**Problem 1** *20 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that $X$ and $Y$ are continuous random variables.

(a) If $f_X(a) \geq f_X(b)$, then $F_X(a) \geq F_X(b)$.

(b) If $\rho_{X,Y} = 0$, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

(c) For $a < b < c$, we have that $\mathbb{P}[a < X < c | X < b] = 1 - \dfrac{F_X(a)}{F_X(b)}$.

(d) Let $X$ and $Y$ are jointly Gaussian with $\text{Var}[X] = \text{Var}[Y]$. Define $U = X + Y$, $V = X - Y$. Then, $U$ and $V$ are independent.

(e) If $X$ and $Y$ are jointly Gaussian, then $\mathbb{E}[X|Y = y]$ is a linear function of $y$ plus a constant.

(continued on the next page)

(f) If $\rho_{X,Y} > 0$, then $X$ and $Y$ have the same sign with probability at least $1/2$.

(g) Let $X$ be a uniformly distributed random variable on [0,2], and let $f_{Y|X}(y|x)$ be such that $\mathbb{E}[Y|X = x] = 3x$. Then, $\mathbb{E}[Y] = 3$.

(h) Assume now that $X$ is a Gaussian random variable with mean 0 and variance 1. Then, $\mathbb{E}[X^4] = 0$.

(i) Let $X$ is an exponential random variable with $\mathbb{E}[X] = 1$. Then, the random variable $Y = 3X$ is also an exponential random variable with $\mathsf{Var}[Y] = 9$.

(j) If $X, Y$ are jointly continuous random variables, and $Z = -5 - Y + X$, then $\mathsf{Var}[Z] = \mathsf{Var}[X] + 2\mathsf{Cov}[X, Y] + \mathsf{Var}[Y]$.

**Problem 2** *16 points*

For each of the following parts, calculate the two requested quantities exactly. You should arrive at a numerical answer in each part. For this particular problem, you may not leave your answer in terms of integrals. Show your steps for partial credit.

(a) Let $X$ be a Uniform$(-2, 3)$ random variable. Calculate $\mathsf{Var}[X]$ and $\mathbb{P}[X^2 > 1]$.

(b) Let $X$ and $Y$ be independent Exponential$(\frac{1}{2})$ random variables. Calculate $\mathbb{E}[2X^2 - X + 3]$ and $\mathbb{E}[XY^2]$.

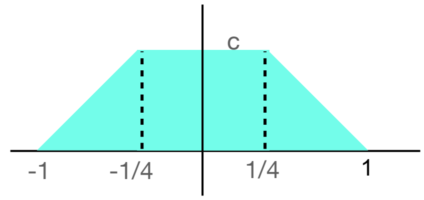
(c) Let $X$ be an Exponential$(3)$ random variable and let $Y$ given that $X = x$ be a Gaussian$(x, 1)$ random variable. Calculate $\mathbb{E}[Y^2 | X = x]$ and $\mathbb{E}[Y^2]$.

(d) Let $X$ and $Y$ be jointly Gaussian random variables with means $\mu_X = 1, \mu_Y = -2$, $\mathsf{Var}[X] = \frac{1}{4}$, $\mathsf{Var}[Y] = 1$, and correlation coefficient $\rho_{X,Y} = \frac{1}{4}$. Let $W = 3X + Y$. Calculate $\mathbb{P}[\{Y \in [-3, 3]\}]$ and $\mathbb{P}[\{W > 0\}]$. Write your answers in terms of the standard normal CDF $\Phi(z)$.

**Problem 3** *16 points*

Let $X$ be a continuous random variable, with probability density $f_X(x)$ given below:

$$f_X(x) = \begin{cases} \frac{4}{3}c(x+1) & -1 \le x \le -\frac{1}{4}, \\ c & -\frac{1}{4} \le x \le \frac{1}{4} \\ \frac{4}{3}c(1-x) & \frac{1}{4} \le x \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$




(a) Determine the value of $c$ that satisfies the normalization property **as a number.** Set $c$ to this value for the remainder of the problem. Note that the two triangular parts of the PDF have the same area, and the figure is a trapezoid.

(b) Calculate $\mathbb{P}[\{X \in B\}]$ for the set $B = (\frac{1}{2}, 1)$. The answer should be a number.

(c) Calculate $f_{X|B}(x|B)$, the conditional density of $X$ given that $B$ is observed

(d) Calculate $\mathbb{E}[X^2|B]$. You can leave the answer in terms of integrals.

**Problem 4** *16 points*

Discrete random variables $X, Y$ have joint probability mass function (PMF) $P_{X,Y}(x, y)$ given in the table described below.

| $y \backslash x$ | x = -1 | x = 0 | x = 2 |
|---|---|---|---|
| y = 2 | 0.1 | 0 | 0.05 |
| y = 0 | 0 | 0.5 | 0 |
| y = -1 | 0.2 | 0.05 | 0.1 |

(a) Compute the marginal probability mass functions $P_X(x)$ and $P_Y(y)$.

(b) Compute $\mathsf{Cov}[X, Y]$.

(c) Compute the conditional probability mass function $P_{Y|X}(y|X = -1)$ for $y = -1, 0, 2$.

(d) Compute $\mathsf{Var}[Y - 2|X = -1]$.

**Problem 5** *16 points*

Let continuous random variables $X, Y$ have joint density $f_{X,Y}(x,y) = \begin{cases} 1/x & 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

(a) Compute $\mathbb{P}[X \leq 2Y]$. You can leave the answer in terms of an integral with the right limits.

(b) Compute the conditional probability density function $f_{Y|X}(y|x)$. The answer must not use any integrals.

(c) Compute $\mathbb{E}[Y|X = x]$ for all $x \in (0,1)$. The answer must be a function of $x$, not an integral.

(d) Compute $\mathsf{Cov}[X,Y]$. You can leave the answer in terms of integrals.

**Problem 6** *16 points*

Assume that $X$, $Y$ are uncorrelated, jointly Gaussian random variables, such that $\mathbb{E}[X] = 1, \mathbb{E}[Y] = 1, \mathsf{Var}[X] = 1, \mathsf{Var}[Y] = 2$. Define $A = 4X + 2Y - 1, B = 2X + 4Y + 1$.

(a) Compute $\mathbb{E}[A], \mathbb{E}[B]$.

(b) Compute $\mathsf{Var}[A], \mathsf{Var}[B]$ and $\mathsf{Cov}[A, B]$.

(c) Compute $\mathbb{E}[A|B = b]$ and $\mathsf{Var}[A|B = b]$.

(d) Let $Z = X^2Y^2$ be a new random variable. Compute $\mathbb{E}[Z]$.

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