Problem 1 *20 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X and Y are continuous random variables.

(a) If $F_X(a) > F_X(b)$, then $a > b$.

(b) If $\mathbb{E}[XY] = 0$, then $\rho_{X,Y} = 0$.

(c) $\mathsf{Cov}[X, X] = \mathsf{Var}[X]$.

(d) $f_{X|Y}(x|y) = f_{Y|X}(y|x)$

(e) If X and Y are independent, then $\mathsf{Var}[aX + bY] = a\mathsf{Var}[X] + b\mathsf{Var}[Y]$.

(f) If $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, then $\mathsf{Var}[X+Y] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY]$.

(g) Let X and Y be independent, each with mean 1 and variance 1. Then, $\mathbb{E}[X^2Y^2] = 4$.

(h) If $\mathbb{E}[X|Y=y] = 0$, then $\mathbb{E}[X] = 0$.

(i) $\mathbb{P}[A \cap B] \leq \mathbb{P}[A]\mathbb{P}[B]$.

(j) $\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$

Problem 2 *12 points*

For each of the following parts, calculate the quantities of interest. Show your steps for partial credit.

(a) (2 pts) Let X be a Uniform $(-2, 4)$ random variable. Calculate $\mathbb{P}\big[X > 2 | X^2 > 1\big]$.

(b) (2 pts) Let X be an Gaussian $(-2, 3)$ random variable. Calculate $\mathbb{E}[2X^2 + 3X - 2]$.

(c) (4 pts) Let X be a Uniform $(1, 3)$ random variable and let Y given that $X = x$ be an Exponential $\left(\frac{2}{x}\right)$ random variable. Calculate $\mathbb{E}[Y|X = x]$ and $\mathbb{E}[Y]$.

(d) (4 pts) Let X and Y be jointly Gaussian random variables with $\mu_X = 2, \mu_Y = -1$, $\mathsf{Var}[X] = \frac{1}{2}$, $\mathsf{Var}[Y] = 2$, and $\rho_{X,Y} = \frac{1}{2}$. Let $V = 2X - Y$. Calculate $\mathbb{P}[\{V > 1\}]$ and $\mathbb{P}[\{Y \in [-2, 1]\}]$. Write your answers in terms of the standard normal CDF $\Phi(z)$.

Problem 3 *20 points*

Consider the following Cumulative Distribution Function (CDF) $F_X(x) = \begin{cases} 0 & x < 1 \\ c\left(\frac{1}{3}x^3 - x + \frac{2}{3}\right) & 1 \le x \le 2 \\ 1 & 2 \ge x\ . \end{cases}$

(a) Determine the value of c that satisfies the normalization property. Set c to this value for the remainder of the problem.

(b) Determine the probability density function (PDF) $f_X(x)$ of X.

(c) What is the expected value of X?

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(d) What is the probability that $X < \frac{4}{3}$?

(e) Calculate $\mathbb{E}[X|B]$ where B is the event that $X < \frac{4}{3}$.

Problem 4 *16 points*

Consider the following joint PDF $f_{X,Y}(x,y) = \begin{cases} \frac{3}{8}y^2 & -1 \le x \le 1 \text{ and } x-1 \le y \le x+1 \\ 0 & \text{otherwise.} \end{cases}$

(a) Calculate $\mathrm{Cov}[X,Y]$. You can leave your answers in terms of integrals. Diagrams help...

(b) Determine the conditional PDF $f_{Y|X}(y|x)$.

(c) Calculate the conditional expected value $\mathbb{E}[Y|X=x]$.

(d) Calculate $\mathrm{Var}[X-Y]$.

Problem 5 *16 points*

Consider the following marginal and conditional PMFs for X and Y:

$$P_Y(y) = \begin{cases} \frac{1}{2} & y = 1 \\ \frac{1}{4} & y = 2, 3. \\ 0 & \text{otherwise} \end{cases} \qquad P_{X|Y}(x|y) = \begin{cases} 1 - \frac{y}{3} & x = -1 \\ \frac{y}{3} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Fill out the joint PMF table for X and Y.

	$P_{X,Y}(x,y)$	$y = 1$	$y = 2$	$y = 3$
x	-1			
	$+1$			

(b) Calculate $\mathbb{P}\big[|X - Y| \leq 2\big]$.

(c) Are X and Y statistically independent? Explain why or why not.

(d) Calculate the covariance of X and Y.

Problem 6 *16 points*

Let X and Y be jointly Gaussian random variables with $\mu_X = 1, \mu_Y = -1$, $\mathsf{Var}[X] = 1$, $\mathsf{Var}[Y] = 2$, and $\rho_{X,Y} = \frac{3}{4}$. Define $A = 3X - 2Y$ and $B = 2X + 1$

(a) What are the expected values of A and B?

(b) What is the covariance $\mathsf{Cov}[A, B]$?

(c) Assume you observe $B = b$. What is $\mathbb{E}[A|B = b]$ as a function of b? What is the conditional variance $\mathsf{Var}[A|B = b]$?

(d) Are A and B independent? Why?