Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

(a) $\mathbb{P}[A|B] + \mathbb{P}[A|B^c] = 1$

False. As a counterexample, let A be a strict subset of B, with $\mathbb{P}[A] = 1/2, \mathbb{P}[B] = 2/3$. Then,

$$\mathbb{P}[A|B] = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}; \quad \mathbb{P}[A|B^c] = 0$$

Hence $\mathbb{P}[A|B] + \mathbb{P}[A|B^c] = \frac{3}{4}$.

(b) If $\mathbb{P}[A] = \mathbb{P}[B]$, then $\mathbb{P}[A|B] = \mathbb{P}[B|A]$.

True. By the definition of conditional probability, $\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$ and $\mathbb{P}[B|A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]}$, which are equal since their denominators are assumed to be equal.

(c) $\mathbb{P}[A|B]\,\mathbb{P}[C|A \cap B] = \mathbb{P}[A \cap C|B]$

True. By the definition of conditional probability,

$$\mathbb{P}[A|B]\,\mathbb{P}[C|A \cap B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \frac{\mathbb{P}[A \cap B \cap C]}{\mathbb{P}[A \cap B]} = \frac{\mathbb{P}[A \cap B \cap C]}{\mathbb{P}[B]} = \mathbb{P}[A \cap C|B].$$

(d) $\mathbb{P}[A|B] + \mathbb{P}[B|A] = \mathbb{P}[A \cap B]$.

False. As a counterexample, assume that $\mathbb{P}[A] = \mathbb{P}[B] = 1/2$. Now, by the definition of conditional probability

$$\mathbb{P}[A|B] + \mathbb{P}[B|A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} + \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = 4\mathbb{P}[A \cap B].$$

(e) $\mathbb{P}[A^c \cap B^c] \geq 1 - \mathbb{P}[A] - \mathbb{P}[B]$.

True. By De Morgan's rules and the inclusion-exclusion property,

$$\mathbb{P}[A^c \cap B^c] = 1 - \mathbb{P}[A \cup B] = 1 - (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) = 1 - \mathbb{P}[A] - \mathbb{P}[B] + \mathbb{P}[A \cap B]$$

Since probability is non-negative, we know that $\mathbb{P}[A \cap B] \geq 0$ and so dropping this can only make the expression smaller. We thus have that

$$\mathbb{P}[A^c \cap B^c] = 1 - \mathbb{P}[A] - \mathbb{P}[B] + \mathbb{P}[A \cap B] \geq 1 - \mathbb{P}[A] - \mathbb{P}[B].$$

Problem 2 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If $a < b$ and $F_X(a) = F_X(b)$, then $\mathbb{P}[X \in [a, b) = 0]$.

False. The right statement is $\mathbb{P}[X \in (a, b]] = 0$. In particular, $P_X(a)$ can be greater than 0.

(b) If $\mathbb{E}[X^2] = \mathsf{Var}[X]$, then $\mathbb{E}[X] = 0$.

True. $\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$ which implies that $\big(\mathbb{E}[X]\big)^2 = \mathbb{E}[X]^2 - \mathsf{Var}[X]$, which is equal to 0 if $\mathbb{E}[X^2] = \mathsf{Var}[X]$. This implies that $\mathbb{E}[X] = 0$ as well.

(c) $\mathbb{E}[\log(X)] = \log\big(\mathbb{E}[X]\big)$

False. As a counterexample, let's work in base-2 with PMF $P_X(x) = \begin{cases} \frac{1}{2} & x = -\frac{1}{2}, 2, \\ 0 & \text{otherwise.} \end{cases}$. Thus, $\mathbb{E}[X] = 3/4$ and $\log_2\big(\mathbb{E}[X]\big) < 0$. However, $\mathbb{E}[\log_2(X)] = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot (+1) = 0$.

(d) If $\mathbb{E}[X] = 0$, then $\mathbb{E}[X^3] = 0$ as well.

False. PMFs are not necessarily symmetric. As a counterexample, take

$$P_X(x) = \begin{cases} \frac{2}{3} & x = -1, \\ \frac{1}{3} & x = 2. \end{cases}$$

Then, $\mathbb{E}[X] = \frac{2}{3} \cdot -1 + \frac{1}{3} \cdot 2 = 0$ but $\mathbb{E}[X^3] = \frac{2}{3} \cdot (-1)^3 + \frac{1}{3} \cdot 2^3 = 2$.

(e) If $\mathsf{Var}[X] = 0$, then $P_X(x) = \begin{cases} 1 & x = a, \\ 0 & \text{otherwise.} \end{cases}$ for some value a.

True. The variance is determined by summing non-negative terms,

$$\text{Var}[X] = \sum_{x \in S_X} \big(x - E[X]\big)^2 P_X(x) \ .$$

Therefore, if any value of $x \neq E[X]$ has $P_X(x) > 0$, then $\text{Var}[X] > 0$. So, if $\text{Var}[X] = 0$, all of the probability must be assigned to $x = E[X]$, which we can call a.

Problem 3 Please complete the following quick calculations. *16 points*

(a) Let X be Discrete Uniform$(1, b)$, and let $\mathbb{E}[X] = 4$. Compute b and $\mathsf{Var}[X]$.

Since $\mathbb{E}[X] = \frac{1+b}{2} = 4$, then $b = 8 - 1 = 7$. From the formula for the variance of a discrete uniform random variable,

$$\mathsf{Var}[X] = \frac{(b-a)(b-a+2)}{12} = \frac{6 \times 8}{12} = 4.$$

(b) Let X be Geometric$(1/3)$. Calculate $\mathbb{E}[X+1]$ and $\mathbb{E}[(X+1)^2]$.

For Geometric(p) RVs, $\mathbb{E}[X] = \frac{1}{p}, \mathsf{Var}[X] = \frac{1-p}{p^2}$. Hence,

$$\mathbb{E}[X+1] = 1 + \mathbb{E}[X] = 4; \mathbb{E}[X^2] = \mathsf{Var}[X] + (\mathbb{E}[X])^2 = \frac{\frac{2}{3}}{\frac{1}{9}} + 9 = 15$$

So,

$$\mathbb{E}[(X+1)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[X] + 1 = 15 + 6 + 1 = 22.$$

(c) Let X be a Bernoulli$(\frac{1}{3})$ random variable. Compute $\mathbb{E}[X^4]$ and $\mathbb{E}[e^X]$.

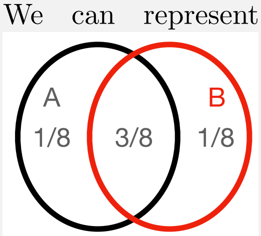
For Bernoulli(p) RVs, $R_X = \{0, 1\}$, and $P_X(x) = p$ if $x = 1$, and 0 elsewhere. Thus

$$\mathbb{E}[X^4] = \sum_{x \in R_X} x^4 P_X(x) = \frac{2}{3}0 + \frac{1}{3} = \frac{1}{3}.$$

$$\mathbb{E}[e^X] = \sum_{x \in R_X} e^x P_X(x) = \frac{2}{3}e^0 + \frac{1}{3}e = \frac{2}{3} + \frac{1}{3}e.$$

(d) Let A and B be events with $\mathbb{P}[A] = \frac{1}{2}, \mathbb{P}[A \cap B] = \frac{3}{8}$ and $\mathbb{P}[A^\mathsf{c} \cap B] = \frac{1}{8}$. Calculate $\mathbb{P}[B^c]$ and $\mathbb{P}[A|B^c]$.

We can represent the information in the problem using the diagram below.



Hence, $\mathbb{P}[B] = 1/2$, so $\mathbb{P}[B^c] = 1/2$. Furthermore,

$$\mathbb{P}[A|B^c] = \frac{\mathbb{P}[A \cap B^c]}{\mathbb{P}[B^c]} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

Problem 4 *16 points*

You are dealt three cards from a well-shuffled standard 52-card deck: four suits (Diamonds, Hearts, Clubs, and Spades), 13 cards of each suit (numbers from 2 to 10, Jack, Queen, King, and Ace). For each of the questions below, you can leave the answer as a ratio of combinations or factorials.

(a) What is the probability that you get an Ace, a King, and a Queen in the 3 cards?

You must choose an Ace out of 4, a King out of 4, a Queen out of 4, and 3 cards out of 52. The probability is $\frac{\binom{4}{1}^3}{\binom{52}{3}} = \frac{4^3}{\binom{52}{3}}$.

(b) What is the probability that you get an Ace, a King, and a Queen of the same suit in the 3 cards?

Since there are 4 suits, there are only 4 ways of getting an Ace, King and Queen in matching suits. The answer is $\frac{4}{\binom{52}{3}}$.

(c) Given that you get an Ace, a King, and a Queen, what is the probability that you get the Ace of Spades, the King of Hearts and the Queen of Clubs?

There is only one way of getting that specific combination, and there are $\binom{4}{1}^3$ ways of getting an Ace, a King and a Queen. Hence, the answer is $\frac{1}{64}$.

(d) Now, suppose you play the game repeatedly, replacing the cards and shuffling well between games. What is the expected number of games until you get an Ace, a King and a Queen of the same suit in that game?

Let $p = \frac{4}{\binom{52}{3}} = \frac{24}{52 \times 51 \times 50} = \frac{1}{13 \times 17 \times 25}$ be the answer to part b above. Then, the number of games to be played until getting an Ace, King, Queen of the same suit is a Geometric (p) random variable, so the expected number of games is $\frac{1}{p} = 13 \times 17 \times 25 = 5525$ games.

Problem 5 *16 points*

Your favorite cereal has started a promotion where every box contains a prize. There are 3 unique prizes, and you would like to collect all of them. Each box of cereal you buy is equally likely to contain one of the three prizes. Clearly, the first box of cereal you buy will contain a prize you don't have yet.

(a) Let X be the number of boxes you buy (after the first one) until you obtain your second type of prize. What kind of random variable is X? (Don't forget the parameters.)

X is a Geometric $(2/3)$ random variable, since we are counting the number of independent trials until the first success, which has probability $2/3$ per trial.

(b) Calculate $\mathbb{E}[X]$.

Since X is Geometric (p) with $p = 2/3$, then its mean is $\mathbb{E}[X] = \dfrac{1}{p} = \dfrac{3}{2}$.

(c) Let Y be the number of boxes you buy after you've found two unique prizes until you find your third prize. What kind of random variable is Y? (Don't forget the parameters.)

Y is a Geometric$(1/3)$ random variable, since we are counting the number of independent trials until the first success, which has probability $1/3$ per trial.

(d) Say that each box costs 5 dollars. How many dollars do you spend on average to get 3 unique prizes?

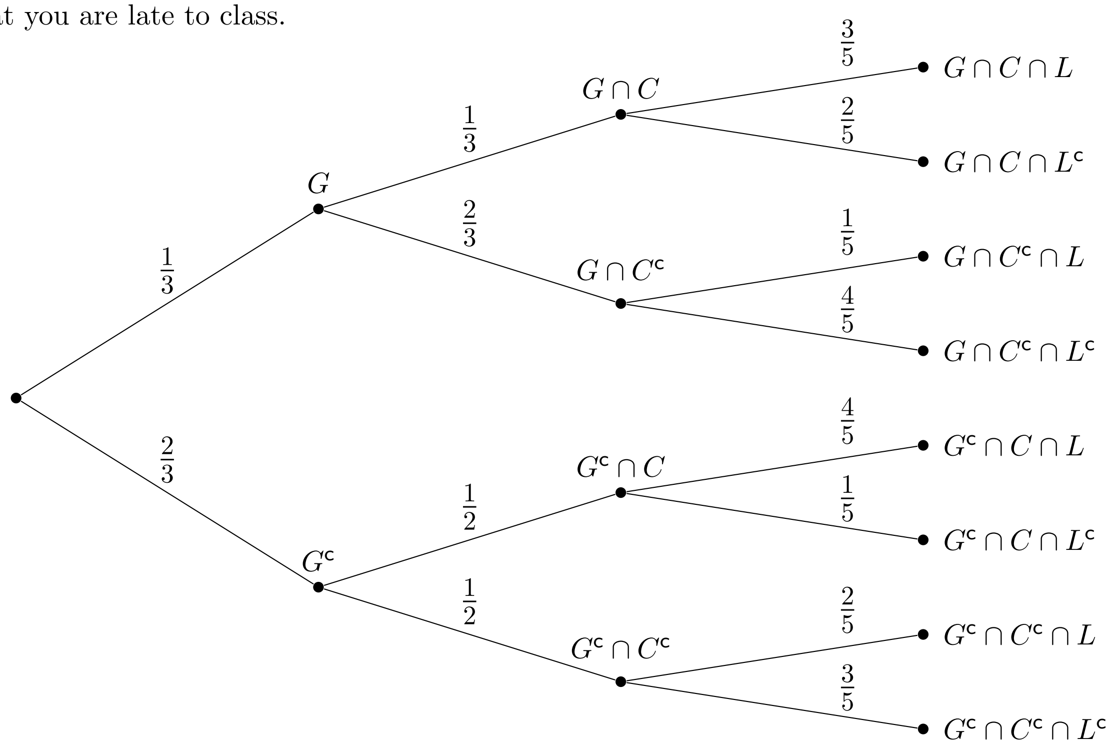
The total number of boxes purchased is $1 + X + Y$ so the number of dollars spent is $5 + 5X + 5Y$. By linearity of expectation, the average number of dollars spent is

$$\mathbb{E}[5 + 5X + 5Y] = 5 + 5\mathbb{E}[X] + 5\mathbb{E}[Y] = 5 + 5 \cdot \frac{3}{2} + 5 \cdot 3 = \frac{55}{2} = 27.5$$

where we used the fact that $\mathbb{E}[Y] = 3$ since Y is Geometric$(1/3)$

Problem 6 *16 points*

You always take the same bus to school and have built a probability model to predict when you will be late. Specifically, you have made the following conditional probability tree where G is the event that the weather is good, C is the event that the bus is crowded, and L the event that you are late to class.



(a) What is the probability that the weather is good and you are late to class?

We can see from the tree that there are two mutually exclusive events $G \cap C \cap L$ and $G \cap C^c \cap L$ that contain $G \cap L$, so we can just add up their probabilities, which we obtain from the tree using the multiplication rule,

$$\mathbb{P}[G \cap L] = \mathbb{P}[G \cap C \cap L] + \mathbb{P}[G \cap C^c \cap L] = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{5} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{5} = \frac{1}{9}.$$

(b) Given that the weather is good, what is the probability of being late to class?

Using part (a), we can use the definition of conditional probability to get

$$\mathbb{P}[L|G] = \frac{\mathbb{P}[L \cap G]}{\mathbb{P}[G]} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}.$$

We can also obtain this directly from the conditional probability tree, starting at G and adding up the probabilities to each of the leaves that contain L:

$$\mathbb{P}[L|G] = \mathbb{P}[C|G]\,\mathbb{P}[L|G \cap C] + \mathbb{P}[C^{\mathsf{c}}|G]\,\mathbb{P}[L|G \cap C^{\mathsf{c}}] = \frac{1}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{1}{5} = \frac{1}{3}.$$

You can also think of this as a conditional version of the Total Probability Theorem.

(c) What is the probability of being late to class?

We can express this using the Total Probability Theorem as

$$\mathbb{P}[L] = \mathbb{P}[G]\,\mathbb{P}[L|G] + \mathbb{P}[G^{\mathsf{c}}]\,\mathbb{P}[L|G^{\mathsf{c}}]$$

so that we can take advantage of the fact that we have $\mathbb{P}[L|G]$ from part (b). Using the same approach as in (a), we obtain

$$\mathbb{P}[L|G^{\mathsf{c}}] = \mathbb{P}[C|G^{\mathsf{c}}]\,\mathbb{P}[L|G^{\mathsf{c}} \cap C] + \mathbb{P}[C^{\mathsf{c}}|G^{\mathsf{c}}]\,\mathbb{P}[L|G^{\mathsf{c}} \cap C^{\mathsf{c}}] = \frac{1}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{5}.$$

Now, plugging everything in, we get $\mathbb{P}[L] = \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{3}{5} = \frac{23}{45}$.

(d) Given that you are late to class, what is the probability that the bus was crowded?

We can get this by directly applying the definition of conditional probability, writing the intersection $C \cap L$ in terms of the leaves of the tree, $C \cap L = (G \cap C \cap L) \cup (G^{\mathsf{c}} \cap C \cap L)$, and multiplying through to evaluate these probabilities (i.e., the multiplication rule):

$$\mathbb{P}[C|L] = \frac{\mathbb{P}[C \cap L]}{\mathbb{P}[L]} = \frac{\mathbb{P}[G \cap C \cap L] + \mathbb{P}[G^{\mathsf{c}} \cap C \cap L]}{\mathbb{P}[L]} = \frac{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{4}{5}}{\frac{23}{45}} = \frac{\frac{1}{3}}{\frac{23}{45}} = \frac{15}{23}.$$

Problem 7 *16 points*

The Black Panther is searching for Vibranium, a rare metal (in the Marvel Universe) that is used for his suit (and Captain America's shield). The number of Vibranium deposits X at a randomly chosen site is well-modeled as Poisson(2) random variable.

(a) What is the probability of finding between 2 and 4 deposits at a site?

$$\mathbb{P}[2 \le X \le 4] = P_X(2) + P_X(3) + P_X(4) = e^{-2}\left(\frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right) = e^{-2}\left(2 + \frac{4}{3} + \frac{2}{3}\right) = 4e^{-2}$$

(b) Given that a site holds at least 2 deposits, what is the probability it has less than 5 deposits?

Using the definition of conditional probability, we get

$$\mathbb{P}[X < 5 | X \ge 2] = \frac{\mathbb{P}[\{X < 5\} \cap \{X \ge 2\}]}{\mathbb{P}[X \ge 2]} = \frac{\mathbb{P}[2 \le X \le 4]}{\mathbb{P}[X \ge 2]}$$

so we can take the numerator from part (a). The denominator can be found using the complement,

$$\begin{aligned}\mathbb{P}[X \ge 2] = 1 - \mathbb{P}[X < 2] &= 1 - \big(P_X(0) + P_X(1)\big) \\ &= 1 - e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!}\right) = 1 - 3e^{-2}.\end{aligned}$$

Overall, we get

$$\mathbb{P}[X < 5 | X \ge 2] = \frac{4e^{-2}}{1 - 3e^{-2}}.$$

(c) The utility of a site increases with larger deposits. It can be modeled by $U = (X+1)^2$. What is the average value of U?

Expanding the square, we get $\mathbb{E}[U] = \mathbb{E}[(X+1)^2] = \mathbb{E}[X^2 + 2X + 1] = \mathbb{E}[X^2] + 2\mathbb{E}[X] + 1$ where the last step uses the linearity of expectation. Since X is Poisson(2), it has mean $\mathbb{E}[X] = 2$ and variance $\mathsf{Var}[X] = 2$. Using the alternate variance formula, we get $\mathbb{E}[X^2] = \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 = 2 + 2^2 = 6$. Plugging back in, we get $\mathbb{E}[U] = 6 + 2 \cdot 2 + 1 = 11$.

(d) In a certain region of Wakanda, you are guaranteed that there is at least one deposit. What is the average number of deposits given that you are looking in this region, given that it must have at least one deposit? To get full credit, the answer must compute the sum explicitly.

Let $B = \{X \ge 1\}$ be the event there is at least 1 deposit in the region. We have that

$$\mathbb{P}[B] = \mathbb{P}[X \ge 1] = 1 - P_X(0) = 1 - e^{-2}.$$

We want the conditional expectation $\mathbb{E}[X|B]$ so first we will need the conditional PMF.

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[B]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \dfrac{\frac{2^x}{x!}e^{-2}}{1 - e^{-2}} & x = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[X|B] = \sum_{x \in B} x P_{X|B}(x) = \sum_{x=1}^{\infty} \frac{x P_X(x)}{\mathbb{P}[B]} = \frac{1}{\mathbb{P}[B]} \sum_{x=1}^{\infty} x P_X(x) = \frac{1}{\mathbb{P}[B]} \sum_{x=0}^{\infty} x P_X(x) = \frac{\mathbb{E}[X]}{1 - e^{-2}} = \frac{2}{1 - e^{-2}}$$