Problem 1 *12 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X and Y are jointly continuous random variables, with probability density functions $f_X(x), f_Y(y)$, joint probability density function $f_{X,Y}(x,y)$, and corresponding cumulative distribution functions $F_X(x), F_Y(y)$, and $F_{X,Y}(x,y)$.

(a) If $\mathbb{P}[X > Y] = 1$, then $f_X(a) > f_Y(a)$ for every choice of a.

(b) If $b > a$, then $F_X(b) > F_X(a)$.

(c) If X and Y are independent, then $\mathsf{Var}[XY] = \mathbb{E}[X^2]\mathbb{E}[Y^2] - (\mathbb{E}[X^2])^2(\mathbb{E}[Y^2])^2$.

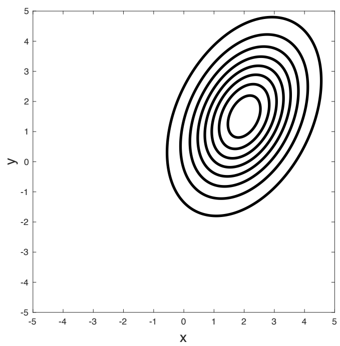
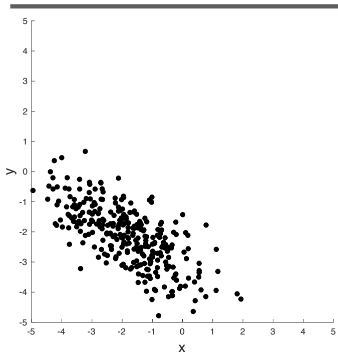
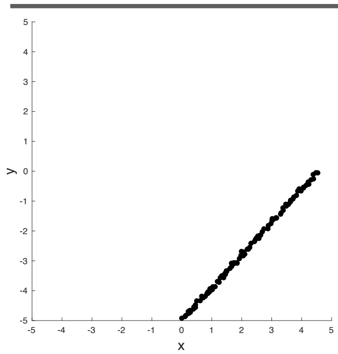
(d) If $\mathsf{Var}[X] = a^2$, and $\mathsf{Var}[Y] = b^2$, then $\mathsf{Cov}[X,Y] \leq ab$.

(e) If $\mathsf{Cov}[X,Y] = 0$, then $\mathsf{Cov}[-X,Y] = 0$.

(f) If $\rho_{X,Y} = \dfrac{1}{4}$, then $Y = \dfrac{X}{4} + b$ from some constant b.

Problem 2 *20 points*

The table below lists four scenarios via contour and scatter plots as well as equations. **Put a checkmark in the boxes in each row that you think are true for that scenario.** No justifications are needed and there may be multiple boxes checked per row and/or column.

	$\mathbb{E}[X]$ noticeably more than 0	$\mathbb{E}[Y]$ noticeably less than 0	$\mathsf{Var}[X]$ noticeably larger than $\mathsf{Var}[Y]$	$\mathsf{Cov}[X,Y]$ noticeably more than 0	$\lvert\rho_{X,Y}\rvert$ close to 1
	☐	☐	☐	☐	☐
	☐	☐	☐	☐	☐
	☐	☐	☐	☐	☐
$Y = -2X - 6$ $\mathbb{E}[X] = -1$ $\mathsf{Var}[X] = 2$	☐	☐	☐	☐	☐

Problem 3 *16 points*

For each of the following parts, calculate the **two requested quantities exactly.** You should arrive at a numerical answer in each part. For this particular problem, you may **not** leave your answer in terms of integrals. Show your steps for partial credit.

(a) Let X be a Uniform $(-2, 2)$ random variable.
Calculate $\mathbb{E}[X]$ and $\mathbb{E}[X|B]$ for $B = (0, 2)$.

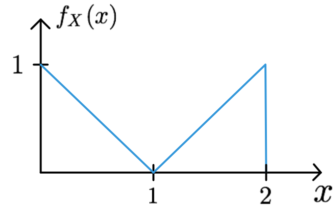
(b) Let X be a Gaussian $(1, 4)$ random variable.
Calculate $\mathbb{P}[X < 3]$ and $\mathbb{P}[X < -1|X < 3]$ in terms of the $\Phi(z)$ function.

(c) Let Y be Geometric $(1/5)$ and X given $Y = y$ is Binomial $(y, 1/3)$.
Calculate $\mathbb{E}[X|Y = y]$ and $\mathbb{E}[X]$.

(d) Let X and Y be jointly Gaussian with $\mathbb{E}[X] = 2$, $\mathbb{E}[Y] = -1$, $\mathsf{Var}[X] = \mathsf{Var}[Y] = 3$, and $\rho_{X,Y} = -\dfrac{1}{2}$. For $W = X + Y$ and $Z = X - Y$, calculate $\mathsf{Var}[W]$ and $\mathsf{Cov}[W, Z]$.

Problem 4 *20 points*

Consider a continuous random variable X with the following PDF:



$$f_X(x) = \begin{cases} 1-x & 0 \le x < 1 \\ x-1 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that the geometry of this PDF will allow you to solve parts (a) - (d) without using integration, but you may leave your answers as integrals if you wish.

(a) Calculate $\mathbb{P}\left[X \le \frac{3}{2}\right]$.

(b) Calculate $\mathbb{E}[X]$.

(c) Calculate $f_{X|B}(x)$ where $B = [0, \frac{3}{2}]$.

(d) Calculate $\mathbb{P}[X > 1 | X \in B]$ where $B = [0, \frac{3}{2}]$.

(e) Calculate $\mathbb{E}[X|B]$ where $B = [0, \frac{3}{2}]$.

Problem 5 *16 points*

Consider the joint probability mass function:

$P_{XY}(x,y)$		$y=1$	$y=2$	$y=3$
x	1	0	0	0.2
	2	0.3	0.1	0
	3	0.1	0.3	0

(a) Calculate the marginal probability mass function $P_Y(y)$.

(b) Calculate $\mathbb{E}[XY]$.

(c) Complete the following table for the conditional probability mass function $P_{X|Y}(x|y)$:

$P_{X\|Y}(x\|y)$		$y=1$	$y=2$	$y=3$
x	1			
	2			
	3			

(d) Calculate $\mathbb{E}[X|Y=y]$ (as a case-by-case function of y).

Problem 6 *16 points*

Consider the joint probability density function

$$f_{XY}(x, y) = \begin{cases} c, & x^2 + y^2 < 4 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the range of X and Y in the x - y plane. (Don't forget to label your axes.)

(b) Find the value of the constant c that satisfies the normalization property so that $f_{XY}(x, y)$ is a valid joint probability density function.

(c) Calculate $\mathbb{P}[Y > 3X]$.

(d) Determine the marginal PDFs $f_X(x)$ and $f_Y(y)$.