Problem 1 *12 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X and Y are jointly continuous random variables, with probability density functions $f_X(x), f_Y(y)$, joint probability density function $f_{X,Y}(x,y)$, and corresponding cumulative distribution functions $F_X(x), F_Y(y)$, and $F_{X,Y}(x,y)$.

(a) $\mathbb{P}[X^2 > a] = \int_{\sqrt{a}}^{\infty} f_X(x)\,dx$

Solution:

False. $\mathbb{P}[X^2 > a] = \mathbb{P}[X < -\sqrt{a}] + \mathbb{P}[X > \sqrt{a}] = \int_{-\infty}^{-\sqrt{a}} f_X(x)\,dx + \int_{\sqrt{a}}^{\infty} f_X(x)\,dx$

(b) If $W = 2X + 3$ and $Z = -Y + 2$, then the correlation coefficients satisfy $\rho_{W,Z} = -\rho_{X,Z}$.

Solution:

True. $\mathsf{Var}[W] = 4\mathsf{Var}[X], \mathsf{Var}[Z] = \mathsf{Var}[Y], \mathsf{Cov}[W,Z] = \mathsf{Cov}[2X, -Y] = -2\mathsf{Cov}[X,Y]$. Then, $\rho_{W,Z} = \dfrac{\mathsf{Cov}[W,Z]}{\sqrt{\mathsf{Var}[W]\mathsf{Var}[Z]}} = \dfrac{-2\mathsf{Cov}[X,Y]}{2\sqrt{\mathsf{Var}[X]\mathsf{Var}[Y]}} = -\dfrac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]\mathsf{Var}[Y]}} = -\rho_{X,Y}$.

(c) If X and Y are independent, then $\mathbb{P}\big[\{X > a\} \cap \{Y > a\}\big] = 1 - F_X(a)F_Y(a)$.

Solution:

False. $\mathbb{P}\big[\{X > a\} \cap \{Y > a\}\big] = \mathbb{P}[X > a]\mathbb{P}[Y > a] = (1 - F_X(a))(1 - F_Y(a))$.

(d) $\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2]$

Solution:

True. By linearity of expectation,

$$\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2 + 2XY + Y^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2]\ .$$

(e) If $\mathsf{Cov}[X,Y] = \mathbb{E}[XY]$, then either $\mathbb{E}[X] = 0$ or $\mathbb{E}[Y] = 0$ or both are zero.

Solution:

True since $\mathsf{Cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$.

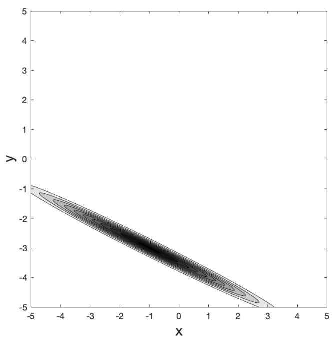
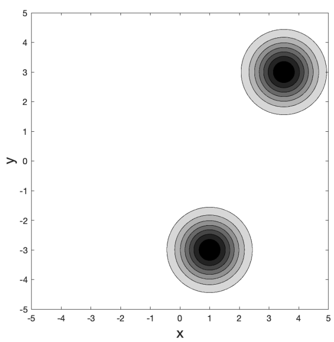
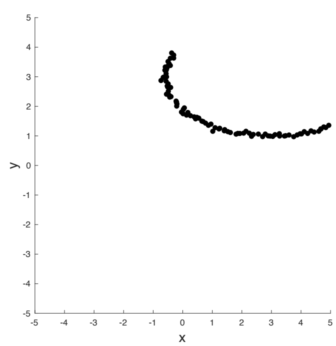
(f) If $\mathbb{E}[X|Y = y] = y^2$, then $\mathbb{E}[X] = \mathsf{Var}[Y] + (\mathbb{E}[Y])^2$.

Solution:

True. $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}[Y^2] = \mathsf{Var}[Y] + (\mathbb{E}[Y])^2$.

Problem 2 *20 points*

The table below lists four scenarios via contour and scatter plots as well as equations. **Put a checkmark in the boxes in each row that you think are true for that scenario.** No justifications are needed and there may be multiple boxes checked per row and/or column.

	$\mathbb{E}[X]$ noticeably more than 0	$\mathbb{E}[Y]$ noticeably less than 0	$\mathsf{Var}[X]$ noticeably larger than $\mathsf{Var}[Y]$	$\mathsf{Cov}[X,Y]$ noticeably more than 0	$\lvert\rho_{X,Y}\rvert$ close to 1
	☐	☑	☑	☐	☑
	☑	☐	☐	☑	☐
	☑	☐	☑	☐	☐
$\mathbb{E}[U]=4 \quad \mathbb{E}[V]=0$ $\mathsf{Var}[U]=1 \quad \mathsf{Var}[V]=1$ $\mathsf{Cov}[U,V]=\frac{1}{2}$ $X=2U-5, \quad Y=-V-3$	☑	☑	☑	☐	☐

Solution:

For the last row, we have that
$\mathbb{E}[X]=2\mathbb{E}[U]-5=8-5=3$, $\mathbb{E}[Y]=-\mathbb{E}[V]-3=0-3=-3$,
$\mathsf{Var}[X]=(2)^2\,\mathsf{Var}[U]=4$, $\mathsf{Var}[Y]=(-1)^2\mathsf{Var}[V]=1$, and
$\mathsf{Cov}[X,Y]=2\cdot(-1)\cdot\mathsf{Cov}[U,V]=-1$.

Problem 3 *16 points*

For each of the following parts, calculate the **two requested quantities exactly.** You should arrive at a numerical answer in each part. For this particular problem, you may **not** leave your answer in terms of integrals. Show your steps for partial credit.

(a) Let X be a Gaussian$(1,3)$ random variable. Calculate $\mathbb{E}[-2X+1]$ and $\mathbb{E}[(-2X+1)^2]$.

> **Solution:**
>
> $\mathbb{E}[-2X+1] = -2\mathbb{E}[X]+1 = -2\cdot 1+1 = -1$
> $\mathbb{E}[X^2] = \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 = 3+1^2 = 4$
> $\mathbb{E}[(-2X+1)^2] = \mathbb{E}[4X^2-4X+1] = 4\mathbb{E}[X^2]-4\mathbb{E}[X]+1 = 4\cdot 4-4\cdot 1+1 = 13$

(b) Let X be an Exponential(3) random variable. Calculate $\mathbb{P}[X<4]$ and $\mathbb{P}[X>2|X<4]$.

> **Solution:**
>
> The CDF of an Exponential(3) random variable is $F_X(x) = 1-e^{-3x}$ for $x\geq 0$. Then, $\mathbb{P}[\{X<4\}] = F_X(4) = 1-e^{-12}$.
>
> $$\mathbb{P}[\{X>2\}|\{X<4\}] = \frac{\mathbb{P}[\{2<X<4\}}{\mathbb{P}[\{X<4\}]} = \frac{F_X(4)-F_X(2)}{F_X(4)} = \frac{e^{-6}-e^{-12}}{1-e^{-12}}.$$

(c) Let Y be an Exponential(5) random variable and let X given $Y=y$ be a Uniform$(y,y+7)$ random variable. Calculate $\mathbb{E}[X|Y=2]$ and $\mathbb{E}[X]$.

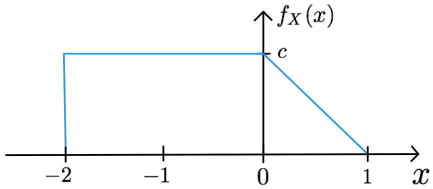
> **Solution:**
>
> For a Uniform(a,b) random variable Z, we know $\mathbb{E}[Z] = \frac{a+b}{2}$. We know $X|Y=y$ is Uniform$(y,y+7)$. Hence, $\mathbb{E}[X|Y=y] = \frac{y+y+7}{2} = y+\frac{7}{2}$ and $\mathbb{E}[X|Y=2] = \frac{11}{2}$. Note that Y is Exponential(5), so $\mathbb{E}[Y] = \frac{1}{5}$. Using the law of iterated expectation,
>
> $$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}\left[Y+\frac{7}{2}\right] = \frac{1}{5}+\frac{7}{2} = \frac{37}{10}\ .$$

(d) Let X and Y be jointly Gaussian with $\mathbb{E}[X]=\mathbb{E}[Y]=2$, $\mathsf{Var}[X]=\mathsf{Var}[Y]=2$, and $\mathsf{Cov}[X,Y]=-1$. Calculate $\mathsf{Var}[X+Y]$ and $\mathbb{P}[X+Y<2]$.
Here, you may leave your second answer in terms of the $\Phi(z)$ function.

> **Solution:**
>
> $\mathsf{Var}[X+Y] = \mathsf{Var}[X]+\mathsf{Var}[Y]+2\mathsf{Cov}[X,Y] = 2+2-2 = 2.$
> Note that $W = X+Y$ is Gaussian, with $\mathbb{E}[W] = 2+2 = 4$, and $\mathsf{Var}[W] = 2$ Thus,
>
> $$\mathbb{P}[X+Y<2] = \mathbb{P}[W<2] = F_W(2) = \Phi\left(\frac{2-4}{\sqrt{2}}\right) = \Phi\left(\frac{-2}{\sqrt{2}}\right) = \Phi(-\sqrt{2})$$

Problem 4 *20 points*

Consider a continuous random variable X with the following PDF:



$$f_X(x) = \begin{cases} c & -2 \leq x \leq 0 \\ c(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of c that satisfies the normalization property. Your answer can be an integral, but you can also take advantage of the simple structure of the PDF.

Solution:

The plot of $f_X(x)$ is a rectangle of height c, base 2, for $x \in [-2, 0]$, and a triangle formed by the points $(0,0), (0,c), (c,0)$, of height c, base 1. The total area is $2c + c/2$, which must be 1. Hence,

$$5/2c = 1 \Rightarrow c = 2/5.$$

(b) Calculate $\mathbb{E}[X]$. Your answer can be an integral.

Solution: $$\mathbb{E}[X] = \int_{-2}^{0} \frac{2x}{5} dx + \int_{0}^{1} \frac{2x(1-x)}{5} dx = -\frac{4}{5} + \frac{1}{5} - \frac{2}{15} = -\frac{11}{15}$$

(c) Calculate $\mathsf{Var}[X]$. Your answer can be an integral.

Solution: $$\mathsf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

where $\mathbb{E}[X]$ is from part (b) and

$$\mathbb{E}[X^2] = \int_{-2}^{0} \frac{2x^2}{5} dx + \int_{0}^{1} \frac{2}{5} x^2(1-x) dx = \frac{16}{15} + \frac{2}{15} - \frac{1}{10} = \frac{11}{10}.$$

(d) Calculate $\mathbb{P}[X \in B]$ for $B = (-\frac{1}{2}, \frac{1}{2})$. Your answer can be an integral, but you can also take advantage of the simple structure of the PDF.

Solution:

$$\mathbb{P}[X \in B] = \int_{-1/2}^{0} \frac{2}{5} dx + \int_{0}^{1/2} \frac{2(1-x)}{5} dx = \frac{1}{5} + \frac{3}{20} = \frac{7}{20}$$

(e) Calculate $\mathbb{P}\big[X < 0 \big| X \in B\big]$ for $B = (-\frac{1}{2}, \frac{1}{2})$. Your answer can be left in terms of integrals, but you can also take advantage of the simple structure of the PDF.

Solution:

Using the structure of the PDF, this is $\frac{1/5}{7/20} = \frac{4}{7}$. Via an integral, this is

$$\mathbb{P}\big[X < 0 \big| X \in B\big] = \frac{\mathbb{P}[-\frac{1}{2} < X < 0]}{\mathbb{P}[X \in B]} = \frac{1}{\mathbb{P}[X \in B]} \int_{-1/2}^{0} \frac{2}{5} dx = \frac{1}{\frac{7}{20}} \frac{1}{5} = \frac{4}{7}.$$

Problem 5 *16 points*

Consider the following **conditional PMF** $P_{X|Y}(x|y)$ for X given Y and **marginal PMF** $P_Y(y)$ for Y:

		y		
$P_{X\mid Y}(x\mid y)$		1	2	3
x	0	$\frac{1}{2}$	$\frac{1}{4}$	0
	1	$\frac{1}{2}$	$\frac{3}{4}$	1

$$P_Y(y) = \begin{cases} \frac{1}{3} & y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Complete the following table for the **joint probability mass function** $P_{X,Y}(x, y)$:

Solution:

Using $P_{X,Y}(x, y) = P_{X|Y}(x|y)\, P_Y(y)$,

		y		
$P_{X,Y}(x, y)$		1	2	3
x	0	$\frac{1}{6}$	$\frac{1}{12}$	0
	1	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$

(b) Calculate the marginal probability mass function $P_X(x)$.

Solution:

$$P_X(x) = \begin{cases} \frac{1}{6} + \frac{1}{12} + 0 = \frac{1}{4}, & x = 0 \\ \frac{1}{6} + \frac{1}{4} + \frac{1}{3} = \frac{3}{4}, & x = 1 \end{cases}$$

(c) Calculate $\mathbb{P}[Y > X]$

Solution:

We only have that $X = Y$ for $(x, y) = (1, 1)$. Otherwise, for all pairs in the range, we have that $y > x$. Therefore, $\mathbb{P}[Y > X] = 1 - P_{X,Y}(1, 1) = 1 - \frac{1}{6} = \frac{5}{6}$.

(d) Calculate $\mathsf{Cov}[X, Y]$.

Solution:

$$\mathbb{E}[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{3}{4}$$

$$\mathbb{E}[Y] = 1 \cdot \left(\frac{1}{6} + \frac{1}{6}\right) + 2 \cdot \left(\frac{1}{12} + \frac{1}{4}\right) + 3 \cdot \left(0 + \frac{1}{3}\right) = 2$$

$$\mathbb{E}[XY] = 1 \cdot 1 \cdot \frac{1}{6} + 1 \cdot 2 \cdot \frac{1}{4} + 1 \cdot 3 \cdot \frac{1}{3} = \frac{5}{3}$$

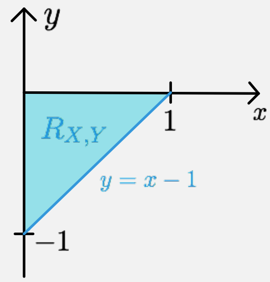
$$\mathsf{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{5}{3} - \frac{3}{4} \cdot 2 = \frac{1}{6}$$

Problem 6 *16 points*

Consider the following joint PDF $f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le 1 \textbf{ and } x-1 \le y \le 0 \\ 0 & \text{otherwise.} \end{cases}$

(a) Sketch the range of X and Y in the x-y plane. (Don't forget to label your axes.)

Solution:



(b) What is the probability that Y is less than X?

Solution:

Note that we have that $Y \le 0$ and $X \ge 0$. Therefore, Y is always less than X and $\mathbb{P}[Y < X] = 1$. (The probability that $X = Y$ is 0 since this only happens if $Y = 0$ and $X = 0$, which has probability 0 for continuous random variables.)

(c) Determine the marginal PDF $f_Y(y)$.

Solution:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^{y+1} 2\,dx & -1 \le y \le 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2y+2 & -1 \le y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) Determine the conditional expected value $\mathbb{E}[X|Y = y]$.

Solution:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{2}{f_Y(y)} & 0 \le x \le 1 \textbf{ and } x-1 \le y \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \dfrac{1}{y+1} & 0 \le x \le 1 \textbf{ and } x-1 \le y \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx = \int_0^{y+1} x f_{X|Y}(x|y)\,dx = \frac{y+1}{2}$$