Problem 1 *20 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X and Y are continuous random variables.

(a) If $F_X(a) > F_X(b)$, then $a > b$.

True. $F_X(x)$ is a non-decreasing function of x.

(b) If $\mathbb{E}[XY] = 0$, then $\rho_{X,Y} = 0$.

False. We know that $\mathsf{Cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Thus, $\mathsf{Cov}[X,Y] = \mathbb{E}[XY]$ only if $\mathbb{E}[X] = 0$ or $\mathbb{E}[Y] = 0$.

(c) $\mathsf{Cov}[X,X] = \mathsf{Var}[X]$.

True. $\mathsf{Cov}[X,X] = \mathbb{E}\big[(X - \mathbb{E}[X])(X - \mathbb{E}[X])\big] = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = \mathsf{Var}[X]$.

(d) $f_{X|Y}(x|y) = f_{Y|X}(y|x)$

False. This is only true if $f_X(x) = f_Y(y)$.

(e) If X and Y are independent, then $\mathsf{Var}[aX + bY] = a\mathsf{Var}[X] + b\mathsf{Var}[Y]$.

False. If X and Y are independent, then $\mathsf{Cov}[X,Y] = 0$ and we have that $\mathsf{Var}[aX + bY] = a^2\mathsf{Var}[X] + b^2\mathsf{Var}[Y]$.

(f) If $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, then $\mathsf{Var}[X + Y] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY]$.

True. Since $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, we have that $\mathsf{Var}[X] = \mathbb{E}[X^2]$, $\mathsf{Var}[Y] = \mathbb{E}[Y^2]$, and $\mathsf{Cov}[X,Y] = \mathbb{E}[XY]$. Therefore, the stated formula is equivalent to our formula for the variance of sums $\mathsf{Var}[X + Y] = \mathsf{Var}[X] + \mathsf{Var}[Y] + 2\mathsf{Cov}[X,Y]$.

(g) Let X and Y be independent, each with mean 1 and variance 1. Then, $\mathbb{E}[X^2Y^2] = 4$.

True. Since X and Y are independent, we can factor the product of expectations, $\mathbb{E}[X^2Y^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2]$. Now, we can use the alternate variance formula to get $\mathbb{E}[X^2] = \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 = 1 + 1^2 = 2$ and $\mathbb{E}[Y^2] = 2$ by the same calculation. Thus, $\mathbb{E}[X^2Y^2] = 4$.

(h) If $\mathbb{E}[X|Y = y] = 0$, then $\mathbb{E}[X] = 0$.

True. By the iterated expectation property, $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}[0] = 0$.

(i) $\mathbb{P}[A \cap B] \leq \mathbb{P}[A]\mathbb{P}[B]$.

False. Consider $A = B$ with $\mathbb{P}[A] < 1$. Then, $\mathbb{P}[A \cap B] = \mathbb{P}[A] > \mathbb{P}[A]\mathbb{P}[B] = \mathbb{P}[A]^2$.

(j) $\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$

True. This is a consequence of the Cauchy-Schwartz inequality used to prove that the correlation coefficient is has magnitude less than or equal to 1.

Can also prove it by brute force computation

$$\begin{aligned}
(\mathsf{Cov}[X,Y] + \mathbb{E}[X]\mathbb{E}[Y])^2 &\leq (\mathsf{Var}[X] + \mathbb{E}[X]^2)(\mathsf{Var}[Y] + \mathbb{E}[Y]^2) \\
\mathsf{Cov}[X,Y]^2 + 2\mathsf{Cov}[X,Y]\mathbb{E}[X]\mathbb{E}[Y] &\leq \mathsf{Var}[X]\mathsf{Var}[Y] + \mathbb{E}[X]^2\mathsf{Var}[Y] + \mathbb{E}[Y]^2\mathsf{Var}[X] \\
\mathsf{Cov}[X,Y]^2 &\leq \mathsf{Var}[X]\mathsf{Var}[Y] + \mathbb{E}[X]^2\mathsf{Var}[Y] - 2\mathsf{Cov}[X,Y]\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[Y]^2\mathsf{Var}[X] \\
&= \mathsf{Var}[X]\mathsf{Var}[Y] + \mathsf{Var}[\mathbb{E}[X]Y - \mathbb{E}[Y]X]
\end{aligned}$$

which makes it true, because we know $\mathsf{Cov}[X,Y]^2 \leq \mathsf{Var}[X]\mathsf{Var}[Y]$.

Problem 2 *12 points*

For each of the following parts, calculate the quantity of interest. Show your steps for partial credit.

(a) (2 pts) Let X be a Uniform$(-2,4)$ random variable. Calculate $\mathbb{P}\big[X > 2 | X^2 > 1\big]$.

Define the events $A = \{X > 2\}$ and $B = \{X^2 > 1\} = \{X < -1\} \cup \{X > 1\}$. Note that $A \cap B = A$. Since $f_X(x) = \begin{cases} 1/6 & -2 \leq x \leq 6 \\ 0 & \text{otherwise.} \end{cases}$, we have that

$$\begin{aligned}
\mathbb{P}[A] &= \int_A f_X(x)\,dx = \int_2^4 \frac{1}{6}\,dx = \frac{1}{3} \\
\mathbb{P}[B] &= \int_B f_X(x)\,dx = \int_{-2}^{-1} \frac{1}{6}\,dx + \int_1^4 \frac{1}{6}\,dx = \frac{2}{3} \\
\mathbb{P}[A|B] &= \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[A]}{\mathbb{P}[B]} = \frac{1/3}{2/3} = \frac{1}{2}.
\end{aligned}$$

(b) (2 pts) Let X be an Gaussian$(-2,3)$ random variable. Calculate $\mathbb{E}[2X^2 + 3X - 2]$.

Using the formula $\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$ we have that $\mathbb{E}[X^2] = \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 = 3 + (-2)^2 = 3 + 4 = 7$. Using the linearity of expectation, we get $\mathbb{E}[2X^2 + 3X - 2] = 2\mathbb{E}[X^2] + 3\mathbb{E}[X] - 2 = 2 \cdot 7 + 3 \cdot (-2) - 2 = 14 - 6 - 2 = 6$.

(c) (4 pts) Let X be a Uniform$(1,3)$ random variable and let Y given that $X = x$ be an Exponential$\left(\frac{2}{x}\right)$ random variable. Calculate $\mathbb{E}[Y]$.

Since Y given $X = x$ is Exponential $\left(\frac{2}{x}\right)$, it has conditional expectation $\mathbb{E}[Y|X = x] = \frac{x}{2}$. Now, using the law of iterated expectation, we find that $\mathbb{E}[Y] = \mathbb{E}\big[\mathbb{E}[Y|X]\big] = \mathbb{E}[X/2] = 2/2 = 1$ where the expectation of X uses the fact that it is Uniform $(1, 3)$.

(d) (4 pts) Let X and Y be jointly Gaussian random variables with $\mu_X = 2, \mu_Y = -1$, $\mathsf{Var}[X] = \frac{1}{2}$, $\mathsf{Var}[Y] = 2$, and $\rho_{X,Y} = \frac{1}{2}$. Let $V = 2X - Y$. Calculate $\mathbb{P}[\{V > 1\}]$ and $\mathbb{P}[\{Y \in [-2, 1]\}]$. Write your answers in terms of the standard normal CDF $\Phi(z)$.

Since X and Y are jointly Gaussian, any linear function of them is itself Gaussian. Our first task is to determine the mean and variance of Z.

$$\begin{aligned}\mathbb{E}[V] &= \mathbb{E}[2X - Y] = 2\mathbb{E}[X] - \mathbb{E}[Y] = 2 \times 2 - (-1) = 5 \\ \mathsf{Var}[V] &= \mathsf{Var}[2X - Y] = 4\mathsf{Var}[X] + \mathsf{Var}[Y] - 4\,\mathsf{Cov}[X, Y] \\ &= 4 \times \frac{1}{2} + 2 + -4\,\rho_{X,Y}\sqrt{\frac{1}{2} \times 2} = 4 - 2 = 2\end{aligned}$$

This tells us that V is Gaussian $(5, 2)$.

Hence, $\mathbb{P}[\{V > 1\}] = 1 - F_V(1) = 1 - \Phi\left(\frac{-4}{\sqrt{2}}\right)$.

Similarly, we know Y is Gaussian(-1,2), so

$$\mathbb{P}[\{Y \in [-2, 1]\}] = F_Y(2) - F_Y(-1) = \Phi\left(\frac{2}{\sqrt{2}}\right) - \Phi\left(\frac{-1}{\sqrt{2}}\right)$$

Problem 3 *20 points*

Consider the following Cumulative Distribution Function (CDF)

$$F_X(x) = \begin{cases} 0 & x < 1 \\ c\left(\frac{1}{3}x^3 - x + \frac{2}{3}\right) & 1 \le x \le 2 \\ 1 & 2 \le x\,. \end{cases}$$

(a) Determine the value of c that satisfies the normalization property. Set c to this value for the remainder of the problem.

We just need to make sure that the CDF reaches 1 at $x = 2$.

$$1 = F_X(2) = c\left(\frac{1}{3}2^3 - 2 + \frac{2}{3}\right) = c\left(\frac{8}{3} - 2 + \frac{2}{3}\right) = c\,\frac{4}{3} \quad \Longrightarrow \quad c = \frac{3}{4}.$$

(b) Determine the PDF of X.

The PDF is just the derivative of the CDF,

$$f_X(x) = \frac{d}{dx}F_X(x) \;=\; \frac{d}{dx}\begin{cases} 0 & x < 1 \\ \frac{3}{4}\left(\frac{1}{3}x^3 - x + \frac{2}{3}\right) & 1 \le x \le 2 \\ 1 & 2 \ge x \ . \end{cases} \;=\; \begin{cases} 0 & x < 1 \\ \frac{3}{4}\left(x^2 - 1\right) & 1 \le x \le 2 \\ 0 & 2 \ge x \ . \end{cases}$$

(c) What is the expected value of X?

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^2 \frac{3}{4}(x^3 - x)\, dx \\ &= \frac{3}{4}\left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Bigg|_1^2 = \frac{3}{4}\left(\left(\frac{2^4}{4} - \frac{2^2}{2}\right) - \left(\frac{1^4}{4} - \frac{1^2}{2}\right)\right) = \frac{3}{4}\left(2 + \frac{1}{4}\right) = \frac{27}{16}\end{aligned}$$

(d) What is the variance of X?

Since $\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$, we only need to calculate $\mathbb{E}[X^2]$.

$$\begin{aligned}\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx = \int_1^2 \frac{3}{4}(x^4 - x^2)\, dx = \frac{3}{4}\left(\frac{1}{5}x^5 - \frac{1}{3}x^3\right)\Bigg|_1^2 \\ &= \frac{3}{4}\left(\left(\frac{2^5}{5} - \frac{2^3}{3}\right) - \left(\frac{1^5}{5} - \frac{1^3}{3}\right)\right) = \frac{3}{4}\left(\left(\frac{32}{5} - \frac{8}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right) = \frac{29}{10} \\ \mathsf{Var}[X] &= \frac{29}{10} - \left(\frac{27}{16}\right)^2 = \frac{67}{1280}\end{aligned}$$

(e) What is the probability that $X < \frac{4}{3}$?

Do it by brute force integration:

$$\mathbb{P}\left[X < \frac{4}{3}\right] = \int_{-\infty}^{4/3} f_X(x)\, dx = \int_1^{4/3} \frac{3}{4}(x^2 - 1)\, dx$$

Or recognize that this is just $F_X(\frac{4}{3})$, and you were given the CDF!

$$\begin{aligned}\mathbb{P}\left[X < \frac{4}{3}\right] &= F_X(\frac{4}{3}) = \frac{3}{4}(\frac{1}{3}(\frac{4}{3})^3 - \frac{4}{3} + \frac{2}{3}) \\ &= \frac{3}{4}(\frac{64}{81} - \frac{2}{3}) = \frac{3}{4} \times \frac{10}{81} = \frac{5}{54}\end{aligned}$$

(f) Calculate $E[X|B]$ where B is the event that $X < \frac{4}{3}$.

First, we need the conditional PDF of X given the event B,

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[B]} & x \in B \\ 0 & \text{o/w} \end{cases} = \begin{cases} \frac{3/4}{5/54}\left(x^2-1\right) & 1 \le x \le \frac{4}{3} \\ 0 & \text{o/w.} \end{cases} = \begin{cases} \frac{81}{10}\left(x^2-1\right) & 1 \le x \le \frac{4}{3} \\ 0 & \text{o/w.} \end{cases}$$

Now, we have that

$$\begin{aligned}\mathbb{E}[X|B] &= \int_{-\infty}^{\infty} x f_{X|B}(x)\,dx = \int_1^{4/3} \frac{81}{10}(x^3 - x)\,dx = \frac{81}{10}\left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Big|_1^{4/3} \\ &= \frac{81}{10}\left(\left(\frac{1}{4}\left(\frac{4}{3}\right)^4 - \frac{1}{2}\left(\frac{4}{3}\right)^2\right) - \left(\frac{1^4}{4} - \frac{1^2}{2}\right)\right) = \frac{81}{10}\left(\frac{64}{81} - \frac{8}{9} + \frac{1}{4}\right) = \frac{81}{40} - \frac{4}{5} = \frac{49}{40}\end{aligned}$$

Problem 4 *16 points*

Consider the following joint PDF $f_{X,Y}(x,y) = \begin{cases} \dfrac{3}{8}y^2 & -1 \le x \le 1 \text{ and } x-1 \le y \le x+1 \\ 0 & \text{otherwise.} \end{cases}$

(a) Calculate $\mathrm{Cov}[X,Y]$. You can leave your answers in terms of integrals. Diagrams help...

$$\begin{aligned}\mathbb{E}[X] &= \int_{-1}^{1}\int_{x-1}^{x+1} x\frac{3}{8}y^2\;dy\;dx = \int_{-1}^{1} x\frac{1}{8}\Big((x+1)^3 - (x-1)^3\Big)\;dx \\ &= \int_{-1}^{1} x\frac{1}{8}\Big(6x^2 + 2)\;dx = 0\end{aligned}$$

where the last equality can be seen by symmetry, or also by realizing you are integrating an odd function of x over a symmetric interval $[-1,1]$.

$$\begin{aligned}\mathbb{E}[Y] &= \int_{-1}^{1}\int_{x-1}^{x+1} y\frac{3}{8}y^2\;dy\;dx = \int_{-1}^{1} \frac{3}{32}\Big((x+1)^4 - (x-1)^4\Big)\;dx \\ &= \int_{-1}^{1} \frac{3}{32}\Big(8x^3 + 8x)\;dx = 0 \\ \mathbb{E}[XY] &= \int_{-1}^{1}\int_{x-1}^{x+1} xy\frac{3}{8}y^2\;dy\;dx = \int_{-1}^{1} x\frac{3}{32}\Big((x+1)^4 - (x-1)^4\Big)\;dx \\ &= \int_{-1}^{1} x\frac{3}{32}\Big(8x^3 + 8x\Big)\;dx = \frac{3}{10} + \frac{1}{2} = \frac{4}{5} \\ \mathsf{Cov}[X,Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{4}{5}\end{aligned}$$

(b) Determine the conditional PDF $f_{Y|X}(y|x)$.

Compute first the marginal density of X, as

$$f_X(x) = \int_{x-1}^{x+1} f_{XY}(x,y)\ dy = \int_{x-1}^{x+1} \frac{3}{8}y^2\ dy = \frac{1}{8}\Big((x+1)^3 - (x-1)^3\Big)$$

Then,

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \begin{cases} \frac{3y^2}{8f_X(x)} & y \in [x-1, x+1] \\ 0 & \text{elsewhere} \end{cases}$$

(c) Calculate the conditional expected value $\mathbb{E}[Y|X=x]$.

$$\mathbb{E}[Y|X=x] = \int_{x-1}^{x+1} y\, \frac{3y^2}{8f_X(x)}\ dy = \frac{3}{32 f_X(x)}\Big((x+1)^4 - (x-1)^4\Big)$$

(d) Calculate $\text{Var}[X-Y]$.

$$\mathbb{E}[X-Y] = \int_{-1}^{1}\int_{x-1}^{x+1} (x-y)\frac{3}{8}y^2\ dy\ dx$$

$$\mathbb{E}[(X-Y)^2] = \int_{-1}^{1}\int_{x-1}^{x+1} (x-y)^2\frac{3}{8}y^2\ dy\ dx$$

$$\mathsf{Var}[X-Y] = \mathbb{E}[(X-Y)^2] - (\mathbb{E}[X-Y])^2$$

Problem 5 *24 points*

Consider the following marginal and conditional PMFs for X and Y:

$$P_Y(y) = \begin{cases} \frac{1}{2} & y = 1 \\ \frac{1}{4} & y = 2, 3. \\ 0 & \text{otherwise} \end{cases} \qquad P_{X|Y}(x|y) = \begin{cases} 1 - \frac{y}{3} & x = -1 \\ \frac{y}{3} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Fill out the joint PMF table for X and Y.

Recall that the joint PMF can be expressed as the product of a marginal and conditional PMF, $P_{X,Y}(x,y) = P_{X|Y}(x|y)P_Y(y)$.

$P_{X,Y}(x,y)$		$y=1$	$y=2$	$y=3$
x	-1	$\frac{1}{3}$	$\frac{1}{12}$	0
	$+1$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$

(b) Calculate $\mathbb{P}\big[|X - Y| \leq 2\big]$.

First, we write the event in terms of pairs $A = \big\{(x,y) \in S_{X,Y} : |x-y| \leq 2\big\} = \big\{(-1,1), (+1,1), (+1,2), (+1,3)\big\}$. Now, we add up the probabilities of these pairs,

$$\mathbb{P}\big[|X-Y| \leq 2\big] = \mathbb{P}[A] = \sum_{(x,y) \in A} P_{X,Y}(x,y) = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{4} = \frac{11}{12}.$$

(c) Are X and Y independent?

No, they are not independent, since the joint PMF table contains a zero for which the associated column and row are both non-zero.

(d) Calculate the covariance of X and Y.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x \in S_X} x P_X(x) = (-1) \cdot \frac{5}{12} + 1 \cdot \frac{7}{12} = \frac{1}{6} \\
\mathbb{E}[Y] &= \sum_{y \in S_Y} y P_Y(y) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{7}{4} \\
\mathbb{E}[XY] &= \sum_{(x,y) \in S_{X,Y}} xy P_{X,Y}(x,y) \\
&= (-1) \cdot 1 \cdot \frac{1}{3} + (-1) \cdot 2 \cdot \frac{1}{12} + (-1) \cdot 3 \cdot 0 + 1 \cdot 1 \cdot \frac{1}{6} + 1 \cdot 2 \cdot \frac{1}{6} + 1 \cdot 3 \cdot \frac{1}{4} = \frac{3}{4} \\
\mathsf{Cov}[X,Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{3}{4} - \frac{1}{6} \cdot \frac{7}{4} = \frac{11}{24}
\end{aligned}$$

Problem 6 *16 points*

Let X and Y be jointly Gaussian random variables with $\mu_X = 1, \mu_Y = -1$, $\mathsf{Var}[X] = 1$, $\mathsf{Var}[Y] = 2$, and $\rho_{X,Y} = \frac{3}{4}$. Define $A = 3X - 2Y$ and $B = 2X + 1$

(a) What are the expected values of A and B?

$$\mathbb{E}[A] = 3\mathbb{E}[X] - 2\mathbb{E}[Y] = 5$$
$$\mathbb{E}[B] = 2\mathbb{E}[X] + 1 = 3$$

(b) What is the covariance $\mathsf{Cov}[A,B]$?

$$\mathsf{Cov}[A,B] = \mathsf{Cov}[3X - 2Y, 2X+1] = 6\mathsf{Var}[X] - 4\mathsf{Cov}[X,Y] = 6 - 4 \times \frac{3}{4} \times \sqrt{2} = 6 - 3\sqrt{2}$$

(c) Assume you observe $B = b$. What is $\mathbb{E}[A|B = b]$ as a function of b? What is the conditional variance $\mathsf{Var}[A|B = b]$?

$$\mathbb{E}[A|B = b] = \mathbb{E}[A] + \frac{\mathsf{Cov}[A, B]}{\mathsf{Var}[B]}(b - \mathbb{E}[B]) = 5 + \frac{6 - 3\sqrt{2}}{4}(b - 3).$$

$$\mathsf{Var}[A|B = b] = \mathsf{Var}[A] - \frac{\mathsf{Cov}[A, B]^2}{\mathsf{Var}[B]}$$

$$\mathsf{Var}[A] = 9\mathsf{Var}[X] + 4\mathsf{Var}[Y] - 12\mathsf{Cov}[X, Y] = 9 + 8 - 12(\frac{3}{4}\sqrt{2}) = 17 - 9\sqrt{2}$$

$$\mathsf{Var}[B] = 4$$

$$\mathsf{Var}[A|B = b] = 17 - 9\sqrt{2} - \frac{(6 - 3\sqrt{2})^2}{4}$$

(d) Are A and B independent? Why?

No, because $\mathsf{Cov}[A, B] \neq 0$.