2.1 Discrete Random Variables

- A **random variable** is a mapping that assigns real numbers to outcomes in the sample space.
- Random variables are denoted by capital letters (such as X) and their specific values are denoted by lowercase letters (such as x).
- The **range** of a random variable X is denoted by R_X.

2.1.1 Probability Mass Function

- The **probability mass function (PMF)** specifies the probability that a discrete random variable X takes the value x:
$$P_X(x) = \mathbb{P}[X = x].$$
- The PMF satisfies the following basic properties:
  1. **Non-negativity:** $P_X(x) \geq 0$ for all x.
  2. **Normalization:** $\displaystyle\sum_{x \in R_X} P_X(x) = 1$.
  3. **Additivity:** For any event $B \subset R_X$, the probability that X falls in B is
$$\mathbb{P}[\{X \in B\}] = \sum_{x \in B} P_X(x).$$

2.1.2 Cumulative Distribution Function

- The **cumulative distribution function (CDF)** returns the probability that a random variable X is less than or equal to a value x:
$$F_X(x) = \mathbb{P}[X \leq x].$$
- The CDF satisfies the following basic properties:
  - **Normalization:** $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.
  - **Non-negativity:** $F_X(x)$ is a non-decreasing function of x.
  - For $b \geq a$, $F_X(b) - F_X(a) = \mathbb{P}[a < X \leq b]$.
  - $F_X(x)$ is piecewise constant and jumps at points x where $P_X(x) > 0$ by height $P_X(x)$.

2.2 Expectation

2.2.1 Expected Value

- The **expected value** of a discrete random variable X is
$$\mathbb{E}[X] = \sum_{x \in R_X} x \, P_X(x).$$
- This is also known as the **mean** or **average.**
- Sometimes denoted as $\mu_X = \mathbb{E}[X]$.

2.2.2 Variance

- The **variance** measures how spread out a random variable is around its mean,

$$\mathsf{Var}[X] = \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)^2\Big] = \sum_{x \in R_X} (x - \mu_X)^2 P_X(x).$$

- Alternate formula: $\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$.
- **Standard Deviation:** $\sigma_X = \sqrt{\mathsf{Var}[X]}$.
- The variance is sometimes written as $\sigma_X^2 = \mathsf{Var}[X]$.

2.2.3 Moments

- n^{th} **Moment:** $\mathbb{E}[X^n] = \sum_{x \in R_X} x^n P_X(x)$.
- n^{th} **Central Moment:** $\mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)^n\Big] = \sum_{x \in R_X} (x - \mu_X)^n P_X(x)$.

2.3 Functions of a Random Variable

- A **function** $Y = g(X)$ of a discrete random variable X is itself a discrete random variable.
- Sometimes referred to as a **derived random variable.**
- Range: $R_Y = \{g(x) : x \in R_X\}$.
- PMF: $P_Y(y) = \sum_{x : g(x) = y} P_X(x)$.
- Expected Value: $\mathbb{E}[Y] = \sum_{y \in R_Y} y\, P_Y(y) = \sum_{x \in R_X} g(x)\, P_X(x)$.
- **Linearity of Expectation:** $\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$.
- **Variance of a Linear Function:** $\mathsf{Var}[aX + b] = a^2\,\mathsf{Var}[X]$.

2.4 Important Families of Discrete Random Variables

2.4.1 Bernoulli Random Variables

- X is a **Bernoulli**(p) random variable if it has PMF

$$P_X(x) = \begin{cases} 1 - p & x = 0, \\ p & x = 1\ . \end{cases}$$

- Range: $R_X = \{0, 1\}$.
- Expected Value: $\mathbb{E}[X] = p$.
- Variance: $\mathsf{Var}[X] = p(1 - p)$.
- Interpretation: single trial with success probability p.

2.4.2 Geometric Random Variables

- X is a **Geometric**(p) random variable if it has PMF

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $R_X = \{1, 2, \ldots\}$.
- Expected Value: $\mathbb{E}[X] = \dfrac{1}{p}$.
- Variance: $\mathsf{Var}[X] = \dfrac{1-p}{p^2}$.
- Interpretation: # of independent Bernoulli(p) trials until first success.

2.4.3 Binomial Random Variables

- X is a **Binomial**(n, p) random variable if it has PMF

$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $R_X = \{0, 1, \ldots, n\}$.
- Expected Value: $\mathbb{E}[X] = np$.
- Variance: $\mathsf{Var}[X] = np(1-p)$.
- Interpretation: # of successes in n independent Bernoulli(p) trials.

2.4.4 Discrete Uniform Random Variables

- X is a **Discrete Uniform**(a, b) random variable if it has PMF

$$P_X(x) = \begin{cases} \dfrac{1}{b-a+1} & x = a, a+1, \ldots, b, \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $R_X = \{a, a+1, \ldots, b\}$.
- Expected Value: $\mathbb{E}[X] = \dfrac{a+b}{2}$.
- Variance: $\mathsf{Var}[X] = \dfrac{(b-a)(b-a+2)}{12} = \dfrac{(b-a+1)^2 - 1}{12}$.
- Interpretation: equally likely to take any (integer) value between a and b.

2.4.5 Poisson Random Variables

- X is a **Poisson**(λ) random variable if it has PMF

$$P_X(x) = \begin{cases} \frac{\lambda^x}{x!} e^{-\lambda} & x = 0, 1, \dots \\ 0 & \text{otherwise.} \end{cases}$$

- Range: $R_X = \{0, 1, \dots\}$.
- Expected Value: $\mathbb{E}[X] = \lambda$.
- Variance: $\mathsf{Var}[X] = \lambda$.
- Interpretation: # of arrivals in a fixed time window.

2.5 Conditioning for Discrete Random Variables

- The **conditional PMF** of X given an event B is

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[\{X \in B\}]} & x \in B \\ 0 & x \notin B \end{cases} \quad \text{where } \mathbb{P}[\{X \in B\}] = \sum_{x \in B} P_X(x)$$

The conditional PMF satisfies the basic PMF properties

1. **Non-negativity:** $P_{X|B}(x) \geq 0$ for all x.
2. **Normalization:** $\sum_{x \in B} P_{X|B}(x) = 1$.
3. **Additivity:** For any event $A \subset R_X$, the conditional probability that X falls in A given that X falls in B is
$$\mathbb{P}\big[\{X \in A\}\big|\{X \in B\}\big] = \sum_{x \in A} P_{X|B}(x).$$

- The **conditional expected value** of X given an event B is

$$\mathbb{E}[X|B] = \sum_{x \in B} x \, P_{X|B}(x) \ .$$

- The **conditional expected value of a function** $g(X)$ given an event B is

$$\mathbb{E}\big[g(X)|B\big] = \sum_{x \in B} g(x) \, P_{X|B}(x) \ .$$

- The **conditional variance** of X given an event B is

$$\begin{aligned} \mathsf{Var}\big[X|B\big] &= \mathbb{E}\Big[\big(X - \mathbb{E}[X|B]\big)^2 \Big| B\Big] = \sum_{x \in B} \big(x - \mathbb{E}[X|B]\big)^2 P_{X|B}(x) \\ &= \mathbb{E}\big[X^2|B\big] - \big(\mathbb{E}[X|B]\big)^2 \end{aligned}$$