**Problem 1** *12 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that $X$ and $Y$ are jointly continuous random variables, with probability density functions $f_X(x), f_Y(y)$, joint probability density function $f_{X,Y}(x,y)$, and corresponding cumulative distribution functions $F_X(x), F_Y(y)$, and $F_{X,Y}(x,y)$.

(a) If $\mathbb{P}[X > Y] = 1$, then $f_X(a) > f_Y(a)$ for every choice of $a$.

**Solution:**

**False.** If $X$ is a Uniform(2,7) continuous random variable and $Y$ is a Uniform(0,1) continuous random variable, then $f_X(a) = 1/5$ for $2 < a < 7$ and $f_Y(a) = 1$ for $0 < a < 1$. Thus, we have that $\mathbb{P}[X > Y] = 1$ but $f_X(a) \not> f_Y(a)$.

(b) If $b > a$, then $F_X(b) > F_X(a)$.

**Solution:**

**False.** If $X$ is a Uniform(0, 1/2) continuous random variable, $b = 2$, and $a = 1$, then $F_X(2) = F_X(1) = 1$.

(c) If $X$ and $Y$ are independent, then $\mathsf{Var}[XY] = \mathbb{E}[X^2]\mathbb{E}[Y^2] - (\mathbb{E}[X^2])^2(\mathbb{E}[Y^2])^2$.

**Solution:**

**False.** Let $W = XY$. Then

$$\begin{aligned}\mathsf{Var}[W] &= \mathbb{E}[W^2] - (\mathbb{E}[W])^2 \\ &= \mathbb{E}[X^2Y^2] - (\mathbb{E}[XY])^2 \\ &= \mathbb{E}[X^2]\mathbb{E}[Y^2] - (\mathbb{E}[X]\mathbb{E}[Y])^2 \quad \text{(using independence)} \\ &= \mathbb{E}[X^2]\mathbb{E}[Y^2] - (\mathbb{E}[X])^2(\mathbb{E}[Y])^2\end{aligned}$$

(d) If $\mathsf{Var}[X] = a^2$, and $\mathsf{Var}[Y] = b^2$, then $\mathsf{Cov}[X,Y] \le ab$.

**Solution:**

**True.**

$$\rho_{X,Y} = \frac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]}\sqrt{\mathsf{Var}[Y]}} = \frac{\mathsf{Cov}[X,Y]}{ab} \le 1$$

so $\mathsf{Cov}[X,Y] \le ab$. If you wrote that this is False and justified it by using the fact that $a$ or $b$ could be negative, then you should receive full credit. The problem should have included the assumption that $a > 0$ and $b > 0$, and this was announced during the exam, but if you finished early it is possible you missed the announcement.

(e) If $\mathsf{Cov}[X,Y] = 0$, then $\mathsf{Cov}[-X,Y] = 0$.

**Solution:**

**True.**

$$\begin{aligned}\mathsf{Cov}[-X,Y] &= \mathbb{E}[-XY] - \mathbb{E}[-X]\mathbb{E}[Y] \\ &= -\mathbb{E}[XY] + \mathbb{E}[X]\mathbb{E}[Y] \\ &= -(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) \\ &= 0.\end{aligned}$$

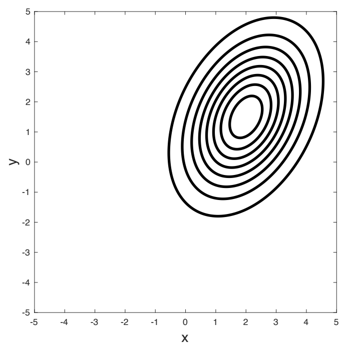
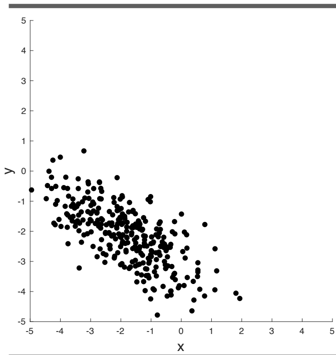
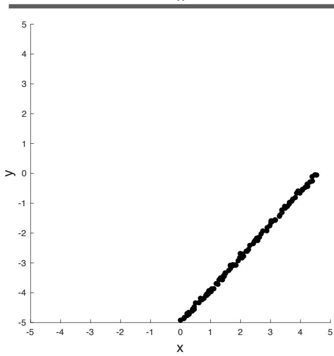
(f) If $\rho_{X,Y} = \dfrac{1}{4}$, then $Y = \dfrac{X}{4} + b$ from some constant $b$.

**Solution:**

**False.** If $Y = \dfrac{X}{4} + b$, then we must have $\rho_{X,Y} = 1$ since $Y$ is a linear function of $X$ with positive slope.

**Problem 2** *20 points*

The table below lists four scenarios via contour and scatter plots as well as equations. **Put a checkmark in the boxes in each row that you think are true for that scenario.** No justifications are needed and there may be multiple boxes checked per row and/or column.

| | $\mathbb{E}[X]$ noticeably more than 0 | $\mathbb{E}[Y]$ noticeably less than 0 | $\mathsf{Var}[X]$ noticeably larger than $\mathsf{Var}[Y]$ | $\mathsf{Cov}[X,Y]$ noticeably more than 0 | $\lvert\rho_{X,Y}\rvert$ close to 1 |
|---|---|---|---|---|---|
|   | ☑ | ☐ | ☐ | ☑ | ☐ |
|   | ☐ | ☑ | ☑ | ☐ | ☐ |
|   | ☑ | ☑ | ☐ | ☑ | ☑ |
| $Y = -2X - 6$<br>$\mathbb{E}[X] = -1$<br>$\mathsf{Var}[X] = 2$ | ☐ | ☑ | ☐ | ☐ | ☑ |

**Solution:**

For the last row, we have that $\mathbb{E}[Y] = -2\mathbb{E}[X] - 6 = -2(-1) - 6 = -4$, $\mathsf{Var}[Y] = (-2)^2\,\mathsf{Var}[X] = 4 \cdot 2 = 8$, and $\mathsf{Cov}[X,Y] = \mathsf{Cov}[X,-2X] = -2\,\mathsf{Var}[X] = -2 \cdot 2 = -4$.

**Problem 3** *16 points*

For each of the following parts, calculate the **two requested quantities exactly.** You should arrive at a numerical answer in each part. For this particular problem, you may **not** leave your answer in terms of integrals. Show your steps for partial credit.

(a) Let $X$ be a Uniform $(-2,2)$ random variable.
Calculate $\mathbb{E}[X]$ and $\mathbb{E}[X|B]$ for $B=(0,2)$.

**Solution:**

$\mathbb{E}[X]=\dfrac{-2+2}{2}=0$.

Note that $X$ given $B$ is Uniform $(0,2)$. Therefore, $\mathbb{E}[X|B]=\dfrac{0+2}{2}=1$.

(b) Let $X$ be a Gaussian $(1,4)$ random variable.
Calculate $\mathbb{P}[X<3]$ and $\mathbb{P}[X<-1|X<3]$ in terms of the $\Phi(z)$ function.

**Solution:**

$$\mathbb{P}[X<3]=\Phi\left(\frac{3-1}{2}\right)=\Phi(1)$$

$$\mathbb{P}[X<-1|X<3]=\frac{\mathbb{P}[\{X<-1\}\cap\{X<3\}]}{\mathbb{P}[X<3]}=\frac{\mathbb{P}[X<-1]}{\mathbb{P}[X<3]}=\frac{\Phi((\frac{-1-1}{2})}{\Phi(1)}=\frac{\Phi(-1)}{\Phi(1)}$$

(c) Let $Y$ be Geometric $(1/5)$ and $X$ given $Y=y$ is Binomial $(y,1/3)$.
Calculate $\mathbb{E}[X|Y=y]$ and $\mathbb{E}[X]$.

**Solution:**

$$\mathbb{E}[X|Y=y]=\frac{y}{3}$$

$$\mathbb{E}[X]=\mathbb{E}\big[E[X|Y]\big]=\mathbb{E}\left[\frac{Y}{3}\right]=\frac{5}{3}$$

(d) Let $X$ and $Y$ be jointly Gaussian with $\mathbb{E}[X]=2$, $\mathbb{E}[Y]=-1$, $\mathsf{Var}[X]=\mathsf{Var}[Y]=3$, and $\rho_{X,Y}=-\dfrac{1}{2}$. For $W=X+Y$ and $Z=X-Y$, calculate $\mathsf{Var}[W]$ and $\mathsf{Cov}[W,Z]$.

**Solution:**

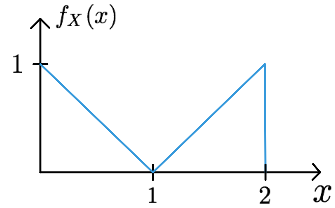
First, note that $\mathsf{Cov}[X,Y]=\rho_{X,Y}\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]}=-\dfrac{1}{2}\sqrt{3\cdot 3}=-\dfrac{3}{2}$.

$$\mathsf{Var}[W]=\mathsf{Var}[X]+\mathsf{Var}[Y]+2\mathsf{Cov}[X,Y]=3+3+2\cdot\left(-\frac{3}{2}\right)=3$$

$$\begin{aligned}\mathsf{Cov}[W,Y]&=1\cdot 1\cdot\mathsf{Var}[X]+1\cdot(-1)\cdot\mathsf{Var}[Y]+\big(1\cdot(-1)+1\cdot 1\big)\mathsf{Cov}[X,Y]\\&=3-3+0\cdot-\frac{3}{2}=0\end{aligned}$$

**Problem 4** *20 points*

Consider a continuous random variable $X$ with the following PDF:




$$f_X(x) = \begin{cases} 1-x & 0 \le x < 1 \\ x-1 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

If you left your answer as an integral, the correct answer is in blue.

(a) Calculate $\mathbb{P}\Big[X \le \dfrac{3}{2}\Big]$. (This can be done without integration, but you may leave your answer as an integral if you wish.)

> **Solution:**
>
> This is the area of two triangles, one with area $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ and the other with area $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. Therefore, $\mathbb{P}[X \le \frac{3}{2}] = \frac{5}{8}$.
> If you left this as an integral,
>
> $$\mathbb{P}\Big[X \le \frac{3}{2}\Big] = \int_0^1 (1-x)\,dx + \int_1^{3/2} (x-1)\,dx$$

(b) Calculate $\mathbb{E}[X]$. (This can be done without integration, but you may leave your answer as an integral if you wish.)

> **Solution:**
>
> By symmetry, $\mathbb{E}[X] = 1$. If you left this as an integral,
>
> $$\mathbb{E}[X] = \int_0^1 x(1-x)\,dx + \int_1^2 x(x-1)\,dx$$

(c) Calculate $f_{X|B}(x)$ where $B = [0, \frac{3}{2}]$. (This can be done without integration, but you may leave your answer as an integral if you wish.)

> **Solution:**
>
> $$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \dfrac{1-x}{\mathbb{P}[X \in B]} & 0 \le x < 1 \\ \dfrac{x-1}{\mathbb{P}[X \in B]} & 1 \le x \le \frac{3}{2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{8}{5}(1-x) & 0 \le x < 1 \\ \frac{8}{5}(x-1) & 1 \le x \le \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$
>
> where $\mathbb{P}[X \in B]$ is the answer from part (a).

(d) Calculate $\mathbb{P}[X > 1 | X \in B]$ where $B = [0, \frac{3}{2}]$. (This can be done without integration, but you may leave your answer as an integral if you wish.)

**Solution:**

We can work this out visually. The "restrict" step leaves us with $1 < X \leq \frac{3}{2}$, which is just a triangle with area $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. The "rescale" step is to divide by the area of the region $B = \{0 \leq X \leq \frac{3}{2}\}$, which we know from part (a) is $\frac{5}{8}$. Overall, we have $\frac{1/8}{5/8} = \frac{1}{5}$.

To work out the integral, we have

$$\mathbb{P}[X < 1 | X \in B] = \int_{-\infty}^{1} f_{X|B}(x)\, dx = \int_0^1 \frac{1-x}{\mathbb{P}[X \in B]}\, dx = \int_0^1 \frac{8}{5}(1-x)\, dx$$

(e) Calculate $\mathbb{E}[X|B]$ where $B = [0, \frac{3}{2}]$. (You can leave your answer as an integral.)

**Solution:**

$$\begin{aligned}\mathbb{E}[X|B] = \int_{-\infty}^{\infty} x\, f_{X|B}(x)\, dx &= \int_0^1 x\, \frac{1-x}{\mathbb{P}[X \in B]}\, dx + \int_1^{3/2} x\, \frac{x-1}{\mathbb{P}[X \in B]}\, dx \\ &= \int_0^1 x\, \frac{8}{5}(1-x)\, dx + \int_1^{3/2} x\, \frac{8}{5}(x-1)\, dx\end{aligned}$$

**Problem 5** *16 points*

Consider the joint probability mass function

| | $P_{XY}(x,y)$ | $y$ = 1 | 2 | 3 |
|---|---|---|---|---|
| $x$ | 1 | 0 | 0 | 0.2 |
| | 2 | 0.3 | 0.1 | 0 |
| | 3 | 0.1 | 0.3 | 0 |

(a) Calculate the marginal probability mass function $P_Y(y)$.

**Solution:**

$$P_Y(y) = \begin{cases} 0 + 0.3 + 0.1 = 0.4, & y = 1 \\ 0 + 0.1 + 0.3 = 0.4, & y = 2 \\ 0.2 + 0 + 0 = 0.2, & y = 3 \end{cases}$$

(b) Calculate $\mathbb{E}[XY]$.

**Solution:**

$$\mathbb{E}[XY] = (0.2)(1)(3) + (0.3)(2)(1) + (0.1)(2)(2) + (0.1)(3)(1) + (0.3)(3)(2) = 3.7$$

(c) Complete the following table for the conditional probability mass function $P_{X|Y}(x|y)$:

| | $P_{X\|Y}(x\|y)$ | $y$ = 1 | 2 | 3 |
|---|---|---|---|---|
| $x$ | 1 | | | |
| | 2 | | | |
| | 3 | | | |

**Solution:**

| $P_{X\|Y}(x\|Y=y)$ | | $y$ | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| $x$ | 1 | 0 | 0 | 1 |
| | 2 | 0.75 | 0.25 | 0 |
| | 3 | 0.25 | 0.75 | 0 |

(d) Calculate $\mathbb{E}[X|Y=y]$ (as a case-by-case function of $y$).

**Solution:**

$$\mathbb{E}[X|Y=y] = \begin{cases} 2 \cdot 0.75 + 3 \cdot 0.25 & y=1 \\ 2 \cdot 0.25 + 3 \cdot 0.75 & y=2 \\ 1 \cdot 1 & y=3 \end{cases} = \begin{cases} 2.25 & y=1 \\ 2.75 & y=2 \\ 1 & y=3 \end{cases}$$

**Problem 6** *16 points*

Consider the joint probability density function

$$f_{XY}(x,y) = \begin{cases} c, & x^2+y^2<4 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the range of $X$ and $Y$ in the $x$-$y$ plane.

**Solution:**

The range of $X$ and $Y$ is a circle with center at the origin and radius 2.

(b) Find the value of the constant $c$ that satisfies the normalization property so that $f_{XY}(x,y)$ is a valid joint probability density function.

**Solution:**

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = c\pi(2)^2 = 1,$$

so $c = 1/4\pi$.

(c) Calculate $\mathbb{P}[Y > 3X]$.

**Solution:**

From the symmetry of $f_{XY}(x,y)$ in the $x$-$y$ plane, $\mathbb{P}[Y > 3X] = 1/2$.

(d) Determine the marginal PDFs $f_X(x)$ and $f_Y(y)$.

**Solution:**

The marginal probability density functions are

$$f_X(x) = \begin{cases} \displaystyle\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} c\,dy & -2<x<2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2c\sqrt{4-x^2} & -2<x<2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \displaystyle\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} c\,dx & -2<y<2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2c\sqrt{4-y^2} & -2<y<2 \\ 0 & \text{otherwise} \end{cases}$$