Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

(a) $\mathbb{P}[A|B] + \mathbb{P}[A|B^c] = 1$

(b) If $\mathbb{P}[A] = \mathbb{P}[B]$, then $\mathbb{P}[A|B] = \mathbb{P}[B|A]$.

(c) $\mathbb{P}[A|B]\,\mathbb{P}[C|A \cap B] = \mathbb{P}[A \cap C|B]$

(d) $\mathbb{P}[A|B] + \mathbb{P}[B|A] = \mathbb{P}[A \cap B]$.

(e) $\mathbb{P}[A^{\mathsf{c}} \cap B^{\mathsf{c}}] \geq 1 - \mathbb{P}[A] - \mathbb{P}[B]$.

Problem 2 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If $a < b$ and $F_X(a) = F_X(b)$, then $\mathbb{P}[X \in [a, b) = 0]$.

(b) If $\mathbb{E}[X^2] = \mathsf{Var}[X]$, then $\mathbb{E}[X] = 0$.

(c) $\mathbb{E}[\log(X)] = \log\big(\mathbb{E}[X]\big)$

(d) If $\mathbb{E}[X] = 0$, then $\mathbb{E}[X^3] = 0$ as well.

(e) If $\mathsf{Var}[X] = 0$, then $P_X(x) = \begin{cases} 1 & x = a, \\ 0 & \text{otherwise.} \end{cases}$ for some value a.

Problem 3 Please complete the following quick calculations. *16 points*

(a) Let X be Discrete Uniform $(1, b)$, and let $\mathbb{E}[X] = 4$. Compute b and $\mathsf{Var}[X]$.

(b) Let X be Geometric $(1/3)$. Calculate $\mathbb{E}[X+1]$ and $\mathbb{E}[(X+1)^2]$.

(c) Let X be a Bernoulli $(\frac{1}{3})$ random variable. Compute $\mathbb{E}[X^4]$ and $\mathbb{E}[e^X]$.

(d) Let A and B be events with $\mathbb{P}[A] = \frac{1}{2}, \mathbb{P}[A \cap B] = \dfrac{3}{8}$ and $\mathbb{P}[A^{\mathsf{c}} \cap B] = \dfrac{1}{8}$.
Calculate $\mathbb{P}[B^c]$ and $\mathbb{P}[A|B^c]$.

Problem 4 *16 points*

You are dealt three cards from a well-shuffled standard 52-card deck: four suits (Diamonds, Hearts, Clubs, and Spades), 13 cards of each suit (numbers from 2 to 10, Jack, Queen, King, and Ace). For each of the questions below, ou can leave the answer as a ratio of combinations or factorials.

(a) What is the probability that you get an Ace, a King, and a Queen in the 3 cards?

(b) What is the probability that you get an Ace, a King, and a Queen of the same suit in the 3 cards?

(c) If you get an Ace, a King, and a Queen, what is the probability that you get the Ace of Spades, the King of Hearts and the Queen of Clubs?

(d) Now, suppose you play the game repeatedly, replacing the cards and shuffling well between games. What is the expected number of games until you get an Ace, a King and a Queen of the same suit in that game?

Problem 5 *16 points*

Your favorite cereal has started a promotion where every box contains a prize. There are 3 unique prizes, and you would like to collect all of them. Each box of cereal you buy is equally likely to contain one of the three prizes. Clearly, the first box of cereal you buy will contain a prize you don't have yet.

(a) Let X be the number of boxes you buy (after the first one) until you obtain your second type of prize. What kind of random variable is X? (Don't forget the parameters.)

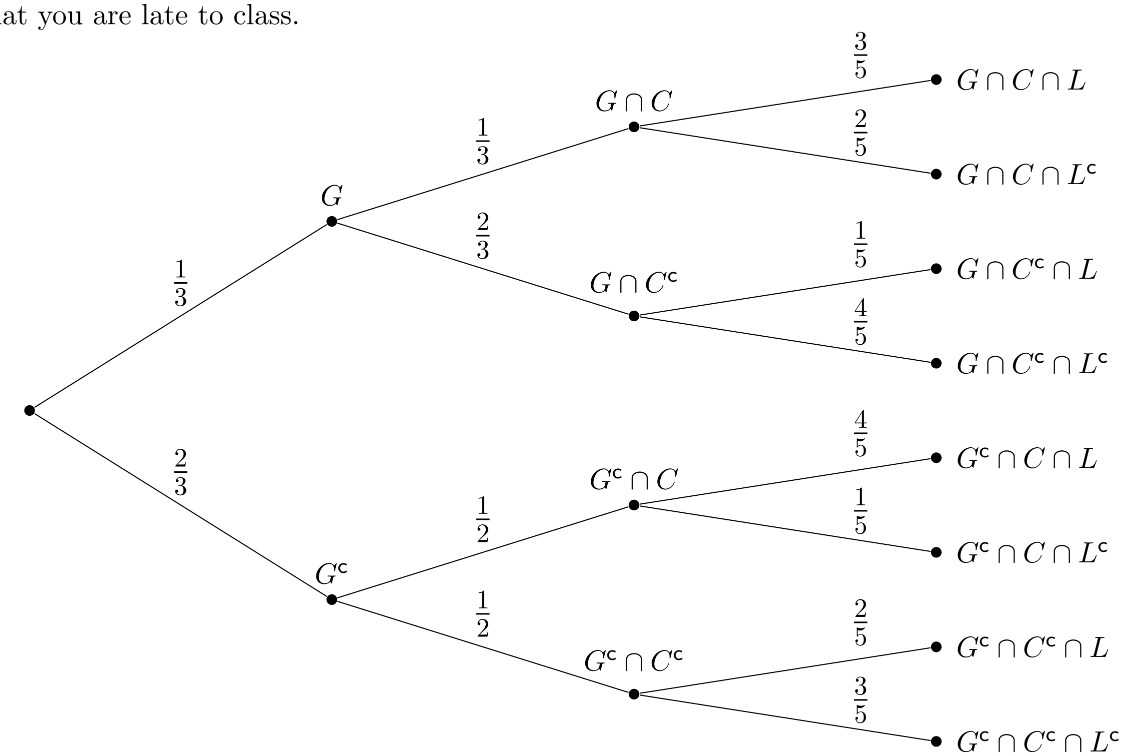
(b) Calculate $\mathbb{E}[X]$.

(c) Let Y be the number of boxes you buy after you've found two unique prizes until you find your third prize. What kind of random variable is Y? (Don't forget the parameters.)

(d) Say that each box costs 5 dollars. How many dollars do you spend on average to get all 3 unique prizes?

Problem 6 *20 points*

You always take the same bus to school and have built a probability model to predict when you will be late. Specifically, you have made the following conditional probability tree where G is the event that the weather is good, C is the event that the bus is crowded, and L the event that you are late to class.



(a) What is the probability that the weather is good and you are late to class?

(b) Given that the weather is good, what is the probability of being late to class?

(continued on next page)

(c) What is the probability of being late to class?

(d) Given that you are late to class, what is the probability that the bus was crowded?

Problem 7 *16 points*

The Black Panther is searching for Vibranium, a rare metal (in the Marvel Universe) that is used for his suit (and Captain America's shield). The number of Vibranium deposits X at a randomly chosen site is well-modeled as $\text{Poisson}(2)$ random variable.

(a) What is the probability of finding 2 or 3 or 4 deposits at a site? You can leave the answer as a sum of terms.

(b) Given that a site holds at least 2 deposits, what is the probability it has less than 5 deposits? You can leave the answer as a sum of terms.

(c) The utility of a site increases with larger deposits. It can be modeled by $U = (X+1)^2$. What is the expected value $\mathbb{E}[U]$? The answer must be a number.

(d) In a certain region of Wakanda, you are guaranteed that there is at least one deposit. What is the average number of deposits given that you are looking in this region, given that it must have at least one deposit? To get full credit, the answer must compute the sum explicitly.