6 Detection

- Probability is a foundation for principled decision making from partial, noisy observations.
- The basic **detection** problem is to decide amongst a few mutually exclusive choices.
- Motivating Example: A radar system trying to decide whether an airplane is present or absent. There are four possibilities:
  - Airplane is absent, detector says it is absent.
  - Airplane is absent, detector says it is present. (This is a **false alarm**.)
  - Airplane is present, detector says it is absent. (This is a **missed detection**.)
  - Airplane is present, detector says it is present.

6.1 Binary Hypothesis Testing

- Two **hypotheses** H_0 and H_1, which are events that form a partition of the sample space Ω.
- We obtain a **measurement** or **observation**, which is a random variable Y whose values are distributed according to
  - Discrete: If H_0 occurs, Y has PMF $P_{Y|H_0}(y)$. If H_1 occurs, X has PMF $P_{Y|H_1}(y)$.
  - Continuous: If H_0 occurs, Y has PDF $f_{Y|H_0}(y)$. If H_1 occurs, Y has PDF $f_{Y|H_1}(y)$.
  - These conditional PMFs/PDFs are known as **likelihoods.**
- There is a **detector** or **decision rule**, which is a function $D(y)$ that takes as input the observation $Y = y$, and outputs 0 if it decides that H_0 occurred and 1 if it decides H_1.
- This decision partitions the range of the observation R_Y into two regions:

$$\underbrace{A_0 = \{y \in R_Y : D(y) = 0\}}_{\text{Decide } H_0} \qquad \underbrace{A_1 = \{y \in R_Y : D(y) = 1\}}_{\text{Decide } H_1}$$

- **Errors** occur when we choose the wrong hypothesis:
  - False Alarm: Choose H_1 when H_0 is true. The probability of false alarm is

$$P_{\text{FA}} = \mathbb{P}\big[\{Y \in A_1 \big| H_0\}\big]$$

  - Missed Detection: Choose H_0 when H_1 is true. The probability of missed detection is

$$P_{\text{MD}} = \mathbb{P}\big[\{Y \in A_0 \big| H_1\}\big]$$

  - An **error** occurs if we decide the wrong hypothesis:

$$\{\text{error}\} = \{A_1 \cap H_0\} \cup \{A_0 \cap H_1\}$$

  - Goal is to minimize the **probability of error** P_e:

$$P_e = \mathbb{P}[\text{error}] = P_{\text{FA}}\,\mathbb{P}[H_0] + P_{\text{MD}}\,\mathbb{P}[H_1]$$

  - Our goal is to find the **optimal decision rule** and the resulting probability of error.

- It is sometimes easier to express decision rules in terms of the **likelihood ratio:**

$$\text{Discrete:} \quad L(y) = \frac{P_{Y|H_1}(y)}{P_{Y|H_0}(y)} \qquad \text{Continuous:} \quad L(y) = \frac{f_{Y|H_1}(y)}{f_{Y|H_0}(y)}$$

- It is sometimes even easier to to use the **log-likelihood ratio:**

$$\text{Discrete:} \quad \ln\big(L(y)\big) = \ln\left(\frac{P_{Y|H_1}(y)}{P_{Y|H_0}(y)}\right) \qquad \text{Continuous:} \quad \ln\big(L(y)\big) = \ln\left(\frac{f_{Y|H_1}(y)}{f_{Y|H_0}(y)}\right)$$

- For **vector observations** $\underline{Y}$, we simply replace all occurrences of Y with $\underline{Y}$. For example, $P_{Y|H_0}(y)$ becomes $P_{\underline{Y}|H_0}(\underline{y})$, $P_{Y|H_1}(y)$ becomes $P_{\underline{Y}|H_1}(\underline{y})$, and $L(y)$ becomes $L(\underline{y}) = \frac{P_{\underline{Y}|H_1}(\underline{y})}{P_{\underline{Y}|H_0}(\underline{y})}$.

6.2 Maximum Likelihood (ML) Decision Rule

- Intuition: Choose the hypothesis that best explains the observation.
- The **maximum likelihood (ML) decision rule** is

$$\text{Discrete: } D^{\text{ML}}(y) = \begin{cases} 1 & P_{Y|H_1}(y) \geq P_{Y|H_0}(y) \\ 0 & P_{Y|H_1}(y) < P_{Y|H_0}(y) \end{cases} \quad \text{Continuous: } D^{\text{ML}}(y) = \begin{cases} 1 & f_{Y|H_1}(y) \geq f_{Y|H_0}(y) \\ 0 & f_{Y|H_1}(y) < f_{Y|H_0}(y) \end{cases}$$

which can be expressed in terms of the likelihood and log-likelihood ratios as follows:

$$D^{\text{ML}}(y) = \begin{cases} 1 & L(y) \geq 1 \\ 0 & L(y) < 1 \end{cases} = \begin{cases} 1 & \ln(L(y)) \geq 0 \\ 0 & \ln(L(y)) < 0 \end{cases}$$

- This is only the optimal decision if $\mathbb{P}[H_0] = \mathbb{P}[H_1] = \frac{1}{2}$. However, it does not require knowledge of $\mathbb{P}[H_0]$ and $\mathbb{P}[H_1]$ to implement the decision rule, only the conditional probability models for the observation Y.

6.3 Maximum a Posteriori (MAP) Rule

- Intuition: Choose the most likely hypothesis given the observation.
- The **maximum a posteriori (MAP) decision rule** is

$$\text{Discrete: } D^{\text{MAP}}(y) = \begin{cases} 1 & P_{Y|H_1}(y)\,\mathbb{P}[H_1] \geq P_{Y|H_0}(y)\,\mathbb{P}[H_0] \\ 0 & P_{Y|H_1}(y)\,\mathbb{P}[H_1] < P_{Y|H_0}(y)\,\mathbb{P}[H_0] \end{cases}$$

$$\text{Continuous: } D^{\text{MAP}}(y) = \begin{cases} 1 & f_{Y|H_1}(y)\,\mathbb{P}[H_1] \geq f_{Y|H_0}(y)\,\mathbb{P}[H_0] \\ 0 & f_{Y|H_1}(y)\,\mathbb{P}[H_1] < f_{Y|H_0}(y)\,\mathbb{P}[H_0] \end{cases}$$

which can be expressed in terms of the likelihood and log-likelihood ratios as follows:

$$D^{\text{MAP}}(y) = \begin{cases} 1 & L(y) \geq \dfrac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]} \\ & \\ 0 & L(y) < \dfrac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]} \end{cases} = \begin{cases} 1 & \ln\left(\dfrac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \geq 0 \\ & \\ 0 & \ln\left(\dfrac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) < 0 \end{cases}$$

- This is the **optimal decision rule** in terms of minimizing the probability of error. However, it requires knowledge of $\mathbb{P}[H_0]$ and $\mathbb{P}[H_1]$ to implement the decision rule.