EK381 Exam 2 Formula Sheet

3. Continuous Random Variables

Probability Density Function (PDF)

- The PDF is the derivative of the CDF: $f_X(x) = \frac{d}{dx} F_X(x)$
- It does not tell us the probability of $X = x$, which is always 0. Instead, it tells us the density of probability around x.
- The PDF satisfies the following properties:
  - Normalization: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.
  - Non-negativity: $f_X(x) \geq 0$.
  - Probability of an interval: $\mathbb{P}[a < X \leq b] = \int_a^b f_X(x)\,dx$.
  - PDF → CDF: $\int_{-\infty}^{x} f_X(u)\,du = F_X(x)$.

Expected Value

- The expected value of a continuous random variable X is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

- The expected value of a function of a continuous random variable X is

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

- **Linearity of Expectation:** $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

Variance

- The variance of a random variable X is

$$\mathsf{Var}[X] = \mathbb{E}\big[\big(X - \mathbb{E}[X]\big)^2\big]$$

- Another useful formula is $\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$
- The standard deviation is the square root of the variance:

$$\sigma_X = \sqrt{\mathsf{Var}[X]}\ .$$

- **Variance of a Linear Function:** $\mathsf{Var}[aX + b] = a^2 \mathsf{Var}[X]$

Important Families of Random Variables

Uniform Random Variables

- X is a Uniform(a, b) random variable if it has PDF

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x < b \\ 0 & \text{otherwise.} \end{cases}$$

- CDF: $F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x \end{cases}$
- Expected Value: $\mathbb{E}[X] = \dfrac{a+b}{2}$.
- Variance: $\mathsf{Var}[X] = \dfrac{(b-a)^2}{12}$.

Exponential Random Variables

- X is an Exponential(λ) random variable if it has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0\ . \end{cases}$$

- CDF: $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0\ . \end{cases}$
- Expected Value: $\mathbb{E}[X] = \dfrac{1}{\lambda}$.
- Variance: $\mathsf{Var}[X] = \dfrac{1}{\lambda^2}$.

Gaussian Random Variables

- X is a Gaussian(μ, σ^2) random variable if it has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- CDF: $F_X(x) = \Phi\left(\dfrac{x-\mu}{\sigma}\right)$
- Standard Normal CDF: $\Phi(z) = \displaystyle\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\,dw$
- Standard Normal Complementary CDF:

$$Q(z) = \Phi(-z) = 1 - \Phi(z)$$

- Expected Value: $E[X] = \mu$.
- Variance: $\mathsf{Var}[X] = \sigma^2$.
- Probability of an Interval:

$$\mathbb{P}[a < X \leq b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

- **A linear function of a Gaussian is Gaussian:** If X is Gaussian(μ, σ^2) and $Y = aX + b$, then Y is Gaussian$(a\mu + b, a^2\sigma^2)$.

Conditioning for Continuous RVs

- The conditional PDF of X given an event B is

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases}$$

  where $\mathbb{P}[X \in B] = \int_B f_X(x)\,dx$.
- The conditional expected value of X given an event B is

$$\mathbb{E}[X|B] = \int_{-\infty}^{\infty} x\, f_{X|B}(x)\,dx\ .$$

- The conditional expected value of a function $g(X)$ given an event B is

$$\mathbb{E}\big[g(X)|B\big] = \int_{-\infty}^{\infty} g(x)\, f_{X|B}(x)\,dx\ .$$

- The conditional variance of X given an event B is

$$\mathsf{Var}\big[X|B\big] = \mathbb{E}\Big[\big(X - \mathbb{E}[X|B]\big)^2\Big|B\Big] = \mathbb{E}[X^2|B] - \big(\mathbb{E}[X|B]\big)^2$$

4. Pairs of Random Variables

- Joint CDF: $F_{X,Y}(x, y) = \mathbb{P}[X \leq x,\ Y \leq y]$

Pairs of Discrete Random Variables

- Joint PMF: $P_{X,Y}(x, y) = \mathbb{P}[X = x,\ Y = y]$.
- Range $R_{X,Y} = \{(x, y) : P_{X,Y}(x, y) > 0\}$.
- Marginal PMFs $P_X(x)$ and $P_Y(y)$ are just the PMFs of the individual random variables X and Y, respectively.

$$P_X(x) = \sum_{y \in R_Y} P_{X,Y}(x, y) \qquad P_Y(y) = \sum_{x \in R_X} P_{X,Y}(x, y)$$

- Conditional PMFs give the probability of one random variable when the other is fixed to a value:

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x, y)}{P_Y(y)} \qquad P_{Y|X}(y|x) = \frac{P_{X,Y}(x, y)}{P_X(x)}$$

  for $(x, y) \in R_{X,Y}$, otherwise the conditional PMF is 0.

Pairs of Continuous Random Variables

- Joint PDF: $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$.
- Range $R_{X,Y} = \{(x, y) : f_{X,Y}(x, y) > 0\}$.
- Marginal PDFs $f_X(x)$ and $f_Y(y)$ are just the PDFs of the individual random variables X and Y, respectively.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx$$

- Conditional PDFs give the probability density of one random variable when the other is fixed to a value:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \qquad f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

  for $(x, y) \in R_{X,Y}$, otherwise the conditional PDF is 0.

Joint PMF/PDF Properties

- Non-negativity: $P_{X,Y}(x, y) \geq 0$

  $f_{X,Y}(x, y) \geq 0$
- Normalization: $\displaystyle\sum_{x \in R_X} \sum_{y \in R_Y} P_{X,Y}(x, y) = 1$

  $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx\,dy = 1$
- Probability of an Event $B \subset R_{X,Y}$:

$$\mathbb{P}[(X, Y) \in B] = \sum_{(x,y) \in B} P_{X,Y}(x, y) \quad \text{(discrete)}$$

$$\mathbb{P}[(X, Y) \in B] = \iint_B f_{X,Y}(x, y)\,dx\,dy \quad \text{(continuous)}$$

Conditional PMF/PDF Properties

- Non-negativity: $P_{X|Y}(x|y) \geq 0 \quad P_{Y|X}(y|x) \geq 0$
  $f_{X|Y}(x|y) \geq 0 \quad f_{Y|X}(y|x) \geq 0$
- Normalization: $\sum_{x \in R_X} P_{X|Y}(x|y) = \sum_{y \in R_Y} P_{Y|X}(y|x) = 1$
  $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\,dy = 1$
- Additivity: For any event $B \subset R_X$, the probability that X falls in B given $Y = y$ is

$$\mathbb{P}[X \in B | Y = y] = \sum_{x \in B} P_{X|Y}(x|y) \quad \text{(discrete)}$$

$$\mathbb{P}[X \in B | Y = y] = \int_B f_{X|Y}(x|y)\,dx \quad \text{(continuous)}$$

- Multiplication Rule:

$$P_{X,Y}(x,y) = P_{X|Y}(x|y)P_Y(y) = P_{Y|X}(y|x)P_X(x)$$
$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x)$$

Independence of Random Variables

- X and Y are independent if and only
  - Discrete: $P_{X,Y}(x,y) = P_X(x)P_Y(y)$.
  - Continuous: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.
- Special cases where X and Y are **not** independent:
  - Discrete: If there is a zero entry in the joint PMF table for which neither the entire column or entire row is zero.
  - Continuous: If the range is not a collection of rectangles parallel to the axes

Expected Value of a Function

- The expected value of a function $W = g(X,Y)$ is

$$\text{Discrete:} \quad \mathbb{E}[W] = \sum_{x \in R_X} \sum_{y \in R_Y} g(x,y) P_{X,Y}(x,y)$$

$$\text{Continuous:} \quad \mathbb{E}[W] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

- **Linearity of Expectation:** $\mathbb{E}[aX + bY + c] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] + c$.
- **Expectation of Products:** If X and Y are independent, then $\mathbb{E}[g(X)\,h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]$.

Conditional Expectation

- The conditional expected value of X given $Y = y$ is

$$\text{Discrete:} \quad \mathbb{E}[X|Y = y] = \sum_{x \in R_X} x\,P_{X|Y}(x|y)$$

$$\text{Continuous:} \quad \mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x\,f_{X|Y}(x|y)\,dx$$

- **Law of Total Expectation:** $\mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}[X]$.

5. Second-Order Analysis

Covariance

- The covariance of random variables X and Y is

$$\mathsf{Cov}[X,Y] = \mathbb{E}\big[\big(X - \mathbb{E}[X]\big)\big(Y - \mathbb{E}[Y]\big)\big]$$

- Another useful formula is $\mathsf{Cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- **Variance of Linear Functions:**
  $\mathsf{Var}[aX + bY + c] = a^2\,\mathsf{Var}[X] + b^2\,\mathsf{Var}[Y] + 2ab\,\mathsf{Cov}[X,Y]$
- **Covariance of Linear Functions:**

$$\mathsf{Cov}[aX + bY + c, dX + eY + f] = ad\,\mathsf{Var}[X] + be\,\mathsf{Var}[Y] + (ae + bd)\mathsf{Cov}[X,Y]$$

- The covariance satisfies the following basic properties:
  - $\mathsf{Cov}[X,Y] = \mathsf{Cov}[Y,X]$
  - $\mathsf{Cov}[X,X] = \mathsf{Var}[X]$
  - $\mathsf{Cov}[X,a] = 0$ for any number a.
- X and Y are uncorrelated if $\mathsf{Cov}[X,Y] = 0$.
  - Independence implies uncorrelatedness.
  - Uncorrelatedness does not imply independence.

Correlation Coefficient

- The correlation coefficient is $\rho_{X,Y} = \dfrac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]\mathsf{Var}[Y]}}$
- The correlation coefficient satisfies the following properties:
  - $-1 \leq \rho_{X,Y} \leq 1$.
  - $\rho_{X,Y} = 1$ if and only if $Y = aX + b$ for some $a > 0$.
  - $\rho_{X,Y} = -1$ if and only if $Y = aX + b$ for some $a < 0$.
  - If $U = aX + b$ and $V = cY + d$, then

$$\rho_{U,V} = \text{sign}(ac)\rho_{X,Y} \quad \text{where} \quad \text{sign}(z) = \begin{cases} +1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases}$$

Jointly Gaussian Random Variables

- U and V are called independent, standard Gaussian random variables if they are independent Gaussian$(0,1)$ random variables.
- X and Y are jointly Gaussian random variables if they can be expressed as linear functions of independent, standard Gaussian random variables

$$X = aU + bV + c \qquad Y = dU + eV + f\ .$$

  However, this representation is usually left implicit, and the joint Gaussian distribution of X and Y is specified by 5 parameters:
  - Means: $\mu_X = \mathbb{E}[X]$, $\mu_Y = \mathbb{E}[Y]$
  - Variances: $\sigma_X^2 = \mathsf{Var}[X]$, $\sigma_Y^2 = \mathsf{Var}[Y]$
  - Covariance: $\mathsf{Cov}[X,Y]$ **or** Correlation Coefficient: $\rho_{X,Y}$.
- Jointly Gaussian X and Y satisfy the following properties:
  - Marginal PDFs are Gaussian: X is Gaussian(μ_X, σ_X^2) and Y is Gaussian(μ_Y, σ_Y^2).
  - Uncorrelated implies Independence: X and Y are uncorrelated if and only if X and Y are independent.
  - **Conditional Expected Value for Gaussians:**

$$\mathbb{E}[X|Y = y] = \mu_X + \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(y - \mu_Y) = \mu_X + \frac{\mathsf{Cov}[X,Y]}{\mathsf{Var}[Y]}(y - \mu_Y)$$

  - **Conditional Variance for Gaussians:** $\sigma_{X|Y}^2 = \mathsf{Var}[X|Y = y] = (1 - \rho_{X,Y}^2)\sigma_X^2 = \mathsf{Var}[X] - \dfrac{\big(\mathsf{Cov}[X,Y]\big)^2}{\mathsf{Var}[Y]}$.
  - Conditional PDF is Gaussian: The conditional PDF $f_{X|Y}(x|y)$ of X given Y is Gaussian$(\mathbb{E}[X|Y = y], \sigma_{X|Y}^2)$.
  - **Linear functions of Gaussians are Gaussian:** If $W = aX + bY + c$ and $Z = dX + eY + f$, then W and Z are jointly Gaussian with parameters that be determined via the linearity of expectation and the variance and covariance of linear functions.

Random Vectors

- A random vector is a (column) vector whose entries are random variables

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

- If the entries are discrete random variables, the random vector has a joint PMF $P_{\underline{X}}(\underline{x}) = P_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$. If the entries are continuous random variables, the random vector has a joint PDF $f_{\underline{X}}(\underline{x}) = f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$.
- Mean Vector: $\underline{\mu}_{\underline{X}} = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix}$
- **Linearity of Expectation:** $\mathbb{E}\big[\mathbf{A}\underline{X} + \underline{b}\big] = \mathbf{A}\,\mathbb{E}[\underline{X}] + \underline{b}$
- Covariance Matrix: $\mathbf{\Sigma}_{\underline{X}} = \mathbb{E}\big[(\underline{X} - \mathbb{E}[\underline{X}])(\underline{X} - \mathbb{E}[\underline{X}])^{\mathsf{T}}\big]$

$$= \begin{bmatrix} \mathsf{Cov}[X_1,X_1] & \cdots & \mathsf{Cov}[X_1,X_n] \\ \vdots & & \vdots \\ \mathsf{Cov}[X_n,X_1] & \cdots & \mathsf{Cov}[X_n,X_n] \end{bmatrix}$$

- **Covariance of a Linear Transform:** If $\underline{Y} = \mathbf{A}\underline{X} + \underline{b}$, then $\mathbf{\Sigma}_{\underline{Y}} = \mathbf{A}\mathbf{\Sigma}_{\underline{X}}\mathbf{A}^{\mathsf{T}}$.

Gaussian Vectors

- A standard Gaussian vector is a random vector $\underline{Z}$ whose entries $Z_1,\ldots,Z_n$ are independent Gaussian$(0,1)$ random variables.
- A Gaussian vector is a random vector $\underline{X}$ that can be written as a linear transform $\underline{X} = \mathbf{A}\underline{Z} + \underline{b}$ of a standard Gaussian vector $\underline{Z}$. It is fully specified by its mean vector $\underline{\mu}_{\underline{X}}$ and covariance matrix $\mathbf{\Sigma}_{\underline{X}}$.
- Shorthand notation: We often write $\underline{X} \sim \mathcal{N}(\underline{\mu}_{\underline{X}}, \mathbf{\Sigma}_{\underline{X}})$ to mean that $\underline{X}$ is a Gaussian vector with mean vector $\underline{\mu}_{\underline{X}}$ and covariance matrix $\mathbf{\Sigma}_{\underline{X}}$.
- A Gaussian vector $\underline{X}$ satisfies the following properties:
  - The entries of $\underline{X}$ are independent if and only if $\mathbf{\Sigma}_{\underline{X}}$ is a diagonal matrix.
  - **A linear transformation is a Gaussian vector:** If $\underline{Y} = \mathbf{B}\underline{X} + \underline{c}$, then $\underline{Y} \sim \mathcal{N}(\mathbf{B}\underline{\mu}_{\underline{X}} + \underline{c}, \mathbf{B}\mathbf{\Sigma}_{\underline{X}}\mathbf{B}^{\mathsf{T}})$.