Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

(a) $\mathbb{P}[A \cap B] + \mathbb{P}[B \cap C] = \mathbb{P}[B]$

> **Solution:**
>
> **False.** B and C may not partition the sample space. The following is true: $\mathbb{P}[A \cap B] + \mathbb{P}[B \cap C] + \mathbb{P}[A \cap (B \cup C)^{\mathsf{c}}] = \mathbb{P}[B]$.

(b) If $\mathbb{P}[A] + \mathbb{P}[B] = \mathbb{P}[A \cup B]$, then $\mathbb{P}[A \cap B] = 0$.

> **Solution:**
>
> **True.** From the inclusion-exclusion property, $\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = \mathbb{P}[A \cup B]$. Plugging in $\mathbb{P}[A] + \mathbb{P}[B] = \mathbb{P}[A \cup B]$, we see that $\mathbb{P}[A \cap B] = 0$.

(c) If A and C are independent, then A and B are conditionally independent given C.

> **Solution:**
>
> **False.** Independence does not imply conditional independence.

(d) If $A \cap B \cap C = \phi$, then $\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C]$.

> **Solution:**
>
> **False.** The inclusion-exclusion property for a union of three events is
>
> $$\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] - \mathbb{P}[A \cap B] - \mathbb{P}[A \cap C] - \mathbb{P}[B \cap C] + \mathbb{P}[A \cap B \cap C]$$
>
> so this would only cross out the last term.

(e) If $\mathbb{P}[A] = 0.4$, $\mathbb{P}[B^{\mathsf{c}}] = 0.4$, and $\mathbb{P}[(A \cup B)^{\mathsf{c}}] = 0.24$, then A and B are independent.

> **Solution:**
>
> **True.** We know that $\mathbb{P}[A^{\mathsf{c}}] = 1 - \mathbb{P}[A] = 0.6$ and $\mathbb{P}[A^{\mathsf{c}} \cap B^{\mathsf{c}}] = \mathbb{P}[(A \cup B)^{\mathsf{c}}] = 0.24$. We can check that A^{c} and B^{c} are independent via $\mathbb{P}[A^{\mathsf{c}}]\,\mathbb{P}[B^{\mathsf{c}}] = \mathbb{P}[A^{\mathsf{c}} \cap B^{\mathsf{c}}]$. This implies A and B are independent.

Problem 2 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If $F_X(a) = 1$, then $\mathbb{E}[X] \leq a$.

Solution:

True. $F_X(a) = \mathbb{P}[X \leq a] = 1$ so we know that X never exceeds a and thus its average must be less than or equal to a as well.

(b) $\mathbb{P}[X^2 > 0] = 1 - P_X(0)$.

Solution:

True. $\mathbb{P}[X^2 > 0] = 1 - \mathbb{P}[X^2 \leq 0] = 1 - P_X(0)$.

(c) If $\mathbb{E}[X] = a$, then $\mathbb{E}[X^2] \leq a^2$.

Solution:

False. As a counterexample, let $P_X(-1) = P_X(+1) = 1/2$ so that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = 1$ (rather than 0 as would be implied by the statement).

(d) If $\mathbb{E}[X] = 2$ and $\mathbb{E}[X^2] = 4$, then $\mathsf{Var}[X] = 0$.

Solution:

True. $\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = 4 - 2^2 = 0$.

(e) $\mathbb{P}[a < X < b] = \mathbb{P}[X > a]\,\mathbb{P}[X < b]$ for $a < b$.

Solution:

False. As a counterexample, let $P_X(-1) = P_X(+1) = 1/2$, $a = -1/2$ and $b = +1/2$. We know that $\mathbb{P}[a < X < b] = 0$ but $\mathbb{P}[X > a]\,\mathbb{P}[X < b] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Problem 3 Please complete the following quick calculations. *16 points*

(a) Let X be Poisson (2).
Calculate $\mathbb{E}[2X+1]$ and $\mathbb{E}[(2X+1)^2]$.

Solution:

$$\begin{aligned}\mathbb{E}[2X+1] &= 2\mathbb{E}[X]+1 = 2\cdot 2+1 = 5\\ \mathbb{E}[X^2] &= \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 = 2+2^2 = 6\\ \mathbb{E}[(2X+1)^2] &= \mathbb{E}[4X^2+4X+1] = 4\mathbb{E}[X^2]+4\mathbb{E}[X]+1 = 4\cdot 6+4\cdot 2+1 = 33\end{aligned}$$

(b) Let X be Bernoulli (1/3).
Calculate $\mathbb{E}[X+1]$ and $\mathbb{E}\Big[\dfrac{1}{X+1}\Big]$.

Solution:

$$\begin{aligned}\mathbb{E}[X+1] &= \mathbb{E}[X]+1 = \frac{1}{3}+1 = \frac{4}{3}\\ \mathbb{E}\Big[\frac{1}{X+1}\Big] &= \sum_{x\in R_X}\frac{1}{1+x}P_X(x)\\ &= \frac{1}{1+0}\cdot P_X(0) + \frac{1}{1+1}\cdot P_X(1)\\ &= 1\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{1}{3} = \frac{5}{6}\end{aligned}$$

(c) Let A, B, and C be independent events with $\mathbb{P}[A]=\mathbb{P}[B]=\mathbb{P}[C]=\dfrac{2}{3}$.
Calculate $\mathbb{P}[A\cup B]$ and $\mathbb{P}[A\cup B\cup C]$.

Solution:

$$\begin{aligned}\mathbb{P}[A\cup B] &= \mathbb{P}[A]+\mathbb{P}[B]-\mathbb{P}[A\cap B] = \mathbb{P}[A]+\mathbb{P}[B]-\mathbb{P}[A]\,\mathbb{P}[B]\\ &= \frac{2}{3}+\frac{2}{3}-\Big(\frac{2}{3}\Big)^2 = \frac{4}{3}-\frac{4}{9} = \frac{8}{9}\\ \mathbb{P}[A\cup B\cup C] &= 1-\mathbb{P}[(A\cup B\cup C)^{\mathsf{c}}] = 1-\mathbb{P}[A^{\mathsf{c}}\cap B^{\mathsf{c}}\cap C^{\mathsf{c}}]\\ &= 1-\mathbb{P}[A^{\mathsf{c}}]\,\mathbb{P}[B^{\mathsf{c}}]\,\mathbb{P}[C^{\mathsf{c}}] = 1-\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{26}{27}\end{aligned}$$

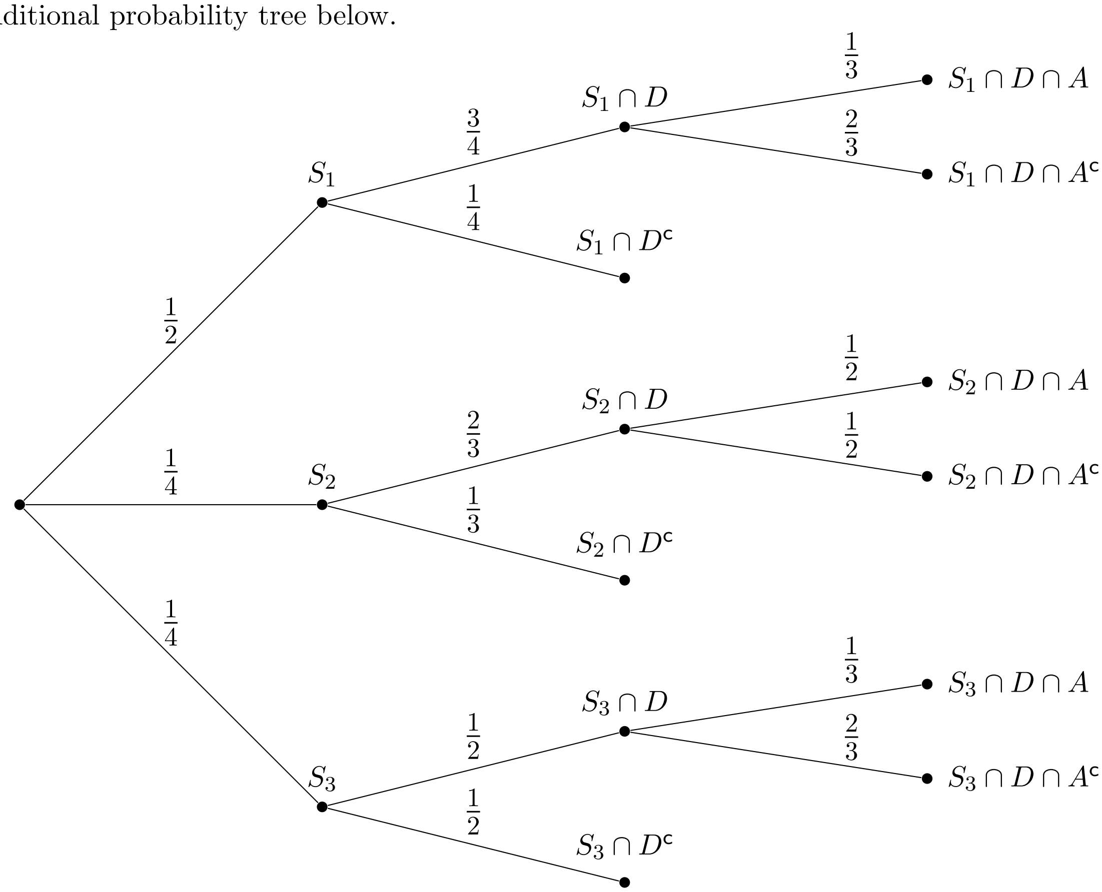
(d) Let A and B be events with $\mathbb{P}[A\cap B]=\dfrac{5}{8}$ and $\mathbb{P}[A^{\mathsf{c}}\cap B]=\dfrac{1}{8}$.
Calculate $\mathbb{P}[B]$ and $\mathbb{P}[A|B]$.

Solution:

$$\begin{aligned}\mathbb{P}[B] &= \mathbb{P}[A\cap B]+\mathbb{P}[A^{\mathsf{c}}\cap B] = \frac{5}{8}+\frac{1}{8} = \frac{3}{4}\\ \mathbb{P}[A|B] &= \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[B]} = \frac{\frac{5}{8}}{\frac{3}{4}} = \frac{5}{6}\end{aligned}$$

Problem 4 *16 points*

You are an archaelogist embarking on an expedition to find a Argentinosaurus dinosaur fossil to complete the local museum's collection. You randomly decide to visit one of three dig sites, where S_i for $i \in \{1, 2, 3\}$ is the event that you visit the i^{th} site. Let D be the event that you discover a dinosaur fossil at your site, and let A be the event that, upon careful examination, it shown to be a Argentinosaurus fossil. The probabilities of these events are modeled via the conditional probability tree below.



(a) What is the probability that you do **not** find a dinosaur fossil?

Solution:

$$\begin{aligned}\mathbb{P}[D^{\mathsf{c}}] &= \mathbb{P}[S_1 \cap D^{\mathsf{c}}] + \mathbb{P}[S_2 \cap D^{\mathsf{c}}] + \mathbb{P}[S_3 \cap D^{\mathsf{c}}] \\ &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} + \frac{1}{12} + \frac{1}{8} = \frac{1}{3}\end{aligned}$$

(b) What is the probability that you visit Site 1 and find a Argentinosaurus fossil?

Solution:

$$\mathbb{P}[S_1 \cap A] = \mathbb{P}[S_1 \cap D \cap A] = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{8}$$

(c) Given that you find a Argentinosaurus fossil, what is the probability that you were **not** at Site 1?

Solution:

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[S_1 \cap D \cap A] + \mathbb{P}[S_2 \cap D \cap A] + \mathbb{P}[S_3 \cap D \cap A] \\ &= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{8} + \frac{1}{12} + \frac{1}{24} = \frac{1}{4} \\ \mathbb{P}[S_1^{\mathsf{c}}|A] = 1 - \mathbb{P}[S_1|A] &= 1 - \frac{\mathbb{P}[S_1 \cap A]}{\mathbb{P}[A]} = 1 - \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}\end{aligned}$$

(d) Given that you do **not** visit Site 1, what is the probability that you find a dinosaur fossil?

Solution:

$$\begin{aligned}\mathbb{P}[D|S_1^{\mathsf{c}}] = \frac{\mathbb{P}[S_1^{\mathsf{c}} \cap D]}{\mathbb{P}[S_1^{\mathsf{c}}]} &= \frac{\mathbb{P}[S_2 \cap D] + \mathbb{P}[S_3 \cap D]}{\mathbb{P}[S_2] + \mathbb{P}[S_3]} \\ &= \frac{\frac{1}{4} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{4} + \frac{1}{4}} = \frac{7}{12}\end{aligned}$$

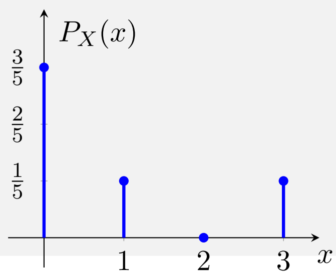
Problem 5 *16 points*

On your first day at the job at the Pear Corporation, you have been asked to work out a few quantities for the random variable X with PMF

$$P_X(x) = \begin{cases} \frac{3}{5} & x = 0 \\ \frac{1}{5} & x = 1 \\ 0 & x = 2 \\ \frac{1}{5} & x = 3 \end{cases}$$

(a) Make a sketch of the PMF. (Don't forget to label the axes.)

Solution:



(b) Calculate $\mathsf{Var}[X]$.

Solution:

$$\mathbb{E}[X] = \sum_{x \in R_X} x\, P_X(x) = 0 \cdot \frac{3}{5} + 1 \cdot \frac{1}{5} + 2 \cdot 0 + 3 \cdot \frac{1}{5} = \frac{4}{5}$$

$$\mathbb{E}[X^2] = \sum_{x \in R_X} x^2\, P_X(x) = 0^2 \cdot \frac{3}{5} + 1^2 \cdot \frac{1}{5} + 2^2 \cdot 0 + 3^2 \cdot \frac{1}{5} = \frac{10}{5}$$

$$\mathsf{Var}[X] = \mathbb{E}[X^2] - \Big(\mathbb{E}[X]\Big)^2 = \frac{10}{5} - \left(\frac{4}{5}\right)^2 = \frac{34}{25}$$

(c) Calculate $\mathbb{E}[2^X]$.

Solution: $$\mathbb{E}[2^X] = \sum_{x \in R_X} 2^x\, P_X(x) = 2^0 \cdot \frac{3}{5} + 2^1 \cdot \frac{1}{5} + 2^2 \cdot 0 + 2^3 \cdot \frac{1}{5} = \frac{13}{5}$$

(d) For $B = \{1, 3\}$, calculate $\mathbb{E}[X|B]$.

Solution:

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\sum_{x \in B} P_X(x)} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \dfrac{P_X(1)}{P_X(1) + P_X(3)} & x = 1 \\ \dfrac{P_X(3)}{P_X(1) + P_X(3)} & x = 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2} & x = 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

Problem 6 *16 points*

You are running a jellybean factory and producing tiny pouches that contain four jellybeans. Each jellybean is equally likely to be one of four colors, red, blue, green, yellow, independently of the other jelly beans.

(a) What is the probability that all four jellybeans in a pouch are green?

Solution:

By independence,

$$\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{256}$$

(b) What is the probability that all four jellybeans in a pouch are the same color?

Solution:

Since there are four choices of color, the probability is

$$4 \cdot \frac{1}{256} = \frac{1}{64}$$

(c) What is the probability of getting all four colors in a single pouch?

Solution:

Let D_i be the event that the i^{th} jellybean selected is a different color from the previous $i-1$ jellybeans.

$$\begin{aligned}
\mathbb{P}[\{\text{four colors}\}] &= \mathbb{P}[D_1 \cap D_2 \cap D_3 \cap D_4] \\
&= \mathbb{P}[D_1]\,\mathbb{P}[D_2|D_1]\,\mathbb{P}[D_3|D_1 \cap D_2]\,\mathbb{P}[D_4|D_1 \cap D_2 \cap D_3] \\
&= 1 \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} = \frac{3}{32}
\end{aligned}$$

(d) Given that there are no blue jellybeans in a pouch, what is the probability that there are no green jellybeans?

Solution:

$$\begin{aligned}
&\mathbb{P}[\{\text{no green jellybeans}\}|\{\text{no blue jellybeans}\}] \\
&= \frac{\mathbb{P}[\{\text{no green jellybeans}\} \cap \{\text{no blue jellybeans}\}]}{\mathbb{P}[\{\text{no blue jellybeans}\}]} \\
&= \frac{\left(\frac{2}{4}\right)^4}{\left(\frac{3}{4}\right)^4} = \frac{2^4}{3^4} = \frac{16}{81}
\end{aligned}$$

Problem 7 *16 points*

In your small hometown, the number of Wikipedia searches for a "CD Player" in a given day can be modeled as a discrete random variable X which is Poisson (2).

(a) What is the probability that X is less than 2?

Solution:

$$\mathbb{P}[X < 2] = P_X(0) + P_X(1) = \frac{2^0}{0!}e^{-2} + \frac{2^1}{1!}e^{-2} = 3e^{-2}$$

(b) Given that X is less than 4, what is the probability that X is less than 2?

Solution:

$$\mathbb{P}[X < 2 | X < 4] = \frac{\mathbb{P}[\{X < 2\} \cap \{X < 4\}]}{\mathbb{P}[X < 4]} = \frac{\mathbb{P}[X < 2]}{\mathbb{P}[X < 4]}$$

$$\mathbb{P}[X < 4] = P_X(0) + P_X(1) + P_X(2) + P_X(3)$$

$$= e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) = e^{-2}\left(1 + 2 + 2 + \frac{4}{3}\right) = \frac{19}{3}e^{-2}$$

$$\mathbb{P}[X < 2 | X < 4] = \frac{3e^{-2}}{\frac{19}{3}e^{-2}} = \frac{9}{19}$$

(c) Given that X is less than 4, what is the conditional expected value of X?

Solution:

Let $B = \{0, 1, 2, 3\}$.

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \frac{e^{-2}}{\frac{19}{3}e^{-2}} & x = 0 \\ \frac{2e^{-2}}{\frac{19}{3}e^{-2}} & x = 1 \\ \frac{2e^{-2}}{\frac{19}{3}e^{-2}} & x = 2 \\ \frac{\frac{4}{3}e^{-2}}{\frac{19}{3}e^{-2}} & x = 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{19} & x = 0 \\ \frac{6}{19} & x = 1 \\ \frac{6}{19} & x = 2 \\ \frac{4}{19} & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X|B] = \sum_{x \in B} x\, P_{X|B}(x) = 0 \cdot \frac{3}{19} + 1 \cdot \frac{6}{19} + 2 \cdot \frac{6}{19} + 3 \cdot \frac{4}{19} = \frac{30}{19}$$

(d) Assume that searches across days are independent from one another. Let Y be the number of days until the first day with 2 or more searches. What kind of random variable is Y? (Don't forget the parameters.)

Solution:

By the complement property, $\mathbb{P}[X \geq 2] = 1 - \mathbb{P}[X < 2] = 1 - 3e^{-2}$.
Y is a Geometric $(1 - 3e^{-2})$ random variable.