X,X] = \mathsf{Var}[X]$
  - $\mathsf{Cov}[X,a] = 0$ for any number $a$.
- $X$ and $Y$ are uncorrelated if $\mathsf{Cov}[X,Y] = 0$.
  - Independence implies uncorrelatedness.
  - Uncorrelatedness does not imply independence.

Variance of Linear Functions:

$$\mathsf{Var}[aX + bY + c] = a^2\mathsf{Var}[X] + b^2\mathsf{Var}[Y] + 2ab\mathsf{Cov}[X,Y]$$

Covariance of Linear Functions:

$$\mathsf{Cov}[aX+bY+c, dX+eY+f] = ad\,\mathsf{Var}[X]+be\mathsf{Var}[Y]+(ae+bd)\mathsf{Cov}[X,Y]$$

**Correlation Coefficient:**

- The correlation coefficient is $\rho_{X,Y} = \dfrac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]\mathsf{Var}[Y]}}$
- This is a “scale-invariant” version of covariance.
- The correlation coefficient satisfies the following properties:
  - $-1 \leq \rho_{X,Y} \leq 1$.
  - $\rho_{X,Y} = 1$ if and only if $Y = aX + b$ for some $a > 0$.
  - $\rho_{X,Y} = -1$ if and only if $Y = aX + b$ for some $a < 0$.
  - If $U = aX + b$ and $V = cY + d$, then

$$\rho_{U,V} = \operatorname{sign}(ac)\rho_{X,Y} \quad \text{where} \quad \operatorname{sign}(z) = \begin{cases} +1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases}$$

## Independent Unit Gaussian Random Variables:

- $U$ and $V$ are independent, standard Gaussian random variables if $U$ is Gaussian$(0, 1)$, $V$ is Gaussian$(0, 1)$, and $U$ and $V$ are independent.

## Jointly Gaussian Random Variables:

- $X$ and $Y$ are jointly Gaussian random variables if they are linear functions of independent, standard Gaussian random variables $U$ and $V$, $X = aU + bV + c$ and $Y = dU + eV + f$. However, this representation is usually left implicit, and the joint Gaussian distribution of $X$ and $Y$ is specified by 5 parameters:
  - Means: $\mu_X = \mathbb{E}[X]$, $\mu_Y = \mathbb{E}[Y]$
  - Variances: $\sigma_X^2 = \mathsf{Var}[X]$, $\sigma_Y^2 = \mathsf{Var}[Y]$
  - Covariance: $\mathsf{Cov}[X, Y]$ **or** Correlation Coefficient: $\rho_{X,Y}$.
- (We will not need to work with the joint PDF directly.)

**Properties of Jointly Gaussian Random Variables:** If $X$ and $Y$ are jointly Gaussian with parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{X,Y}$, then

- Marginal PDFs are Gaussian: $X$ is Gaussian$(\mu_X, \sigma_X^2)$ and $Y$ is Gaussian$(\mu_Y, \sigma_Y^2)$.
- Uncorrelatedness implies Independence: $X$ and $Y$ are uncorrelated if and only if $X$ and $Y$ are independent.
- Conditional Expected Value for Gaussians:
  $$\mathbb{E}[X|Y=y] = \mu_X + \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(y-\mu_Y) = \mu_X + \frac{\mathsf{Cov}[X,Y]}{\mathsf{Var}[Y]}(y-\mu_Y)$$
- Conditional Variance for Gaussians:
  $\sigma_{X|Y}^2 = \mathsf{Var}[X|Y=y] = (1-\rho_{X,Y}^2)\sigma_X^2$.
- Conditional PDF is Gaussian: The conditional PDF $f_{X|Y}(x|y)$ of $X$ given $Y$ is Gaussian$(\mathbb{E}[X|Y=y], \sigma_{X|Y}^2)$.
- **Linear functions of Gaussians are Gaussian:** If $W = aX + bY + c$ and $Z = dX + eY + f$, then $W$ and $Z$ are jointly Gaussian with parameters that be determined via the linearity of expectation and the variance and covariance of linear functions.

**Random Vectors:**

- Random Vector: a column vector of random variables $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$.
- If the entries are discrete random variables, the random vector has a joint PMF $P_{\underline{X}}(\underline{x}) = P_{X_1,\dots,X_n}(x_1,\dots,x_n)$.
  If the entries are continuous random variables, the random vector has a joint PDF $f_{\underline{X}}(\underline{x}) = f_{X_1,\dots,X_n}(x_1,\dots,x_n)$.
- Mean Vector: $\underline{\mu}_{\underline{X}} = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix}$
- Linearity of Expectation: $\mathbb{E}\big[\mathbf{A}\underline{X} + \underline{b}\big] = \mathbf{A}\,\mathbb{E}[\underline{X}] + \underline{b}$
- Covariance Matrix:

$$\mathbf{\Sigma}_{\underline{X}} = \mathbb{E}\Big[(\underline{X} - \mathbb{E}[\underline{X}])(\underline{X} - \mathbb{E}[\underline{X}])^{\mathsf{T}}\Big] = \begin{bmatrix} \mathsf{Cov}[X_1, X_1] & \cdots & \mathsf{Cov}[X_1, X_n] \\ \vdots & \ddots & \vdots \\ \mathsf{Cov}[X_n, X_1] & \cdots & \mathsf{Cov}[X_n, X_n] \end{bmatrix}$$

- Covariance of a Linear Transform:
  If $\underline{Y} = \mathbf{A}\underline{X} + \underline{b}$, then $\mathbf{\Sigma}_{\underline{Y}} = \mathbf{A}\mathbf{\Sigma}_{\underline{X}}\mathbf{A}^{\mathsf{T}}$.

**Gaussian Vectors:**

- A standard Gaussian vector is a random vector $\underline{Z}$ whose entries $Z_1, \ldots, Z_n$ are independent Gaussian$(0,1)$ random variables.
- A (jointly) Gaussian vector is a random vector $\underline{X}$ that can be written as a linear transform $\underline{X} = \mathbf{A}\underline{Z} + \underline{b}$ of a standard Gaussian vector $\underline{Z}$. However, this representation is usually left implicit, and the Gaussian vector distribution is specified by its mean vector $\underline{\mu}_{\underline{X}}$ and covariance matrix $\mathbf{\Sigma}_{\underline{X}}$.
- Shorthand notation: $\underline{X} \sim \mathcal{N}(\underline{\mu}_{\underline{X}}, \mathbf{\Sigma}_{\underline{X}})$
- A Gaussian vector $\underline{X}$ satisfies the following properties:
  - The entries of $\underline{X}$ are independent if and only if $\mathbf{\Sigma}_{\underline{X}}$ is a diagonal matrix. (Uncorrelatedness implies independence and vice versa.)
  - **A linear transformation is a Gaussian vector:**
    If $\underline{Y} = \mathbf{B}\underline{X} + \underline{c}$, then $\underline{Y} \sim \mathcal{N}(\mathbf{B}\underline{\mu}_{\underline{X}} + \underline{c}, \mathbf{B}\mathbf{\Sigma}_{\underline{X}}\mathbf{B}^{\mathsf{T}})$.

## *True/False Practice Questions*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that $X$ and $Y$ are continuous random variables.

# *True/False #1*

For every $x$, $f_X(x) \leq F_X(x)$.

False.

For example, say $X$ is an Exponential$(10)$ random variable. Then, $f_X(0) = 10$ whereas $F_X(x) \leq 1$ for all $x$.

## *True/False #2*

$\mathsf{Var}[X]\,\mathsf{Var}[Y] \geq \big(\mathsf{Cov}[X,Y]\big)^2$

True.

The correlation coefficient $\rho_{X,Y} = \frac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]}}$ satisfies $\rho_{X,Y} \leq 1$.

Therefore, by squaring both sides, we get $\frac{(\mathsf{Cov}[X,Y])^2}{\mathsf{Var}[X]\,\mathsf{Var}[Y]} \leq 1$ from which we can get the desired inequality.

# *True/False #3*

If $Y = aX + b$, then $\rho_{X,Y} = 1$.

False.

If $a < 0$, then $\rho_{X,Y} = -1$.

## *True/False #4*

If $g(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise,} \end{cases}$ then $\mathbb{E}[g(X)] = \mathbb{P}[A]$.

True.

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_A f_X(x)\, dx = \mathbb{P}[A]$$

## *True/False #5*

$f_{X|Y}(x|y) \geq f_{X,Y}(x,y)$ for all $(x,y)$.

False.

If $f_Y(y) > 0$, then $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ and $f_Y(y)$ can be larger than $1$.

## *True/False #6*

If $X$ and $Y$ are independent and $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, then $\mathbb{E}\big[(X+Y)^3\big] = \mathbb{E}[X^3] + \mathbb{E}[Y^3]$.

True.

$$\begin{aligned}
\mathbb{E}\big[(X+Y)^3\big] &= \mathbb{E}[X^3 + 3X^2Y + 3XY^2 + Y^3] \\
&= \mathbb{E}[X^3] + 3\mathbb{E}[X^2Y] + 3\mathbb{E}[XY^2] + \mathbb{E}[Y^3] \\
&= \mathbb{E}[X^3] + 3\mathbb{E}[X^2]\mathbb{E}[Y] + 3\mathbb{E}[X]\mathbb{E}[Y^2] + \mathbb{E}[Y^3] \\
&= \mathbb{E}[X^3] + \mathbb{E}[Y^3]
\end{aligned}$$

# *True/False #7*

If $Y = X^2$, then $F_Y(y) = F_X(\sqrt{y})$.

False.

$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[X^2 \leq y] = \mathbb{P}[-\sqrt{y} \leq X \leq \sqrt{y}]$. The relationship would hold if we assumed $X$ was non-negative so that we could ignore the "$-\sqrt{y} \leq X$" term.

# *Practice Question #1(a)*

Let $X$ be a Uniform$(1,3)$ random variable. Calculate $\mathbb{E}[X^3]$.

- First, we determine the PDF $f_X(x) = \begin{cases} \frac{1}{2} & 1 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$
- Next, we evaluate the expected value

$$\begin{aligned} \mathbb{E}[X^3] &= \int_{-\infty}^{\infty} x^3 f_X(x)\,dx \\ &= \int_1^3 \frac{1}{2} x^3\,dx \\ &= \left(\frac{x^4}{8}\right)\Big|_1^3 = \frac{81-1}{8} = 10 \end{aligned}$$

## *Practice Question #1(b)*

Let $X$ be a Gaussian random variable with $\mathbb{E}[X] = 2$ and $\mathbb{E}[X^2] = 9$. Determine the probability that $X$ falls between $2$ and $7$. Write your answer only in terms of the standard normal CDF $\Phi(z)$.

- First, we determine the variance $\text{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = 9 - 2^2 = 5$.
- Probability of an Interval for a Gaussian$(\mu, \sigma^2)$ is

$$\mathbb{P}[a \leq X \leq b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\Big(\frac{a-\mu}{\sigma}\Big)$$

$$\begin{aligned}\mathbb{P}[2 \leq X \leq 7] &= \Phi\left(\frac{7-2}{\sqrt{5}}\right) - \Phi\left(\frac{2-2}{\sqrt{5}}\right) \\ &= \Phi(\sqrt{5}) - \Phi(0) \\ &= \Phi(\sqrt{5}) - \frac{1}{2}\end{aligned}$$

# *Practice Question #1(c)*

Let $X$ be an Exponential$(3)$ random variable. Calculate $\mathbb{P}[9X + 2 \leq 8]$.

- The CDF of $X$ is $F_X(X) = \begin{cases} 1 - e^{-3x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$
- Using this, we find that

$$\mathbb{P}[9X + 2 \leq 8] = \mathbb{P}\left[X \leq \frac{8-2}{9}\right] = \mathbb{P}\left[X \leq \frac{2}{3}\right] = F_X\left(\frac{2}{3}\right) = 1 - e^{-2}$$

## *Practice Question #1(d)*

Let $X$ be a Discrete Uniform$(2, 4)$ random variable and let $Y$ given that $X = x$ be a Poisson$(1/x)$ random variable. Calculate $\mathbb{E}[Y]$.

- This is a scenario where the iterated expectation property $\mathbb{E}[Y] = \mathbb{E}\big[\mathbb{E}[Y|X]\big]$ is useful.
- Recall that the mean of a Poisson$(\alpha)$ random variable is $\alpha$. Therefore, since $Y$ given that $X = x$ is a Poisson$(1/x)$ random variable, $\mathbb{E}[Y|X = x] = \frac{1}{x}$.
- Now, applying the iterated expectation property, we get

$$\mathbb{E}[Y] = \mathbb{E}\big[\mathbb{E}[Y|X]\big] = \mathbb{E}\left[\frac{1}{X}\right] = \sum_{x \in S_X} \frac{1}{x} P_X(x) = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3} = \frac{13}{36}$$

where we used the fact that $P_X(x) = \begin{cases} \frac{1}{3} & x = 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases}$

## *Practice Question #1(e)*

Let $X$ and $Y$ be jointly Gaussian with $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = -1$, $\text{Var}[X] = 4$, $\text{Var}[Y] = 2$, and $\text{Cov}[X, Y] = -1$. Let $W = X + Y$ and $Z = 2X - Y$.

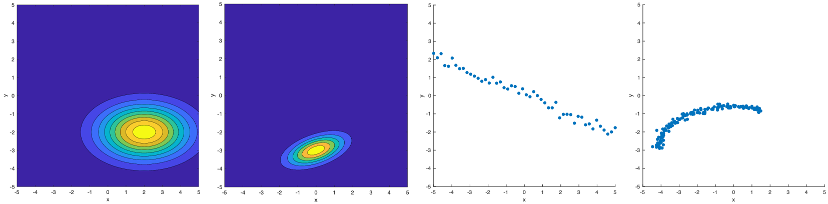
Calculate $\mathbb{E}[W + Z]$ and $\text{Cov}[W, Z]$.

- Linearity of Expectation: $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$
- $$\begin{aligned} \mathbb{E}[W + Z] &= \mathbb{E}[X + Y + 2X - Y] \\ &= \mathbb{E}[3X] = 3\mathbb{E}[X] = 3 \cdot 1 = 3 \end{aligned}$$
- Covariance of Linear Functions:
  $\text{Cov}[aX + bY + c, dX + eY + f]$
  $= ad\,\text{Var}[X] + be\text{Var}[Y] + (ae + bd)\text{Cov}[X, Y]$
- $\text{Cov}[W, Z]$
  $= \text{Cov}[X + Y, 2X - Y]$
  $= 1 \cdot 2 \cdot \text{Var}[X] + 1 \cdot (-1) \cdot \text{Var}[Y] + (1 \cdot (-1) + 1 \cdot 2)\text{Cov}[X, Y]$
  $= 2 \cdot 4 - 1 \cdot 2 + 1 \cdot (-1) = 8 - 2 - 1 = 5$

## *Practice Question #2*

List the scenarios that satisfy the specified criteria.

- a $\mathbb{E}[X]$ noticeably more than $0$:
- b $\mathbb{E}[Y]$ noticeably less than $0$:
- c $\mathrm{Var}[X]$ noticeably larger than $\mathrm{Var}[Y]$:
- d $\mathrm{Cov}[X, Y]$ noticeably more than $0$:
- e $\left|\rho_{X,Y}\right|$ close to $1$:

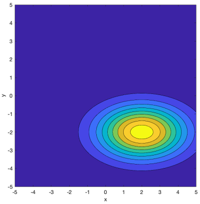


**Scenario 1** **Scenario 2** **Scenario 3** **Scenario 4**

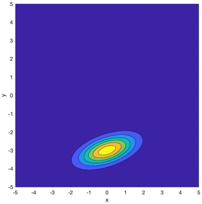
**Scenario 5:** $Y = \dfrac{X}{2} - 3$ where $\mathbb{E}[X] = 2$ and $\mathrm{Var}[X] = 4$.

List the scenarios that satisfy the specified criteria.

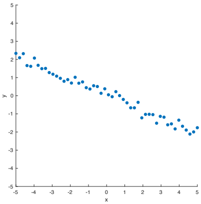
a. $\mathbb{E}[X]$ noticeably more than $0$: 1,5
b. $\mathbb{E}[Y]$ noticeably less than $0$: 1,2,4,5
c. Var$[X]$ noticeably larger than Var$[Y]$: 1,2,3,4,5
d. Cov$[X, Y]$ noticeably more than $0$: 2,4,5
e. $\left|\rho_{X,Y}\right|$ close to $1$: 3,5



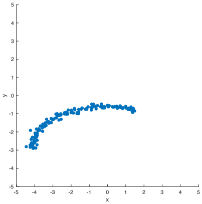
**Scenario 1**



**Scenario 2**



**Scenario 3**



**Scenario 4**

**Scenario 5:** $Y = \dfrac{X}{2} - 3$ where $\mathbb{E}[X] = 2$ and Var$[X] = 4$.

## *Practice Question #3*

Consider the following PDF $f_X(x) = \begin{cases} c(x+1) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$

- a Determine the value of $c$ that satisfies the normalization property. Set $c$ to this value for the remainder of the problem.
- b What is the expected value of $X$?
- c What is the variance of $X$?
- d What is the probability that the absolute value $|X|$ exceeds $1/2$?
- e Calculate $\mathbb{E}[X|B]$ where $B$ is the event that the absolute value $|X|$ exceeds $1/2$.
- f Determine the CDF of $X$.

# *Practice Question #3(a) Solution*

Consider the following PDF $f_X(x) = \begin{cases} c(x+1) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

a. Determine the value of $c$ that satisfies the normalization property. Set $c$ to this value for the remainder of the problem.

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\,dx &= \int_{-1}^{1} c\,(x+1)\,dx \\ &= c\left(\frac{1}{2}x^2 + x\right)\Bigg|_{-1}^{1} \\ &= c\left(\frac{1}{2}\cdot 1^2 + 1\right) - c\left(\frac{1}{2}\cdot(-1)^2 - 1\right) = 2c \end{aligned}$$

so we should set $c = 1/2$ to ensure the total probability is $1$.

# *Practice Question #3(b),(c) Solution*

Consider the following PDF $f_X(x) = \begin{cases} c(x+1) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$

**b** What is the expected value of $X$?

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-1}^{1} \frac{1}{2}(x^2 + x)\, dx$$

$$= \frac{1}{2}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Bigg|_{-1}^{1} = \frac{1}{2}\left(\left(\frac{1}{3} + \frac{1}{2}\right) - \left(-\frac{1}{3} + \frac{1}{2}\right)\right) = \frac{1}{3}$$

**c** What is the variance of $X$?

Since $\text{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$, we only need to calculate $\mathbb{E}[X^2]$.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx = \int_{-1}^{1} \frac{1}{2}(x^3 + x^2)\, dx$$

$$= \frac{1}{2}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\Bigg|_{-1}^{1} = \frac{1}{2}\left(\left(\frac{1}{4} + \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right) = \frac{1}{3}$$

$$\text{Var}[X] = \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{2}{9}$$

# *Practice Question #3(d) Solution*

Consider the following PDF $f_X(x) = \begin{cases} c(x+1) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$

**d** What is the probability that the absolute value $|X|$ exceeds $1/2$?

We start by writing this event in terms of intervals,
$B = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$. Now, we have that

$$\begin{aligned}
\mathbb{P}[X \in B] &= \int_B f_X(x)\,dx \\
&= {\color{blue}\int_{-1}^{-1/2} \frac{1}{2}(x+1)\,dx + \int_{1/2}^{1} \frac{1}{2}(x+1)\,dx} \\
&= \frac{1}{2}\left(\frac{1}{2}x^2 + x\right)\Bigg|_{-1}^{-1/2} + \frac{1}{2}\left(\frac{1}{2}x^2 + x\right)\Bigg|_{1/2}^{1} \\
&= \frac{1}{2}\left(\left(\frac{1}{8} - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)\right) + \frac{1}{2}\left(\left(\frac{1}{2} + 1\right) - \left(\frac{1}{8} + \frac{1}{2}\right)\right) \\
&= \frac{1}{2}
\end{aligned}$$

Consider the following PDF $f_X(x) = \begin{cases} c(x+1) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

- (e) Calculate $\mathbb{E}[X|B]$ where $B$ is the event that the absolute value $|X|$ exceeds $1/2$. We need the conditional PDF of $X$ given the event $B$,

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} x+1 & -1 \leq x < -\frac{1}{2},\ \frac{1}{2} < x \leq 1 \\ 0 & \text{o/w.} \end{cases}$$

Now, we have that

$$\begin{aligned} \mathbb{E}[X|B] &= \int_{-\infty}^{\infty} x f_{X|B}(x)\,dx = \int_{-1}^{-1/2} (x^2+x)\,dx + \int_{1/2}^{1} (x^2+x)\,dx \\ &= \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_{-1}^{-1/2} + \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_{1/2}^{1} \\ &= \left(-\frac{1}{24} + \frac{1}{8}\right) - \left(-\frac{1}{3} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{2}\right) - \left(\frac{1}{24} + \frac{1}{8}\right) = \frac{7}{12} \end{aligned}$$

Consider the following PDF $f_X(x) = \begin{cases} c(x+1) & -1 \leq x \leq 1 \\ 0 & \text{otherwise}. \end{cases}$

f. Determine the CDF of $X$.
   Since the range is $R_X = [-1, 1]$, we know that $F_X(x)$ is equal to $0$ before $x = -1$ and equal to $1$ after $x = 1$. In between, we have that

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = \int_{-1}^{x} \frac{1}{2}(u+1)\, du = \frac{1}{2}\left(\frac{1}{2}u^2 + u\right)\Big|_{-1}^{x}$$
$$= \frac{1}{2}\left(\left(\frac{1}{2}x^2 + x\right) - \left(\frac{1}{2} - 1\right)\right) = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}$$

   Putting everything together, the CDF is

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4} & -1 \leq x < 1 \\ 1 & 1 \leq x\,. \end{cases}$$

# *Practice Question #4*

Consider the following joint PMF for $X$ and $Y$:

| $P_{X,Y}(x,y)$ | | $y = -2$ | $y = -1$ | $y = 1$ | $y = 2$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | $1/8$ | $1/8$ | $0$ | $0$ |
| | $-1$ | $1/8$ | $0$ | $1/8$ | $0$ |
| | $1$ | $0$ | $1/8$ | $0$ | $1/8$ |
| | $2$ | $0$ | $0$ | $1/8$ | $1/8$ |

a. What is the probability that $X$ is less than or equal to $2Y$?
b. Are $X$ and $Y$ independent?
c. Determine the marginal PMFs of $X$ and $Y$.
d. Write down the conditional PMF of $X$ given $Y$ as a table.
e. Calculate the covariance of $X$ and $Y$.
f. Calculate the correlation coefficient of $X$ and $Y$.
g. Determine the conditional expectation $\mathbb{E}[X|Y=y]$.

# *Practice Question #4(a),(b) Solution*

| $P_{X,Y}(x,y)$ | | $y=-2$ | $y=-1$ | $y=1$ | $y=2$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | $1/8$ | $1/8$ | $0$ | $0$ |
| | $-1$ | $1/8$ | $0$ | $1/8$ | $0$ |
| | $1$ | $0$ | $1/8$ | $0$ | $1/8$ |
| | $2$ | $0$ | $0$ | $1/8$ | $1/8$ |

(a) What is the probability that $X$ is less than or equal to $2Y$?
First, we write the event in terms of pairs
$A = \{(x,y) \in R_{X,Y} : x \leq 2y\} =$
$\{(-2,-1), (-2,1), (-2,2), (-1,1), (-1,2), (1,1), (1,2), (2,1), (2,2)\}$.
Now, we add up the probabilities of these pairs,

$$\mathbb{P}[X \leq 2Y] = \mathbb{P}[X \in A] = \sum_{(x,y)\in A} P_{X,Y}(x,y)$$
$$= \frac{1}{8} + 0 + 0 + \frac{1}{8} + 0 + 0 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{8} \; .$$

(b) Are $X$ and $Y$ independent?
No, they are not independent, since the joint PMF table contains a zero for which the associated column and row are both non-zero.

# *Practice Question #4(c),(d) Solution*

| $P_{X,Y}(x,y)$ | | $y=-2$ | $y=-1$ | $y=1$ | $y=2$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | $1/8$ | $1/8$ | $0$ | $0$ |
| | $-1$ | $1/8$ | $0$ | $1/8$ | $0$ |
| | $1$ | $0$ | $1/8$ | $0$ | $1/8$ |
| | $2$ | $0$ | $0$ | $1/8$ | $1/8$ |

- (c) Determine the marginal PMFs of $X$ and $Y$.

  Summing up the columns, we obtain the marginal PMF of $X$ and, summing up the rows, we obtain the marginal PMF of $Y$,

$$P_X(x) = \begin{cases} 1/4 & x = -2, -1, 1, 2 \\ 0 & \text{otherwise.} \end{cases} \quad P_Y(y) = \begin{cases} 1/4 & y = -2, -1, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

- (d) Write down the conditional PMF of $X$ given $Y$ as a table.

  Using the fact that $P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$, we get

| $P_{X\|Y}(x\|y)$ | | $y=-2$ | $y=-1$ | $y=1$ | $y=2$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | $1/2$ | $1/2$ | $0$ | $0$ |
| | $-1$ | $1/2$ | $0$ | $1/2$ | $0$ |
| | $1$ | $0$ | $1/2$ | $0$ | $1/2$ |
| | $2$ | $0$ | $0$ | $1/2$ | $1/2$ |

| $P_{X,Y}(x,y)$ | | $y=-2$ | $y=-1$ | $y=1$ | $y=2$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | $1/8$ | $1/8$ | $0$ | $0$ |
| | $-1$ | $1/8$ | $0$ | $1/8$ | $0$ |
| | $1$ | $0$ | $1/8$ | $0$ | $1/8$ |
| | $2$ | $0$ | $0$ | $1/8$ | $1/8$ |

e. Calculate the covariance of $X$ and $Y$.

$$\mathbb{E}[X] = \sum_{x \in R_X} x P_X(x) = (-2 - 1 + 1 + 2) \cdot \frac{1}{4} = 0$$

$$\mathbb{E}[Y] = \sum_{y \in R_Y} y P_Y(y) = (-2 - 1 + 1 + 2) \cdot \frac{1}{4} = 0$$

$$\mathbb{E}[XY] = \sum_{(x,y) \in R_{X,Y}} xy P_{X,Y}(x,y)$$

$$= \big((-2)(-2) + (-2)(-1) + (-1)(-2) + (-1)(1) + (1)(-1) + (1)(2) + (2)(1) + (2)(2)\big) \cdot \frac{1}{8} = \frac{7}{4}$$

$$\mathsf{Cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{7}{4}$$

# *Practice Question #4(f) Solution*

| $P_{X,Y}(x,y)$ | | $y=-2$ | $y=-1$ | $y=1$ | $y=2$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | $1/8$ | $1/8$ | $0$ | $0$ |
| | $-1$ | $1/8$ | $0$ | $1/8$ | $0$ |
| | $1$ | $0$ | $1/8$ | $0$ | $1/8$ |
| | $2$ | $0$ | $0$ | $1/8$ | $1/8$ |

(f) Calculate the correlation coefficient of $X$ and $Y$.

$$\mathbb{E}[X^2] = \sum_{x \in R_X} x^2 P_X(x) = \left((-2)^2 + (-1)^2 + 1^2 + 2^2\right) \cdot \frac{1}{4} = \frac{5}{2}$$

$$\mathbb{E}[Y^2] = \sum_{y \in R_Y} y^2 P_Y(y) = \left((-2)^2 + (-1)^2 + 1^2 + 2^2\right) \cdot \frac{1}{4} = \frac{5}{2}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2 = \frac{5}{2}$$

$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - \left(\mathbb{E}[Y]\right)^2 = \frac{5}{2}$$

$$\rho_{X,Y} = \frac{\mathsf{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} = \frac{\frac{7}{4}}{\sqrt{\frac{5}{2} \cdot \frac{5}{2}}} = \frac{7}{10}$$

# Practice Question #4(g) Solution

| $P_{X,Y}(x,y)$ | | $y=-2$ | $y=-1$ | $y=1$ | $y=2$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | 1/8 | 1/8 | 0 | 0 |
| | $-1$ | 1/8 | 0 | 1/8 | 0 |
| | 1 | 0 | 1/8 | 0 | 1/8 |
| | 2 | 0 | 0 | 1/8 | 1/8 |

- (g) Determine the conditional expectation $\mathbb{E}[X|Y=y]$.
  This can be viewed as working out the expectation under $P_{X|Y}(x|y)$ for each possible $y$,

$$\mathbb{E}[X|Y=y] = \sum_{x \in R_X} x P_{X|Y}(x|y) = \begin{cases} (-2-1)\cdot\frac{1}{2} & y=-2 \\ (-2+1)\cdot\frac{1}{2} & y=-1 \\ (-1+2)\cdot\frac{1}{2} & y=1 \\ (1+2)\cdot\frac{1}{2} & y=2 \end{cases}$$

$$= \begin{cases} -\frac{3}{2} & y=-2 \\ -\frac{1}{2} & y=-1 \\ \frac{1}{2} & y=1 \\ \frac{3}{2} & y=2 \end{cases}$$

## *Practice Question #5*

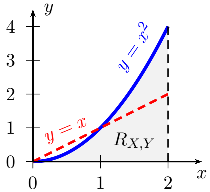
Consider the joint PDF $f_{X,Y}(x,y) = \begin{cases} \dfrac{3\,xy}{16} & 0 \leq y \leq x^2 \leq 4,\ \ x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

- a Draw a sketch of the range $R_{X,Y}$.
- b Are $X$ and $Y$ independent?
- c What is the probability that $Y$ is greater than $X$?
- d Determine the marginal PDFs $f_X(x)$ and $f_Y(y)$.
- e Calculate Cov$[X,Y]$.
- f Determine the conditional PDF $f_{Y|X}(y|x)$.
- g Calculate the conditional expected value $\mathbb{E}[Y|X=x]$.
- h Calculate Var$[7X-5Y+8]$.

# *Practice Question #5(a),(b),(c) Solution*

Consider the joint PDF $f_{X,Y}(x,y) = \begin{cases} \dfrac{3\,xy}{16} & 0 \le y \le x^2 \le 4,\ x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$

a. Draw a sketch of the range $R_{X,Y}$.




b. Are $X$ and $Y$ independent?
No, they are not independent, since the range is not a rectangle.

c. What is the probability that $Y$ is greater than $X$?
First, we can write this as an event
$A = \{(x,y) \in R_{X,Y} : y \ge x\} = \{1 \le X \le 2,\ X \le Y \le X^2\}$. See above for an illustration.

$$\mathbb{P}[(X,Y) \in A] = \iint_A f_{X,Y}(x,y)\,dy\,dx = \int_1^2 \int_x^{x^2} \frac{3\,xy}{16}\,dy\,dx$$

# *Practice Question #5(d) Solution*

Consider the joint PDF $f_{X,Y}(x,y) = \begin{cases} \dfrac{3\,xy}{16} & 0 \le y \le x^2 \le 4,\ x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$

**d** Determine the marginal PDFs $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \displaystyle\int_0^{x^2} \frac{3\,xy}{16}\,dy & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \displaystyle\int_{\sqrt{y}}^{2} \frac{3\,xy}{16}\,dx & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Consider the joint PDF $f_{X,Y}(x,y) = \begin{cases} \dfrac{3\,xy}{16} & 0 \le y \le x^2 \le 4,\ x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$

(e) Calculate Cov$[X, Y]$.

$$\text{Cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^2 \frac{3x^6}{32}\, dx$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_0^4 \left(\frac{3y}{8} - \frac{3y^2}{32}\right) dy$$

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dy\, dx = \int_0^2 \int_0^{x^2} \frac{3\,x^2y^2}{16}\, dy\, dx$$

# *Practice Question #5(f),(g) Solution*

Consider the joint PDF $f_{X,Y}(x,y) = \begin{cases} \dfrac{3\,xy}{16} & 0 \le y \le x^2 \le 4,\ \ x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$

- (f) Determine the conditional PDF $f_{Y|X}(y|x)$.

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_X(x)} & (x,y) \in R_{X,Y} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{\frac{3\,xy}{16}}{f_X(x)} & 0 \le y \le x^2 \le 4,\ \ x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

where $f_X(x)$ was determined in part (d).

- (g) Calculate the conditional expected value $\mathbb{E}[Y|X=x]$.

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy = \int_{0}^{x^2} y\, f_{Y|X}(y|x)\, dy$$

where $f_{Y|X}(y|x)$ was determined in part (f).

Consider the joint PDF $f_{X,Y}(x,y) = \begin{cases} \dfrac{3\,xy}{16} & 0 \le y \le x^2 \le 4,\ \ x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$

**h** Calculate $\mathrm{Var}[7X - 5Y + 8]$.
We can use the formula
$\mathrm{Var}[aX + bY + c] = a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y] + 2ab\mathrm{Cov}[X,Y]$, but first we need to calculate the individual variances.

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$$

$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - \big(\mathbb{E}[Y]\big)^2$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx = \int_0^2 x^2\, f_X(x)\, dx$$

$$\mathbb{E}[Y^2] = \int_{-\infty}^{\infty} y^2\, f_Y(y)\, dy = \int_0^4 y^2\, f_Y(y)\, dy$$

$$\mathrm{Var}[7X - 5Y + 8] = 7^2 \cdot \mathrm{Var}[X] + (-5)^2 \cdot \mathrm{Var}[Y] + 2 \cdot 7 \cdot (-5) \cdot \mathrm{Cov}[X,Y]$$

where $f_X(x)$ and $f_Y(y)$ were determined in part (d), $\mathbb{E}[X]$, $\mathbb{E}[Y]$, and $\mathrm{Cov}[X,Y]$ were determined in part (e).

## *Practice Question #6 and Solution*

Let $\underline{X}$ be a Gaussian vector with mean vector and covariance matrix

$$\underline{\mu}_{\underline{X}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \mathbf{\Sigma}_{\underline{X}} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$

Let $\underline{Y} = [\,1 \;\; 1\,]\underline{X} - 3$.

a. Determine the mean of $Y$.

Linearity of Expectation: $\mathbb{E}[\mathbf{A}\underline{X} + \underline{b}] = \mathbf{A}\mathbb{E}[\underline{X}] + \underline{b}$

$\mathbb{E}[Y] = \mathbb{E}\big[[\,1 \;\; 1\,]\underline{X} - 3\big]$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} - 3 = 1 \cdot (-2) + 1 \cdot 1 - 3 = -4$$

b. Determine the variance of $Y$.

Covariance of a Linear Transform: $\mathbf{\Sigma}_{\underline{Y}} = \mathbf{A}\mathbf{\Sigma}_{\underline{X}}\mathbf{A}^{\mathsf{T}}$

Viewing $Y$ as a length-one vector, $\mathbf{\Sigma}_{\underline{Y}} = [\mathrm{Var}[Y]]$ is a $1 \times 1$ matrix.

$$\mathrm{Var}[Y] = \mathbf{\Sigma}_{\underline{Y}} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 4 - 2 \\ -2 + 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2$$