Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

(a) If $\mathbb{P}[A|B] = \mathbb{P}[B|A]$, then $\mathbb{P}[A] = \mathbb{P}[B]$.

Solution:

False. If $\mathbb{P}[A|B] = \mathbb{P}[B|A] = 0$, then $\mathbb{P}[A]$ may not be equal to $\mathbb{P}[B]$.

(b) If $A \cap B = \phi$, $A \cap C = \phi$, and $B \cap C = \phi$, then $\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + P[C]$.

Solution:

True. The events A, B, and C are mutually exclusive so we can apply the Additivity Axiom to get $\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + P[C]$.

(c) If A and B are independent events, then A^{c} and B^{c} are independent events.

Solution:

True.

$$\begin{aligned} \mathbb{P}[A^{\mathsf{c}} \cap B^{\mathsf{c}}] &= 1 - \mathbb{P}[A \cup B] \\ &= 1 - (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) \\ \text{(by independence)} \quad &= 1 - \mathbb{P}[A] - \mathbb{P}[B] + \mathbb{P}[A]\,\mathbb{P}[B] \\ &= (1 - \mathbb{P}[A])(1 - \mathbb{P}[B]) \\ &= \mathbb{P}[A^{\mathsf{c}}]\,\mathbb{P}[B^{\mathsf{c}}] \implies A^{\mathsf{c}}, B^{\mathsf{c}} \text{ are independent events} \end{aligned}$$

(d) It is possible that $\mathbb{P}[B|A] > \mathbb{P}[B]$ (for certain choices of A and B).

Solution:

True. If $A \subset B$, then $\mathbb{P}[B|A] = 1$, but $\mathbb{P}[B]$ could be less than 1.

(e) If $\mathbb{P}[A] = 0.7$, $\mathbb{P}[B] = 0.4$, and $\mathbb{P}[A \cup B] = 0.8$, then $\mathbb{P}[A^{\mathsf{c}} \cap B^{\mathsf{c}}] = 0.2$.

Solution:

True. $\mathbb{P}[A^{\mathsf{c}} \cap B^{\mathsf{c}}] = \mathbb{P}[A \cup B]^{\mathsf{c}} = 1 - \mathbb{P}[A \cup B] = 0.2$

Problem 2 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) $\mathbb{P}[X^2 > 0] = \mathbb{P}[X > 0]$.

Solution:

False since X can be negative. For example, if $P_X(-1) = P_X(+1) = 1/2$, then $\mathbb{P}[X^2 > 0] = 1$ while $\mathbb{P}[X > 0] = 1/2$.

(b) For a Poisson(λ) random variable, $\mathbb{E}[X^2] = \lambda(1+\lambda)$.

Solution:

True. We know that for $\mathbb{E}[X] = \lambda$ and $\mathsf{Var}[X] = \lambda$. By the alternate variance formula, $\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$ so $\mathbb{E}[X^2] = \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 = \lambda + \lambda^2 = \lambda(1+\lambda)$.

(c) If the range of X is integer-valued, then the mean $\mathbb{E}[X]$ is integer-valued as well.

Solution:

False. As a counterexample, let $X \sim \text{Bernoulli}(1/3)$. Note that the range $R_X = \{0, 1\}$ is integer-valued but the mean $\mathbb{E}[X] = 1/3$ is not.

(d) For a subset B of the range R_X, we have that $P_{X|B}(x) \geq P_X(x)$ for every $x \in B$.

Solution:

True. $P_{X|B}(x) = \dfrac{P_X(x)}{\mathbb{P}[X \in B]} \geq P_X(x)$ since the denominator $\mathbb{P}[X \in B] \leq 1$.

(e) $\mathbb{E}[X] = \mathbb{E}[X|B]\,\mathbb{P}[X \in B] + \mathbb{E}[X|B^{\mathsf{c}}]\,\mathbb{P}[X \in B^{\mathsf{c}}]$

Solution:

True.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x \in R_X} x\, P_X(x) \\
&= \sum_{x \in B} x\, P_X(x) + \sum_{x \in B^{\mathsf{c}}} x\, P_X(x) \\
&= \mathbb{P}[X \in B] \sum_{x \in B} x\, \frac{P_X(x)}{\mathbb{P}[X \in B]} + \mathbb{P}[X \in B^{\mathsf{c}}] \sum_{x \in B^{\mathsf{c}}} x\, \frac{P_X(x)}{\mathbb{P}[X \in B^{\mathsf{c}}]} \\
&= \mathbb{P}[X \in B] \sum_{x \in B} x\, P_{X|B}(x) + \mathbb{P}[X \in B^{\mathsf{c}}] \sum_{x \in B^{\mathsf{c}}} x\, P_{X|B^{\mathsf{c}}}(x) \\
&= \mathbb{E}[X|B]\,\mathbb{P}[X \in B] + \mathbb{E}[X|B^{\mathsf{c}}]\,\mathbb{P}[X \in B^{\mathsf{c}}]
\end{aligned}$$

Problem 3 Please complete the following quick calculations. *16 points*

(a) Let the events B_1, B_2, B_3 be a partition, with $\mathbb{P}[B_1] = 0.4$, $\mathbb{P}[B_2] = 0.1$, and $\mathbb{P}[B_3] = 0.5$. We know that for event A we have $\mathbb{P}[A|B_1] = 0.2$, $\mathbb{P}[A|B_2] = 0.7$, and $\mathbb{P}[A|B_3] = 0.5$. Calculate $\mathbb{P}[A]$ and $\mathbb{P}[B_3|A]$.

Solution:

By the Law of Total Probability,

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[A|B_1]\,\mathbb{P}[B_1] + \mathbb{P}[A|B_2]\,\mathbb{P}[B_2] + \mathbb{P}[A|B_3]\,\mathbb{P}[B_3] \\ &= 0.2 \cdot 0.4 + 0.7 \cdot 0.1 + 0.5 \cdot 0.5 = 0.08 + 0.07 + 0.25 = 0.4 = \frac{2}{5}\end{aligned}$$

By Bayes' Rule, $\mathbb{P}[B_3|A] = \dfrac{\mathbb{P}[A|B_3]\mathbb{P}[B_3]}{\mathbb{P}[A]} = \dfrac{0.5 \cdot 0.5}{0.4} = \dfrac{\frac{1}{4}}{\frac{2}{5}} = \dfrac{5}{8} = 0.625$.

(b) Let $\mathbb{P}[A] = 0.4$, $\mathbb{P}[B] = 0.5$, $\mathbb{P}[C] = 0.3$, $\mathbb{P}[A|B] = 0.3$, $\mathbb{P}[A|C] = 0.2$, $\mathbb{P}[B|C] = 0.5$, and $\mathbb{P}[A \cap B|C] = 0.1$. Are A and B independent? Are A and B conditionally independent given C?

Solution:

A and B are not independent since $\mathbb{P}[A|B] \neq \mathbb{P}[A]$. A and B are conditionally independent given C since $\mathbb{P}[A|C]\,\mathbb{P}[B|C] = 0.2 \cdot 0.5 = 0.1 = \mathbb{P}[A \cap B|C]$.

(c) Let Y be Geometric$(1/4)$. Calculate $\mathsf{Var}[Y + 4]$ and $\mathbb{E}[3Y + 0.2]$.

Solution:

The variance of a linear function is $\mathsf{Var}[aY + b] = a^2\mathsf{Var}[Y]$.
Therefore, $\mathsf{Var}[Y + 4] = \mathsf{Var}[Y] = \frac{1-p}{p^2} = \frac{3/4}{(1/4)^2} = 12$.
By linearity of expectation, $\mathbb{E}[3Y + 0.2] = 3\mathbb{E}[Y] + 0.2 = 3 \cdot \frac{1}{p} + 0.2 = 3 \cdot 4 + 0.2 = 12.2$.

(d) Let X be Discrete Uniform$(1, 3)$. Calculate $\mathbb{E}[X^3]$ as well as the conditional expected value $\mathbb{E}[X|B]$ of X given that $\{X \in B\}$ for $B = \{1, 2\}$.

Solution:

$$\mathbb{E}[X^3] = \sum_{x \in R_X} x^3\, P_X(x) = 1^3 \cdot \frac{1}{3} + 2^3 \cdot \frac{1}{3} + 3^3 \cdot \frac{1}{3} = \frac{1}{3} + \frac{8}{3} + 9 = 12$$

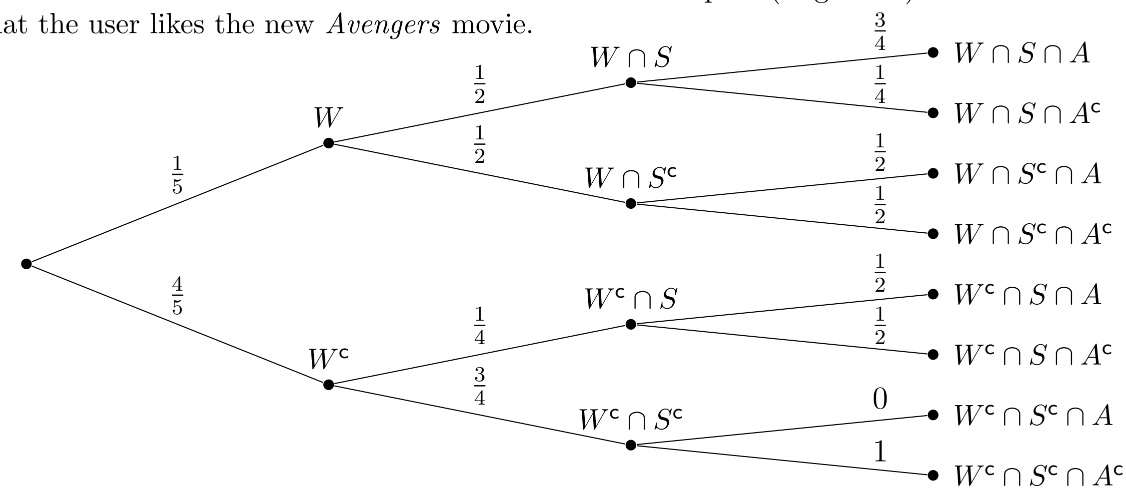
The conditional PMF is $P_{X|B}(x) = \begin{cases} 1/2 & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$ so

$$\mathbb{E}[X|B] = \sum_{x \in B} x\, P_{X|B}(x) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2}.$$

(You could have also recognized this conditional PMF as a Discrete Uniform(1,2) random variable, for which the mean is $\frac{1+2}{2} = \frac{3}{2}$.)

Problem 4 *16 points*

You are trying to predict whether a new user of your movie recommendation system will like the latest *Avengers* release, "The Avengers Sequel 23." Let W be the event that the user has a widescreen TV. Let S be the event that the user likes sequels (in general). Let A be the event that the user likes the new *Avengers* movie.



(a) What is the probability that a new user likes sequels **and** likes the new *Avengers* movie?

Solution:

$$\begin{aligned}\mathbb{P}[S \cap A] &= \mathbb{P}[W \cap S \cap A] + \mathbb{P}[W^c \cap S \cap A] \\ &= \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{3}{4} + \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{40} + \frac{4}{40} = \frac{7}{40}\end{aligned}$$

(b) What is the probability that a new user likes the new *Avengers* movie?

Solution:

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[W \cap S \cap A] + \mathbb{P}[W \cap S^c \cap A] + \mathbb{P}[W^c \cap S \cap A] + \mathbb{P}[W^c \cap S^c \cap A] \\ &= \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{1}{2} + \frac{4}{5} \cdot \frac{3}{4} \cdot 0 \\ &= \frac{3}{40} + \frac{1}{20} + \frac{4}{40} + 0 = \frac{9}{40}\end{aligned}$$

(c) Given that a new user has a widescreen, what is the probability that they **do not** like the new *Avengers* movie?

Solution:

$$\mathbb{P}[A^c | W] = \mathbb{P}[S \cap A^c | W] + \mathbb{P}[S^c \cap A^c | W] = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

(d) Given that a person likes the new *Avengers* movie, what is the probability that they have a widescreen?

Solution:

Using Bayes' Rule,

$$\mathbb{P}[W | A] = \frac{\mathbb{P}[A|W]\,\mathbb{P}[W]}{\mathbb{P}[A]} = \frac{(1 - \mathbb{P}[A^c|W])\,\mathbb{P}[W]}{\mathbb{P}[A]} = \frac{(1 - \frac{3}{8}) \cdot \frac{1}{5}}{\frac{9}{40}} = \frac{\frac{5}{40}}{\frac{9}{40}} = \frac{5}{9}$$

Problem 5 *16 points*

Consider the following PMF where β is a constant greater than 0.

$$P_X(x) = \begin{cases} 1/8 & x = 1 \\ 1/2 & x = 2 \\ \beta & x = 5 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of β that makes $P_X(x)$ a valid PMF. Use this specific value for the rest of the problem. (If you cannot figure this out, just pick any value and proceed.)

Solution:

By the normalization property, we must have that $\sum_{x \in R_X} P_X(x) = 1$. Therefore,

$$P_X(1) + P_X(2) + P_X(5) = 1 \implies \frac{1}{8} + \frac{1}{2} + \beta = 1 \implies \beta = 1 - \frac{1}{8} - \frac{1}{2} = \frac{3}{8}$$

(b) Calculate $\mathsf{Var}[X]$

Solution:

$$\mathbb{E}[X] = \sum_{x \in R_X} x\, P_X(x) = 1 \cdot \frac{1}{8} + 2 \cdot \frac{1}{2} + 5 \cdot \frac{3}{8} = 3$$

$$\mathbb{E}[X^2] = \sum_{x \in R_X} x^2\, P_X(x) = 1^2 \cdot \frac{1}{8} + 2^2 \cdot \frac{1}{2} + 5^2 \cdot \frac{3}{8} = \frac{92}{8} = \frac{23}{2}$$

$$\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = \frac{23}{2} - 3^2 = \frac{23}{2} - \frac{18}{2} = \frac{5}{2}$$

(c) Calculate $\mathbb{E}\left[\frac{1}{X}\right]$.

Solution:

$$\mathbb{E}\left[\frac{1}{X}\right] = \sum_{x \in R_X} \frac{1}{x} \cdot P_X(x) = \frac{1}{1} \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{3}{8} = \frac{5}{40} + \frac{10}{40} + \frac{3}{40} = \frac{9}{20}$$

(d) Calculate the conditional expected value $\mathbb{E}[X|B]$ given that X falls into $B = \{1, 5\}$.

Solution:

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\sum_{x \in B} P_X(x)} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \frac{P_X(1)}{P_X(1) + P_X(5)} & x = 1 \\ \frac{P_X(5)}{P_X(1) + P_X(5)} & x = 5 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{4} & x = 1 \\ \frac{3}{4} & x = 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X|B] = \sum_{x \in B} x\, P_{X|B}(x) = 1 \cdot \frac{1}{4} + 5 \cdot \frac{3}{4} = 4$$

Problem 6 *16 points*

You are playing a (slightly boring) lottery game where you press a button five times. Each time you press the button, the screen displays a red icon with probability $2/3$ or a green icon with probability $1/3$. Each button press is independent. If after five presses, you get exactly three green and two red icons, you win.

(a) What is the probability of seeing the following sequence: G R G G R?

> **Solution:**
>
> Button presses are independent so the probability is
>
> $$\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{3^5} = \frac{4}{243}$$

(b) What is the probability of winning? (i.e., of getting three greens and two reds)?

> **Solution:**
>
> There are $\binom{5}{2} = \frac{5!}{2!\,3!} = 10$ ways of choosing two reds out of the five spots. Each of these sequences has the same probability $(\frac{1}{3})^3(\frac{2}{3})^2 = \frac{4}{243}$. Overall, the probability of winning is
>
> $$\binom{5}{2}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{40}{243}.$$

(c) Given that your first two icons are red, what is the probability of winning?

> **Solution:**
>
> The only way to win is to get three greens for the remaining three icons: $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$

(d) If it costs $\$1$ to play and you get $\$10$ if you win, what is the average amount of money you will receive by playing this game?

> **Solution:**
>
> Let X be a Bernoulli$(\frac{40}{243})$ random variable that is 0 if you lose and 1 if you win. Then, the average money you receive by playing is
>
> $$\mathbb{E}[10X - 1] = 10\mathbb{E}[X] - 1 = 10 \cdot \frac{40}{243} - 1 = \frac{400 - 243}{243} = \frac{157}{243}$$
>
> (This is not a wise game for the casino to host.)

Problem 7 *16 points*

You are measuring the number of spikes from a neuron in a one-second window. The resulting random variable X is Poisson(λ).

(a) After careful study, you have determined that the average number of spikes observed from this neuron in one second is $\mathbb{E}[X] = 2$. What is the probability that you see no spikes at all in a one-second window?

Solution:

For a Poisson(λ) random variable, we know that $\mathbb{E}[X] = \lambda$. Therefore, $\lambda = 2$ and we have that $\mathbb{P}[X = 0] = P_X(0) = \frac{2^0}{0!}e^{-2} = e^{-2}$.

(b) What is the probability that the number of spikes you see in a one-second window is *less than or equal* to average? (Recall from (a) that the average is 2.)

Solution:

$$P[X \leq 2] = P_X(0) + P_X(1) + P_X(2) = \left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right)e^{-2} = 5e^{-2}$$

(c) Calculate $\mathbb{E}[3X^2 + 2X + 1]$.

Solution:

Using the alternate variance formula, we know that $\mathbb{E}[X^2] = \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 = \lambda + \lambda^2 = 2 + 2^2 = 6$. By linearity of expectation, we have

$$\mathbb{E}[3X^2 + 2X + 1] = 3\mathbb{E}[X^2] + 2\mathbb{E}[X] + 1 = 3 \cdot 6 + 2 \cdot 2 + 1 = 23$$

(d) Given that the number of spikes in a one-second window is *less than or equal* to average, what is the conditional expected value of X?

Solution:

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\sum_{x \in B} P_X(x)} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \frac{e^{-2}}{5e^{-2}} & x = 0 \\ \frac{2e^{-2}}{5e^{-2}} & x = 1 \\ \frac{2e^{-2}}{5e^{-2}} & x = 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{5} & x = 0 \\ \frac{2}{5} & x = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X|B] = \sum_{x \in B} x\, P_{X|B}(x) = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 2 \cdot \frac{2}{5} = \frac{6}{5}$$