Problem 1 *20 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X and Y are continuous random variables.

(a) If $f_X(a) \geq f_X(b)$, then $F_X(a) \geq F_X(b)$.

False. Consider an Exponentia(1) random variable. Then, $f_X(0) > f_X(1)$, but clearly $F_X(0) = 0 < F_X(1)$.

(b) If $\rho_{X,Y} = 0$, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

True. Since $\rho_{X,Y} = 0$, we know that $\text{Cov}[X, Y] = 0$. Combining this with $\text{Cov}[X, Y] = E[XY] - E[X]E[Y]$, we get the desired result (even without independence).

(c) For $a < b < c$, we have that $\mathbb{P}[a < X < c|X < b] = 1 - \dfrac{F_X(a)}{F_X(b)}$.

True. Using the definition of conditional probability,

$$\mathbb{P}[a < X < c|X < b] = \frac{P\big[\{a < X < c\} \cap \{X < b\}\big]}{P[X < b]} = \frac{P[a < X < b]}{P[X < b]} = \frac{F_X(b) - F_X(a)}{F_X(b)} = 1 - \frac{F_X(a)}{F_X(b)}$$

(d) Let X and Y are jointly Gaussian with $\text{Var}[X] = \text{Var}[Y]$. Define $U = X + Y$, $V = X - Y$. Then, U and V are independent.

True. Note that

$$\mathsf{Cov}[U, V] = \mathsf{Cov}[X, X] - \mathsf{Cov}[Y, Y] = \mathsf{Var}[X] - \mathsf{Var}[Y] = 0.$$

Hence, they are uncorrelated, and jointly Gaussian, hence independent.

(e) If X and Y are jointly Gaussian, then $\mathbb{E}[X|Y = y]$ is a linear function of y plus a constant.

True. For joint Gaussians,

$$\mathbb{E}[X|Y = y] = \mathbb{E}[X] + \frac{\mathsf{Cov}[X, Y]}{\mathsf{Var}[Y]}(y - \mathbb{E}[Y]) = \frac{\mathsf{Cov}[X, Y]}{\mathsf{Var}[Y]}y + \left(\mathbb{E}[X] - \frac{\mathsf{Cov}[X, Y]}{\mathsf{Var}[Y]}\mathbb{E}[Y]\right)$$

(f) If $\rho_{X,Y} > 0$, then X and Y have the same sign with probability at least $1/2$.

False. For positive $\rho_{X,Y} > 0$, we can say that X and Y have the same sign "often" but not necessarily with probability at least $1/2$.

(g) Let X be a uniformly distributed random variable on [0,2], and let $f_{Y|X}(y|x)$ be such that $\mathbb{E}[Y|X = x] = 3x$. Then, $\mathbb{E}[Y] = 3$.

True. This is an application of iterated expectation:

$$\mathbb{E}[Y] = \mathbb{E}\Big[\mathbb{E}[Y|X]\Big] = \mathbb{E}\Big[3X\Big] = 3\mathbb{E}[X] = 3$$

(h) Assume now that X is a Gaussian random variable with mean 0 and variance 1. Then, $\mathbb{E}[X^4] = 0$.

False. $X^4 > 0$ almost everywhere, so its weighted average $\mathbb{E}[X^4] > 0$. Doing integration by parts, we can compute $\mathbb{E}[X^4] = 3$.

(i) Let X is an exponential random variable with $\mathbb{E}[X] = 1$. Then, the random variable $Y = 3X$ is also an exponential random variable with $\mathsf{Var}[Y] = 9$.

True. By scaling $Y = 3X$, the rate $\lambda_Y = 1/3$, so $\mathbb{E}[Y] = 3$, and $\mathsf{Var}[Y] = \frac{1}{\lambda_Y^2} = 9$.

(j) If X, Y are jointly continuous random variables, and $Z = -5 - Y + X$, then $\mathsf{Var}[Z] = \mathsf{Var}[X] + 2\mathsf{Cov}[X,Y] + \mathsf{Var}[Y]$.

False. $\mathsf{Var}[Z] = \mathsf{Cov}[X - Y, X - Y] = \mathsf{Var}[X] - 2\mathsf{Cov}[X,Y] + \mathsf{Var}[Y]$

Problem 2 *16 points*

For each of the following parts, calculate the two requested quantities exactly. You should arrive at a numerical answer in each part. For this particular problem, you may not leave your answer in terms of integrals. Show your steps for partial credit.

(a) Let X be a Uniform $(-2, 3)$ random variable. Calculate $\mathsf{Var}[X]$ and $\mathbb{P}[X^2 > 1]$.

$$\mathsf{Var}[X] = \frac{(3 - (-2))^2}{12} = \frac{25}{12}.$$

$$\mathbb{P}[X^2 > 1] = \mathbb{P}[X > 1] + \mathbb{P}[X < -1] = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}.$$

(b) Let X and Y be independent Exponential $(\frac{1}{2})$ random variables. Calculate $\mathbb{E}[2X^2 - X + 3]$ and $\mathbb{E}[XY^2]$.

$$\mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \mathsf{Var}[X] = 4 + 4 = 8.$$

$$\mathbb{E}[2X^2 - X + 3] = 2\mathbb{E}[X^2] - \mathbb{E}[X] + 3 = 16 - 2 + 3 = 17.$$

$$\mathbb{E}[XY^2] = \mathbb{E}[X]\mathbb{E}[Y^2] = (2)(4 + 4) = 16.$$

(c) Let X be an Exponential (3) random variable and let Y given that $X = x$ be a Gaussian $(x, 1)$ random variable. Calculate $\mathbb{E}[Y^2|X = x]$ and $\mathbb{E}[Y^2]$.

$$\mathbb{E}[Y^2|X=x] = (\mathbb{E}[Y|X=x])^2 + \mathsf{Var}[Y|X=x] = x^2+1.$$

$$\mathbb{E}[Y^2] = \mathbb{E}\Big[\mathbb{E}[Y^2|X]\Big] = \mathbb{E}[X^2+1] = 1 + (\mathbb{E}[X])^2 + \mathsf{Var}[X] = \frac{1}{9} + \frac{1}{9} + 1 = \frac{11}{9}$$

(d) Let X and Y be jointly Gaussian random variables with means $\mu_X = 1, \mu_Y = -2$, $\mathsf{Var}[X] = \frac{1}{4}$, $\mathsf{Var}[Y] = 1$, and correlation coefficient $\rho_{X,Y} = \frac{1}{4}$. Let $W = 3X + Y$. Calculate $\mathbb{P}[\{Y \in [-3,3]\}]$ and $\mathbb{P}[\{W > 0\}]$. Write your answers in terms of the standard normal CDF $\Phi(z)$.

Let $Z_Y = \frac{Y-\mu_Y}{\sqrt{\mathsf{Var}[Y]}} = Y+2$. Then Z_Y is a Gaussian random variable with mean 0, variance 1. The event $\{Y \in [-3,3]\}$ is the same event as $\{Z_Y \in [-1,5]\}$, which has probability $\Phi(5) - \Phi(-1)$.

$W = 3X + Y$ is a Gaussian random variable with mean $\mathbb{E}[W] = 3\mathbb{E}[X] + \mathbb{E}[Y] = 3 - 2 = 1$. Its variance is

$$\mathsf{Var}[W] = \mathsf{Cov}[3X+Y, 3X+Y] = 9\mathsf{Var}[X] + 6\mathsf{Cov}[X,Y] + \mathsf{Var}[Y].$$

Note that $\mathsf{Cov}[X,Y] = \rho_{X,Y} \times \sqrt{\mathsf{Var}[X]\mathsf{Var}[Y]} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$. Then,

$$\mathsf{Var}[W] == 9 \times \frac{1}{4} + 6 \times \frac{1}{8} + 1 = 4.$$

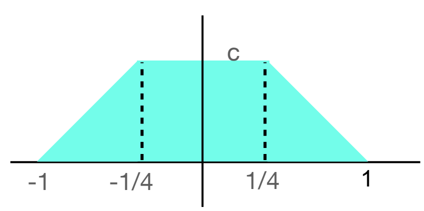
The variable $Z_W = \frac{W-1}{2}$ is a zero-mean, unit variance Gaussian random variable. The event $\{W > 0\}$ is equivalent to the event $\{Z_W > -\frac{1}{2}\}$. Hence,

$$\mathbb{P}[\{W > 0\}] = 1 - \Phi(-\frac{1}{2}).$$

Problem 3 *16 points*

Let X be a continuous random variable, with probability density $f_X(x)$ given below:

$$f_X(x) = \begin{cases} \frac{4}{3}c(x+1) & -1 \le x \le -\frac{1}{4}, \\ c & -\frac{1}{4} \le x \le \frac{1}{4} \\ \frac{4}{3}c(1-x) & \frac{1}{4} \le x \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$



(a) Determine the value of c that satisfies the normalization property **as a number.** Set c to this value for the remainder of the problem. Note that the two triangular parts of the PDF have the same area, and the figure is a trapezoid.

The area of the trapezoid is $c \times \frac{2+0.5}{2} = \frac{5}{4}c$. Hence, $c = \frac{4}{5}$ for normalization.

(b) Calculate $\mathbb{P}[\{X \in B\}]$ for the set $B = (\frac{1}{2}, 1)$. The answer should be a number.

We are looking at the area of a triagle with base length $1/2$, and height $f_X(1/2) = \frac{4}{3}c(1 - \frac{1}{2}) = \frac{16}{30} = \frac{8}{15}$. The area is $\frac{1}{2} \times \frac{1}{2} \times \frac{8}{15} = \frac{2}{15}$.

(c) Calculate $f_{X|B}(x|B)$, the conditional density of X given that B is observed

$$f_{X|B}(x) = \begin{cases} \frac{1}{\mathbb{P}[B]} \frac{16}{15}(1-x) = 8(1-x) & \frac{1}{2} \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(d) Calculate $\mathbb{E}[X^2|B]$. You can leave the answer in terms of integrals.

$$\mathbb{E}[X^2|B] = \int_{\frac{1}{2}}^{1} x^2 \, 8(1-x) \, dx$$

Problem 4 *16 points*

Discrete random variables X, Y have joint pmf $P_{X,Y}(x, y)$ given in the table described below.

$P_{X,Y}(x,y)$	x = -1	x = 0	x = 2
y = 2	0.1	0	0.05
y = 0	0	0.5	0
y = -1	0.2	0.05	0.1

(a) Compute the marginal probability mass functions $P_X(x)$ and $P_Y(y)$.

Summing rows and columns, we obtain

$$P_X(x) = \begin{cases} 0.3 & x = -1, \\ 0.55 & x = 0, \\ 0.15 & x = 2. \end{cases} \qquad P_Y(y) = \begin{cases} 0.35 & y = -1, \\ 0.5 & y = 0, \\ 0.15 & y = 2. \end{cases}$$

(b) Compute $\mathsf{Cov}[X, Y]$.

First, compute $\mathbb{E}[X] = 0.15 \times 2 + 0.3 \times (-1) = 0$. We don't need to compute $\mathbb{E}[Y]$, so we know $\mathsf{Cov}[X, Y] = \mathbb{E}[XY]$.

$\mathbb{E}[XY] = 0.1 \times (-1) \times 2 + 0.05 \times 2 \times 2 + 0.2 \times (-1) \times (-1) + 0.1 \times (-1) \times (2) = -0.2 + 0.2 + 0.2 - 0.2 = 0.$

(c) Compute $P_{Y|X}(y|X = -1)$ for $y = -1, 0, 2$.

Restrict to the column $x = -1$, and rescale to get:

$$P_{Y|X}(y|x = -1) = \begin{cases} \frac{2}{3} & y = -1, \\ 0 & y = 0, \\ \frac{1}{3} & y = 2. \end{cases}$$

(d) Compute $\mathsf{Var}[Y - 2|X = -1]$.

$$\mathbb{E}[Y|X = -1] = \frac{2}{3} \times (-1) + \frac{1}{3} \times 2 = 0.$$

Hence,

$$\mathsf{Var}[Y - 2|X = -1] = \mathsf{Var}[Y|X = -1] = \mathbb{E}[Y^2|X = -1] = \frac{2}{3} \times (-1)^2 + \frac{1}{3} \times 2^2 = 2.$$

Problem 5 *16 points*

Let continuous random variables X, Y have joint density $f_{X,Y}(x, y) = \begin{cases} 1/x & 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

(a) Compute $\mathbb{P}[X \leq 2Y]$. You can leave the answer in terms of an integral with the right limits.

$$\mathbb{P}[X \leq 2Y] = \int_0^1 \int_{x/2}^x \frac{1}{x} \, dy \, dx = \int_0^1 \frac{x}{2} \frac{1}{x} \, dx = \frac{1}{2}$$

$$\mathbb{E}[X] = \int_0^1 \int_0^x x(1/x) dy dx = 1/2$$

$$\mathbb{E}[Y] = \int_0^1 \int_0^x y(1/x) dy dx = \int_0^1 x/2 dx = 1/4$$

(b) Compute the conditional probability density function $f_{Y|X}(y|x)$. The answer must not use any integrals.

Compute the marginal density of X:

$$f_X(x) = \int_0^x (1/x) dy = 1, x \in (0, 1)$$

Then,

$$f_{Y|X}(y|x) = 1/x, y \in (0, x)$$

Could do this by inspection by noting the joint density is uniform in y for each x.

(c) Compute $\mathbb{E}[Y|X = x]$ for all $x \in (0,1)$. The answer must be a function of x, not an integral.

Trivial given above: $\mathbb{E}[Y|X = x] = x/2$.

(d) Compute $\mathsf{Cov}[X,Y]$. You can leave the answer in terms of integrals.

One integral required:

$$\mathbf{E}[XY] = \int_0^1 \int_0^x xy(1/x)dydx = \int_0^1 \int_0^x ydydx = 1/6$$

$$\mathsf{Cov}[X,Y] = 1/6 - 1/8 = 1/24$$

Problem 6 *16 points*

Assume that X, Y are uncorrelated, jointly Gaussian random variables, such that $\mathbb{E}[X] = 1, \mathbb{E}[Y] = 1, \mathsf{Var}[X] = 1, \mathsf{Var}[Y] = 2$. Define $A = 4X + 2Y - 1, B = 2X + 4Y + 1$.

(a) Compute $\mathbb{E}[A], \mathbb{E}[B]$.

$$\mathbb{E}[A] = \mathbb{E}[4X + 2Y - 1] = 4\mathbb{E}[X] + 2\mathbb{E}[Y] - 1 = 5.$$

$$\mathbb{E}[B] = \mathbb{E}[2X + 4Y + 1] = 2\mathbb{E}[X] + 4\mathbb{E}[Y] + 1 = 7.$$

(b) Compute $\mathsf{Var}[A], \mathsf{Var}[B]$ and $\mathsf{Cov}[A,B]$.

$$\mathsf{Var}[A] = \mathsf{Cov}[4X+2Y-1, 4X+2Y-1] = 16\mathsf{Var}[X]+16\mathsf{Cov}[X,Y]+4\mathsf{Var}[Y] = 16+0+8 = 24.$$

$$\mathsf{Var}[B] = \mathsf{Cov}[2X+4Y+1, 2X+4Y+1] = 4\mathsf{Var}[X]+16\mathsf{Cov}[X,Y]+16\mathsf{Var}[Y] = 4+0+32 = 36.$$

$$\mathsf{Cov}[A,B] = \mathsf{Cov}[4X+2Y-1, 2X+4Y+1] = 8\mathsf{Var}[X]+20\mathsf{Cov}[X,Y]+8\mathsf{Var}[Y] = 8+0+16 = 24.$$

(c) Compute $\mathbb{E}[A|B = b]$ and $\mathsf{Var}[A|B = b]$.

Using the Gaussian formulas,

$$\mathbb{E}[A|B = b] = \mathbb{E}[A] + \frac{\mathsf{Cov}[A,B]}{\mathsf{Var}[B]}(b - \mathbb{E}[B] = 5 + \frac{24}{36}(b - 7) = \frac{2}{3}b + \frac{1}{3}.$$

$$\mathsf{Var}[A|B = b] = \mathsf{Var}[A] - \frac{\mathsf{Cov}[A,B]^2}{\mathsf{Var}[B]} = 24 - \frac{24^2}{36} = 24(1 - \frac{2}{3}) = 8.$$

(d) Let $Z = X^2Y^2$ be a new random variable. Compute $\mathbb{E}[Z]$.

Note that X, Y are independent because they are jointly Gaussian and uncorrelated. Then,

$$\mathbb{E}[Z] = \mathbb{E}[X^2Y^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2] = (1^2 + 1) \times (1^2 + 2) = 6.$$