3.1 Continuous Random Variables

- A random variable is **continuous** if it has a continuous CDF (see below) which is differentiable almost everywhere.
- The probability mass function (PMF) is not defined since, for a continuous random variable, the probability of taking an exact value is always zero, $\mathbb{P}[X = x] = 0$. Therefore, open vs. closed interval has no effect: $\mathbb{P}[a < X < b] = \mathbb{P}[a < X \leq b] = \mathbb{P}[a \leq X < b] = \mathbb{P}[a \leq X \leq b]$.

3.1.1 Cumulative Distribution Function

- The **cumulative distribution function (CDF)** returns the probability that a random variable X is less than or equal to a value x:

$$F_X(x) = \mathbb{P}[X \leq x].$$

- The CDF satisfies the following basic properties:
  - **Non-negativity:** $F_X(x)$ is a non-decreasing function of x.
  - **Normalization:** $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.
  - **Probability of an interval:** $\mathbb{P}[a < X \leq b] = F_X(b) - F_X(a)$.
  - **CDF → PDF:** $\frac{d}{dx}F_X(x) = f_X(x)$.

3.1.2 Probability Density Function

- The **probability density function (PDF)** is the derivative of the CDF:

$$f_X(x) = \frac{d}{dx}F_X(x).$$

- The PDF plays the role of the PMF for continuous random variables in many ways, but it does not tell us the probability of $X = x$, which is always 0 for continuous random variables.
- The PDF satisfies the following basic properties:
  - **Non-negativity:** $f_X(x) \geq 0$.
  - **Normalization:** $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.
  - **Probability of an interval:** $\mathbb{P}[a < X \leq b] = \int_a^b f_X(x)\,dx$.
  - **PDF → CDF:** $\int_{-\infty}^{x} f_X(u)\,du = F_X(x)$.

3.2 Expectation

3.2.1 Expected Value

- The **expected value** of a continuous random variable X is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx.$$

- This is also known as the **mean** or **average.**
- Sometimes denoted as $\mu_X = \mathbb{E}[X]$.

3.2.2 Expected Value of a Function of a Random Variable

- A function $Y = g(X)$ of a continuous random variable X might be continuous or not, depending on the function $g(x)$.
- However, we can always calculate the expected value using only the function and PDF,

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

- **Linearity of Expectation:** $\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$.

3.2.3 Variance

- The **variance** measures how spread out a random variable is around its mean,

$$\mathsf{Var}[X] = \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)^2\Big] = \int_{-\infty}^{\infty} (x - \mu_X)^2\, f_X(x)\, dx.$$

- Alternate formula: $\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$.
- **Standard Deviation:** $\sigma_X = \sqrt{\mathsf{Var}[X]}$.
- The variance is sometimes written as $\sigma_X^2 = \mathsf{Var}[X]$.
- **Variance of a Linear Function:** $\mathsf{Var}[aX + b] = a^2\,\mathsf{Var}[X]$.

3.2.4 Moments

- n^{th} **Moment:** $\mathbb{E}[X^n] = \displaystyle\int_{-\infty}^{\infty} x^n\, f_X(x)\, dx$.
- n^{th} **Central Moment:** $\mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)^n\Big] = \displaystyle\int_{-\infty}^{\infty} (x - \mu_X)^n\, f_X(x)\, dx$.

3.3 Important Families of Discrete Random Variables

3.3.1 Uniform Random Variables

- X is a **Uniform**(a, b) random variable if it has PDF $f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x < b \\ 0 & \text{otherwise.} \end{cases}$
- CDF: $F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x \end{cases}$
- Expected Value: $\mathbb{E}[X] = \dfrac{a+b}{2}$.
- Variance: $\mathsf{Var}[X] = \dfrac{(b-a)^2}{12}$.
- Interpretation: Equally likely to take any value between a and b.

3.3.2 Exponential Random Variables

- X is an **Exponential**(λ) random variable if it has PDF $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$
- CDF: $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$.
- Expected Value: $\mathbb{E}[X] = \dfrac{1}{\lambda}$.
- Variance: $\mathsf{Var}[X] = \dfrac{1}{\lambda^2}$.
- Interpretation: Continuous waiting time. "Continuous version" of geometric RV.

3.3.3 Gaussian Random Variables

- X is a **Gaussian**(μ, σ^2) random variable if it has PDF $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \dfrac{(x-\mu)^2}{2\sigma^2} \Big)$.
- CDF: $F_X(x) = \Phi\Big(\dfrac{x-\mu}{\sigma}\Big)$ where $\Phi(z) = \displaystyle\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\Big(- \frac{w^2}{2} \Big)\, dw$.
- $\Phi(z)$ is called the standard normal CDF. (Evaluated via MATLAB, lookup table, etc.)
- $Q(z) = 1 - \Phi(z)$ is the standard normal complementary CDF.
- $\Phi(-z) = 1 - \Phi(z) = Q(z)$.
- Expected Value: $\mathbb{E}[X] = \mu$.
- Variance: $\mathsf{Var}[X] = \sigma^2$.
- Probability of an Interval: $\mathbb{P}[a \leq X \leq b] = \Phi\Big(\dfrac{b-\mu}{\sigma}\Big) - \Phi\Big(\dfrac{a-\mu}{\sigma}\Big)$
- **Linear function of a Gaussian is Gaussian:** If X is a Gaussian(μ, σ^2) random variable, then $Y = aX + b$ is a Gaussian$(a\mu + b, a^2\sigma^2)$ random variable.
- Interpretation: Sum (or average) of many small random effects.

3.4 Conditional Probability Models

- The **conditional PDF** of X given an event B is

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} \quad \text{where} \quad \mathbb{P}[X \in B] = \int_B f_X(x)\, dx \ .$$

- The conditional PDF satisfies the following basic properties:
  - **Non-negativity:** $f_{X|B}(x) \geq 0$.

- ○ **Normalization:** $\int_{-\infty}^{\infty} f_{X|B}(x)\,dx = 1$.
- ○ **Conditional probability of an interval:**

$$\mathbb{P}\big[\{a \leq X \leq b\}\big|\{X \in B\}\big] = \int_a^b f_{X|B}(x)\,dx \ .$$

- The **conditional expected value** of X given an event B is

$$\mathbb{E}[X|B] = \int_{-\infty}^{\infty} x\, f_{X|B}(x)\,dx \ .$$

- The **conditional expected value of a function** $g(X)$ given an event B is

$$\mathbb{E}\big[g(X)|B\big] = \int_{-\infty}^{\infty} g(x)\, f_{X|B}(x)\,dx \ .$$

- The **conditional variance** of X given an event B is

$$\begin{aligned}\mathsf{Var}\big[X|B\big] &= \mathbb{E}\Big[\big(X - \mathbb{E}[X|B]\big)^2\Big|B\Big] = \int_B \big(x - \mathbb{E}[X|B]\big)^2 f_{X|B}(x)\,dx \\ &= \mathbb{E}\big[X^2|B\big] - \big(\mathbb{E}[X|B]\big)^2\end{aligned}$$