**Problem 1** *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that $A$, $B$, and $C$ are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

(a) $\mathbb{P}[A \cap B] = \mathbb{P}[A] - \mathbb{P}[A \cap B^{\mathsf{c}}]$

**Solution:**

**True.** Since $A \cap B$ and $A \cap B^{\mathsf{c}}$ are mutually exclusive, we have by additivity that $\mathbb{P}[A] = \mathbb{P}[(A \cap B) \cup (A \cap B^{\mathsf{c}})] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap B^{\mathsf{c}}]$. The statement follows by solving for $\mathbb{P}[A \cap B]$.

(b) $\mathbb{P}[A|B] + \mathbb{P}[A|B^{\mathsf{c}}] = 1$

**Solution:**

**False.** Consider the following counterexample: $\Sigma = \{0, 1, 2\}$ with $\mathbb{P}[\{0\}] = \mathbb{P}[\{1\}] = \mathbb{P}[\{2\}] = 1/3$ and $A = \{0, 1\}$ and $B = \{1, 2\}$. Then, it can be shown that $\mathbb{P}[A|B] = 1/2$ and $\mathbb{P}[A|B^{\mathsf{c}}] = 1$ so $\mathbb{P}[A|B] + \mathbb{P}[A|B^{\mathsf{c}}] = 3/2$. Note that the complement property gives that $\mathbb{P}[A|B] + \mathbb{P}[A^{\mathsf{c}}|B] = 1$ but does not work on the other side of the conditioning bar.

(c) If $\mathbb{P}[A] = 0.6$ and $\mathbb{P}[B] = 0.5$, then $\mathbb{P}[A \cap B] > 0$.

**Solution:**

**True.** We know that probability cannot exceed 1 so $1 \geq \mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$, which implies that $\mathbb{P}[A \cap B] \geq \mathbb{P}[A] + \mathbb{P}[B] - 1 = 0.6 + 0.5 - 1 = 0.1$.

(d) If $A$ and $B$ are conditionally independent given $C$, then $\mathbb{P}[A \cup B|C] = \mathbb{P}[A|C] + \mathbb{P}[B|C] - \mathbb{P}[A|C]\,\mathbb{P}[B|C]$.

**Solution:**

**True.** The inclusion-exclusion property applies to conditional probability, yielding $\mathbb{P}[A \cup B|C] = \mathbb{P}[A|C] + \mathbb{P}[B|C] - \mathbb{P}[A \cap B|C]$, and conditional independence lets us factor the last term as $\mathbb{P}[A \cap B|C] = \mathbb{P}[A|C]\,\mathbb{P}[B|C]$.

(e) If $\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A]\,\mathbb{P}[B]\,\mathbb{P}[C]$, then the events $A$, $B$, and $C$ are independent.

**Solution:**

**False.** We also need that $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$, $\mathbb{P}[A \cap C] = \mathbb{P}[A]\,\mathbb{P}[C]$, and $\mathbb{P}[B \cap C] = \mathbb{P}[B]\,\mathbb{P}[C]$.

**Problem 2** *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that $X$ is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If $F_X(2) - F_X(0) = 0$, then $P_X(1) = 0$.

> **Solution:**
>
> **True.** Since $\mathbb{P}[a < X \leq b] = F_X(b) - F_X(a)$, the statement $F_X(2) - F_X(0) = 0$ tells us that $\mathbb{P}[0 < X \leq 2] = 0$, so $\mathbb{P}[X = 1] = P_X(1) = 0$ as well.

(b) If X is Bernoulli $(p)$, then $\mathbb{E}\left[\frac{1}{X+1}\right] = \frac{1}{p+1}$.

> **Solution:**
>
> **False.**
> $$\mathbb{E}\left[\frac{1}{X+1}\right] = \sum_{x \in R_X} \frac{1}{x+1} P_X(x) = \frac{1}{0+1} P_X(0) + \frac{1}{1+1} P_X(1) = 1 - p + \frac{p}{2} = 1 - \frac{p}{2}$$

(c) For $a < b$, we have that $\mathbb{P}[X > a | X \leq b] = \frac{F_X(b) - F_X(a)}{F_X(b)}$.

> **Solution:**
>
> **True.** From the definition of conditional probability,
> $$\mathbb{P}[X > a | X \leq b] = \frac{\mathbb{P}\big[\{X > a\} \cap \{X \leq b\}\big]}{\mathbb{P}[X \leq b]} = \frac{\mathbb{P}[a < X \leq b]}{\mathbb{P}[X \leq b]} = \frac{F_X(b) - F_X(a)}{F_X(b)}$$
> where the last step uses the fact that $\mathbb{P}[a < X \leq b] = F_X(b) - F_X(a)$.

(d) If $\mathsf{Var}[X] > 0$, then $\mathsf{Var}[g(X)] > 0$ for any choice of function $g(x)$.

> **Solution:**
>
> **False.** Consider the function $g(x) = 2$. Then, $\mathbb{E}[(g(X))^2] = 4$ and $\mathbb{E}[g(X)] = 2$ so $\mathsf{Var}[g(X)] = \mathbb{E}[(g(X))^2] - \big(\mathbb{E}[g(X)]\big)^2 = 4 - 2^2 = 0$, even if $\mathsf{Var}[X] > 0$.

(e) If $\mathbb{E}[X^2 | B] > 0$ for some subset $B$ of the range $R_X$, then $\mathbb{E}[X^2] > 0$ as well.

> **Solution:**
>
> **True.** $0 < \mathbb{E}[X^2|B] = \sum_{x \in B} x^2 \, P_{X|B}(x) = \frac{1}{\mathbb{P}[X \in B]} \sum_{x \in B} x^2 \, P_X(x)$
> $$\leq \frac{1}{\mathbb{P}[X \in B]} \sum_{x \in R_X} x^2 \, P_X(x) = \frac{\mathbb{E}[X^2]}{\mathbb{P}[X \in B]} \leq \mathbb{E}[X^2]$$
> where the last step uses the fact that $\mathbb{P}[X \in B] \leq 1$.

**Problem 3** Please complete the following quick calculations. *16 points*

(a) Let the events $A_1, A_2, A_3$ be independent with $\mathbb{P}[A_1] = \mathbb{P}[A_2] = \mathbb{P}[A_3] = 1/4$. Calculate $\mathbb{P}[A_1^c \cap A_2^c \cap A_3^c]$ and $\mathbb{P}[A_1 \cup A_2 \cup A_3]$.

**Solution:**

By independence, $\mathbb{P}[A_1^c \cap A_2^c \cap A_3^c] = \mathbb{P}[A_1^c]\,\mathbb{P}[A_2^c]\,\mathbb{P}[A_3^c] = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$.
By De Morgan's rules, $\mathbb{P}[A_1 \cup A_2 \cup A_3] = 1 - \mathbb{P}[A_1^c \cap A_2^c \cap A_3^c] = 1 - \frac{27}{64} = \frac{37}{64}$.

(b) Let $\mathbb{P}[B|A] = 1/2$, $\mathbb{P}[A|B] = 1/3$, and $\mathbb{P}[B] = 1/4$. Calculate $\mathbb{P}[A]$ and determine whether the events $A$ and $B$ are independent.

**Solution:**

We can use Bayes' Rule to solve for $\mathbb{P}[A]$,

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[B|A]\,\mathbb{P}[A]}{\mathbb{P}[B]} \implies \mathbb{P}[A] = \frac{\mathbb{P}[A|B]\,\mathbb{P}[B]}{\mathbb{P}[B|A]} = \frac{\frac{1}{3} \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{1}{6}$$

$A$ and $B$ are not independent since $\mathbb{P}[B|A] \neq \mathbb{P}[B]$.

(c) Let $X$ be Poisson$(1)$. Calculate $\mathbb{E}[2X + 1]$ and $\mathbb{E}[3X^2 + 2]$.

**Solution:**

$\mathbb{E}[2X + 1] = 2\mathbb{E}[X] + 1 = 2 \cdot 1 + 1 = 3$. Using the alternate variance formula, $\mathbb{E}[X^2] = \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 = 1 + 1^2 = 2$. It follows that $\mathbb{E}[3X^2 + 2] = 3\mathbb{E}[X^2] + 2 = 3 \cdot 2 + 2 = 8$.

(d) Let $X$ be Discrete Uniform$(-2, 2)$. Calculate $\mathbb{P}[X \in B]$ as well as the conditional expected value $\mathbb{E}[X|B]$ of $X$ given that $\{X \in B\}$ for $B = \{-1, 0, 2\}$.

**Solution:**

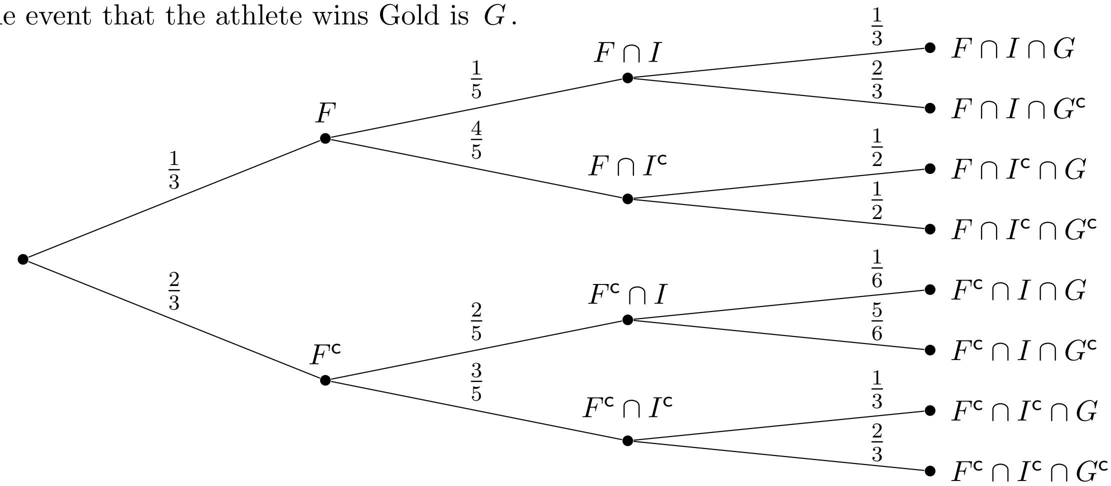
$$\mathbb{P}[X \in B] = \sum_{x \in B} P_X(x) = P_X(-1) + P_X(0) + P_X(2) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$$

The conditional PMF is $P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/3 & x = -1, 0, 2 \\ 0 & \text{otherwise} \end{cases}$ so

$$\mathbb{E}[X|B] = \sum_{x \in B} x\, P_{X|B}(x) = (-1) \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = \frac{1}{3}.$$

**Problem 4** *16 points*

You randomly turn on the TV and watch the final Winter Olympics performance for an athlete. The athlete may be Favored ($F$) to win and might have been Injured ($I$) in a preliminary trial. The event that the athlete wins Gold is $G$.




(a) What is the probability that the athlete you are watching is injured and wins gold?

**Solution:**

$$\mathbb{P}[I \cap G] = \mathbb{P}[F \cap I \cap G] + \mathbb{P}[F^{\mathsf{c}} \cap I \cap G] = \frac{1}{3} \cdot \frac{1}{5} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{5} \cdot \frac{1}{6} = \frac{3}{45} = \frac{1}{15}.$$

(b) Given that the athlete is favored, what is the probability that they **do not** win gold?

**Solution:**

$$\mathbb{P}[G^{\mathsf{c}}|F] = \mathbb{P}[G^{\mathsf{c}} \cap I|F] + \mathbb{P}[G^{\mathsf{c}} \cap I^{\mathsf{c}}|F] = \frac{1}{5} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{1}{2} = \frac{4+12}{30} = \frac{8}{15}.$$

(c) What is the probability that the athlete you are watching **does not** win gold?

**Solution:**

$$\begin{aligned}\mathbb{P}[G^{\mathsf{c}}] &= \mathbb{P}[F \cap I \cap G^{\mathsf{c}}] + \mathbb{P}[F \cap I^{\mathsf{c}} \cap G^{\mathsf{c}}] + \mathbb{P}[F^{\mathsf{c}} \cap I \cap G^{\mathsf{c}}] + \mathbb{P}[F^{\mathsf{c}} \cap I^{\mathsf{c}} \cap G^{\mathsf{c}}] \\ &= \frac{1}{3} \cdot \frac{1}{5} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{4}{5} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{2}{5} \cdot \frac{5}{6} + \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{2}{3} \\ &= \frac{2}{45} + \frac{3}{45} + \frac{10}{45} + \frac{12}{45} = \frac{27}{45} = \frac{3}{5}\end{aligned}$$

(d) Given that the athlete wins gold, what is the probability that they were favored?

**Solution:**

$$\mathbb{P}[F|G] = \frac{\mathbb{P}[G|F]\,\mathbb{P}[F]}{\mathbb{P}[G]} = \frac{\big(1 - \mathbb{P}[G^{\mathsf{c}}|F]\big)\mathbb{P}[F]}{1 - \mathbb{P}[G^{\mathsf{c}}]} = \frac{\frac{7}{15} \cdot \frac{1}{3}}{\frac{2}{5}} = \frac{7}{18}.$$

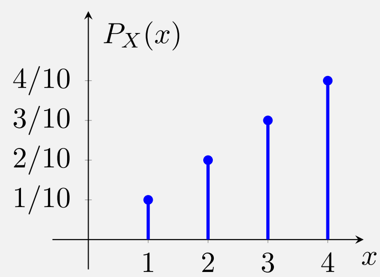
**Problem 5** *16 points*

Consider the following PMF:

$$P_X(x) = \begin{cases} \dfrac{x}{10} & x = 1, 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Make a sketch of the PMF $P_X(x)$. (Don't forget to label the axes.)

**Solution:**




(b) Calculate $\mathbb{E}[X]$.

**Solution:**

$$\mathbb{E}[X] = \sum_{x \in R_X} x\, P_X(x) = 1 \cdot \frac{1}{10} + 2 \cdot \frac{2}{10} + 3 \cdot \frac{3}{10} + 4 \cdot \frac{4}{10} = \frac{1+4+9+16}{10} = 3$$

(c) Calculate $\mathbb{P}[X < 3]$.

**Solution:**

$$\mathbb{P}[X < 3] = P_X(1) + P_X(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$$

(d) Calculate $\mathbb{E}[X^3|B]$ given that $X$ falls into $B = \{1, 2\}$.

**Solution:**

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \frac{P_X(1)}{P_X(1)+P_X(5)} & x = 1 \\ \frac{P_X(2)}{P_X(1)+P_X(2)} & x = 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{3} & x = 1 \\ \frac{2}{3} & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X^3|B] = \sum_{x \in B} x^3\, P_{X|B}(x) = 1^3 \cdot \frac{1}{3} + 2^3 \cdot \frac{2}{3} = \frac{17}{3}$$

**Problem 6** *16 points*

You are designing a new collectible card game. There are 3 unique Moon cards, 4 unique Earth cards, and 3 unique Sun cards. Each pack of cards contains 6 unique cards chosen from these 10 available cards, with all possible configurations equally likely.

(a) How many different configurations of 6-card packs are possible?

**Solution:**

We are selecting $k = 6$ out of $n = 10$ cards without replacements, and order does not matter. Therefore, there are $\binom{n}{k} = \binom{10}{6} = \frac{10!}{6!4!} = 210$ configurations.

(b) How many configurations have exactly 2 Moon cards, 2 Earth cards, and 2 Sun cards?

**Solution:**

We are selecting 2 out of 3 Moon cards, 2 out of 4 Earth cards, and 2 out of 3 Sun cards without replacements, and order does not matter. Therefore, there are $\binom{3}{2}\binom{4}{2}\binom{3}{2} = 3 \cdot 6 \cdot 3 = 54$ configurations.

(c) What is the probability that a pack has exactly 2 Moon cards, 2 Earth cards, and 2 Sun cards?

**Solution:**

$$\mathbb{P}[\{2 \text{ Moon, } 2 \text{ Earth, } 2 \text{ Sun}\}] = \frac{\#\ 2 \text{ Moon, } 2 \text{ Earth, } 2 \text{ Sun configurations}}{\#\ \text{total configurations}} = \frac{54}{210} = \frac{9}{35}$$

(d) What is the probability that a pack has no Earth cards?

**Solution:**

We are selecting 3 out of 3 Moon cards, 0 out of 4 Earth cards, and 3 out of 3 Sun cards without replacements, and order does not matter. Therefore, there is $\binom{3}{3}\binom{4}{0}\binom{3}{0} = 1$ configuration. The probability is thus

$$\mathbb{P}[\{\text{no Earth}\}] = \frac{\#\ 3 \text{ Moon, } 0 \text{ Earth, } 3 \text{ Sun configurations}}{\#\ \text{total configurations}} = \frac{1}{210}$$

**Problem 7** *16 points*

There are two birds that sometimes visit your windowsill: a cardinal and a blue jay. Every day the cardinal visits with probability $1/2$ and the blue jay visits with probability $2/3$, independently of each other. Visits are also independent across days.

(a) You are keeping track of days when **both** birds visit your window. Let $X$ be the total number of days out of $4$ when **both** birds visited. What kind of random variable is $X$? (Don't forget the parameters.)

> **Solution:**
>
> Binomial$(4, 1/3)$.

(b) What is the probability that **both** birds visited **at least 3 out of 4 days**?

> **Solution:**
>
> $$P[X \geq 3] = P_X(3) + P_X(4) = \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 + \binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0 = \frac{8}{81} + \frac{1}{81} = \frac{1}{9}$$

(c) Calculate $\mathsf{Var}[3X - 2]$.

> **Solution:**
>
> By variance of a linear function, we have
>
> $$\mathsf{Var}[3X - 2] = 3^2\mathsf{Var}[X] = 9np(1-p) = 9 \cdot 4 \cdot \frac{1}{3} \cdot \frac{2}{3} = 8$$

(d) Now, you are counting the number of days that you wait until the first day that **no birds** visit. What is the average number of days that you wait?

> **Solution:**
>
> Let $Y$ be the number of days you wait. $Y$ is a Geometric$(1/6)$ random variable so the average is $\mathbb{E}[Y] = \frac{1}{\frac{1}{6}} = 6$ days.