Set Theory

- Set theory provides a mathematical foundation for probability theory.
- A set is a collection of elements.
  - → Elements can be anything you want:
    - * Ex: Numbers $A = \{1, 3, 5\}$ $B = [-2.32, 1.45)$
    - * Ex: Words $C = \{\text{Win}, \text{Lose}\}$
    - * Ex: Animals D = { ,  }
  - → A set can be empty, which we call the empty set or null set ϕ.
  - → The universal set Ω is the set of all elements.

    for the specific context
    - * Ex: Six-sided die  $\Omega = \{1, 2, 3, 4, 5, 6\}$

- Notation: $x \in A$ means "x is an element of A"

  $x \notin A$ means "x is not an element of A"

- There are several ways to describe a set, including:

  → List its elements. <u>Ex</u>: $A = \{3, 4, 5, \ldots\}$

  → Give a rule. (in words) <u>Ex</u>: $A = \{$ natural numbers larger than $2\}$

  $\mathbb{N}$ = set of natural numbers

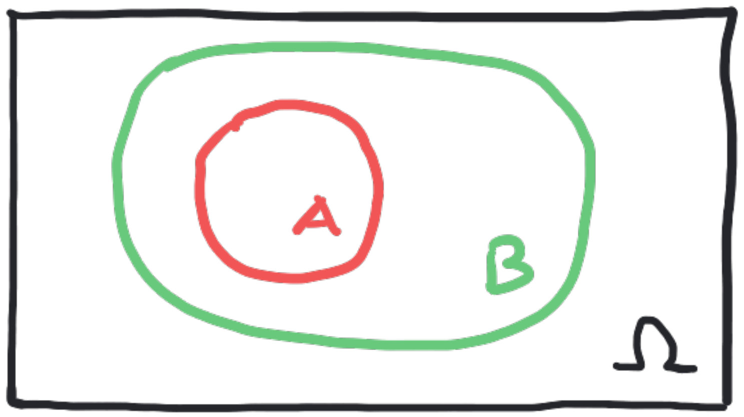
  → Give a rule. (in mathematical symbols) <u>Ex</u>: $A = \{\underset{\text{variable}}{x} \in \mathbb{N} \underset{\text{"such that"}}{:} \underbrace{x > 2}_{\text{condition}}\}$

  "set-builder notation"

- Pick the description method (or combination of methods) that is the clearest for the example at hand.

- a set A is a subset of a set B, $A \subset B$, if all of the elements in A are also in B.
  - → Ex: $A = \{1, 4\}$ $B = \{1, 2, 3, 4\}$ $A \subset B$
  - → A and B are equal if $A \subset B$ and $B \subset A$.
  - → $\phi \subset A$ for any set A.
  - → $A \subset \Omega$ for any set A.
- a Venn diagram can be used to illustrate the relationship between sets.

$$A \subset B \subset \Omega$$

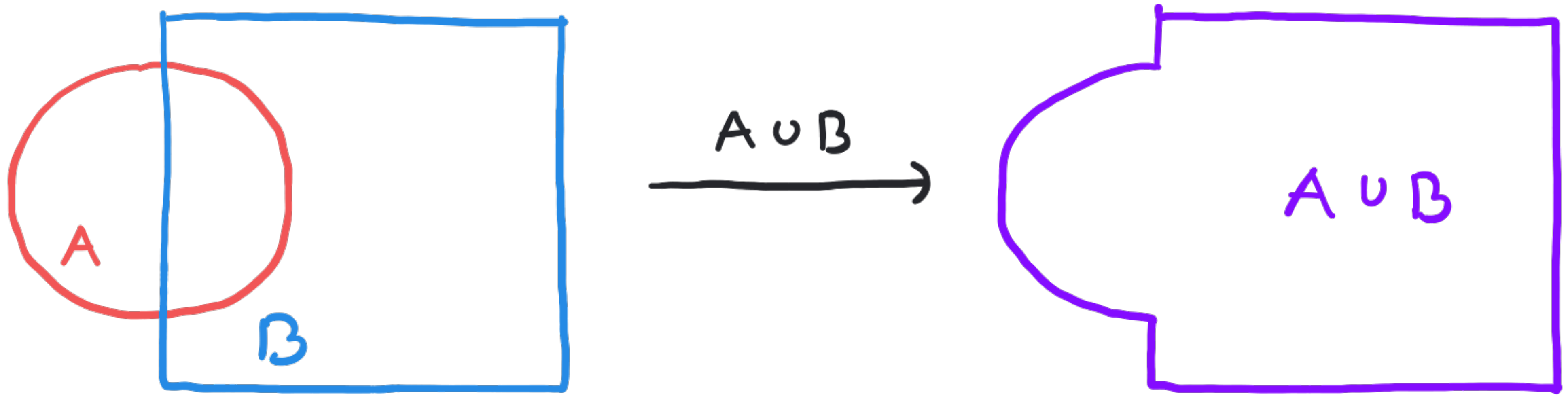


Set Operations

- The union $A \cup B$ of sets A and B is the set consisting of the elements of Ω that belong to A or belong to B.

$$\boxed{A \cup B = \{x \in \Omega : x \in A \text{ or } x \in B\}}$$

→ Set theory version of the logical "OR" operation.



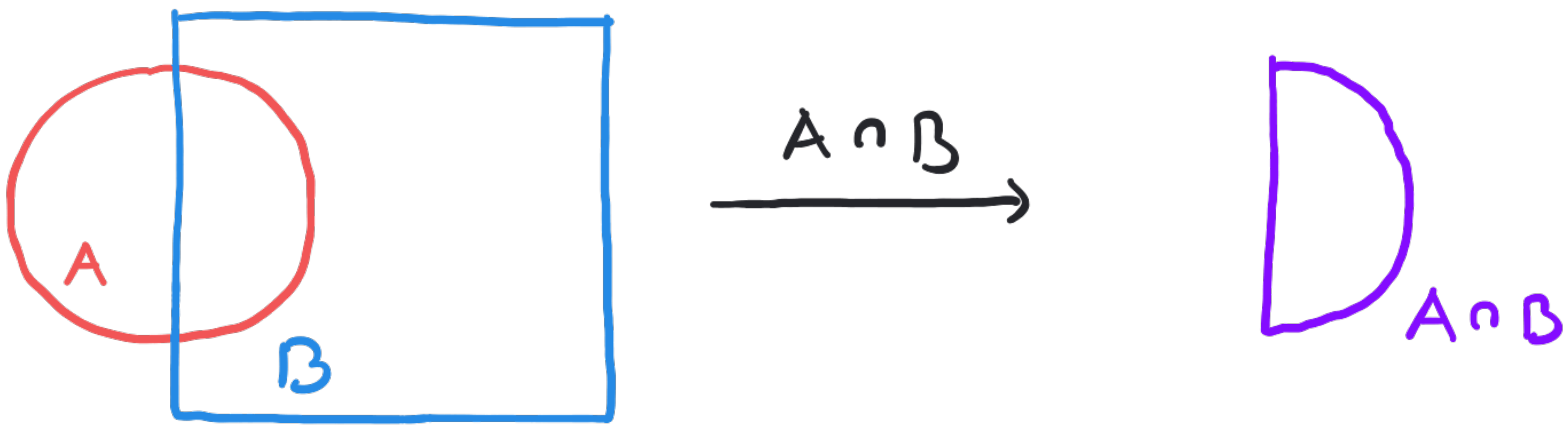
→ Notation: $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$

$$= \{x \in \Omega : x \in A_1 \text{ or } x \in A_2 \text{ or } \cdots \text{ or } x \in A_n\}$$

- The intersection $A \cap B$ of sets A and B is the set consisting of the elements of Ω that belong to A and belong to B.

$$\boxed{A \cap B = \{x \in \Omega : x \in A \text{ and } x \in B\}}$$

→ Set theory version of the logical "AND" operation.



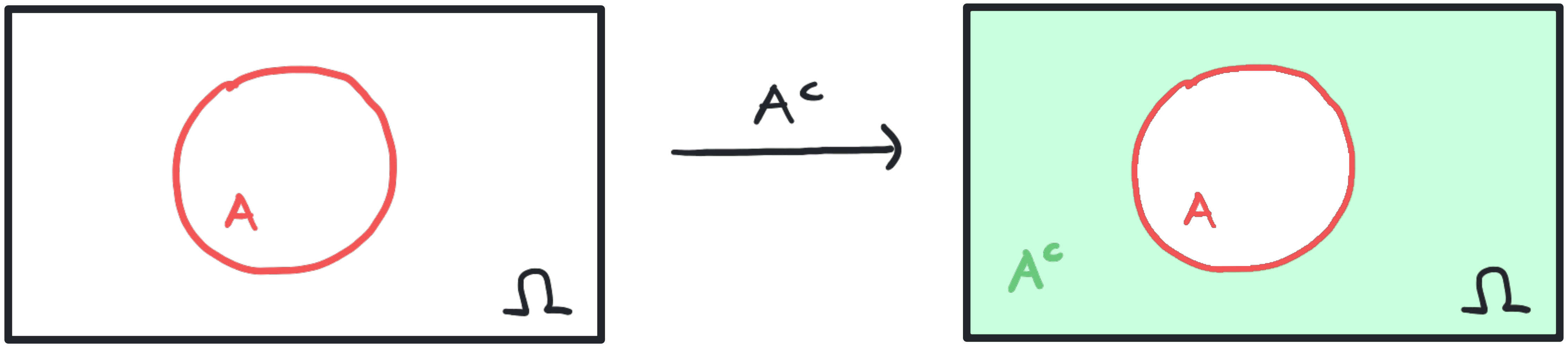
→ Notation: $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$

$$= \{x \in \Omega : x \in A_1 \text{ and } x \in A_2 \text{ and } \cdots \text{ and } x \in A_n\}$$

- The complement A^c of a set A is the set consisting of the elements of Ω that do not belong to A.

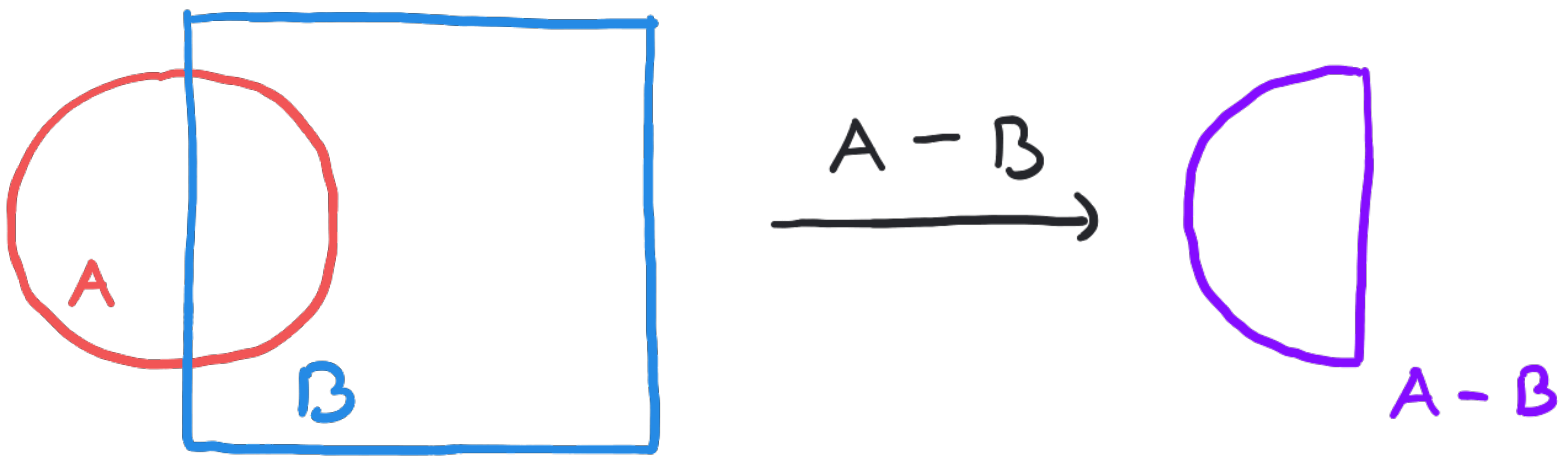
$$\boxed{A^c = \{x \in \Omega : x \notin A\}}$$

→ Set theory version of the logical "NOT" operation.



- The set difference $A - B$ of sets A and B is the set consisting of elements in Ω that belong to A and do not belong to B.

$$\boxed{\begin{aligned} A - B &= \{x \in \Omega : x \in A \text{ and } x \notin B\} \\ &= A \cap B^c \end{aligned}}$$



→ We will not use this as often as the preceding three operations.

De Morgan's Laws

- $(A \cup B)^c = A^c \cap B^c$

  → "not in (A or B)" = "(not in A) and (not in B)"

  → For more than 2 sets, $\left(\bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c$

- $(A \cap B)^c = A^c \cup B^c$

  → "not in (A and B)" = "(not in A) or (not in B)"

  → For more than 2 sets, $\left(\bigcap_{i=1}^{n} A_i \right)^c = \bigcup_{i=1}^{n} A_i^c$

- Recall the notation for an interval of the real line $\mathbb{R}$

$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$ closed interval

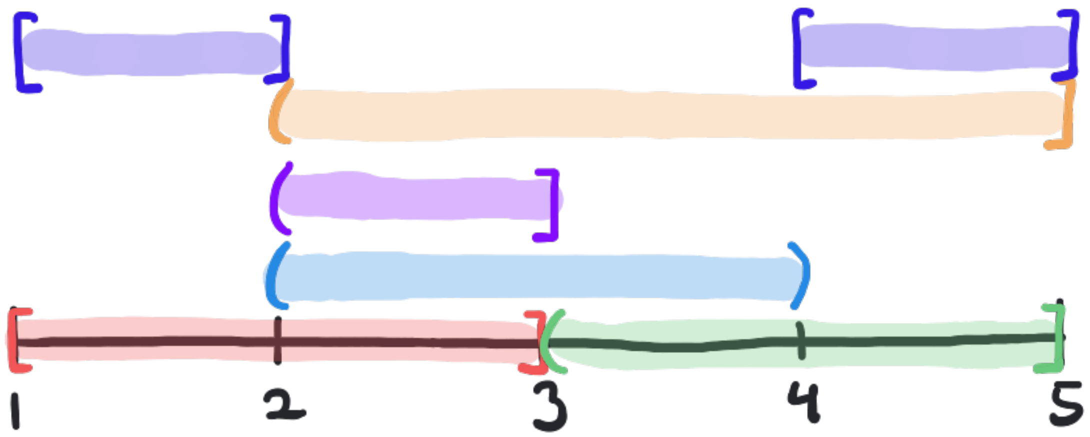
$$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$$

half-open intervals

$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}$$ open interval

- Example: $\Omega = [1,5]$ $A = [1,3]$ $B = (2,4)$ $C = (3,5]$

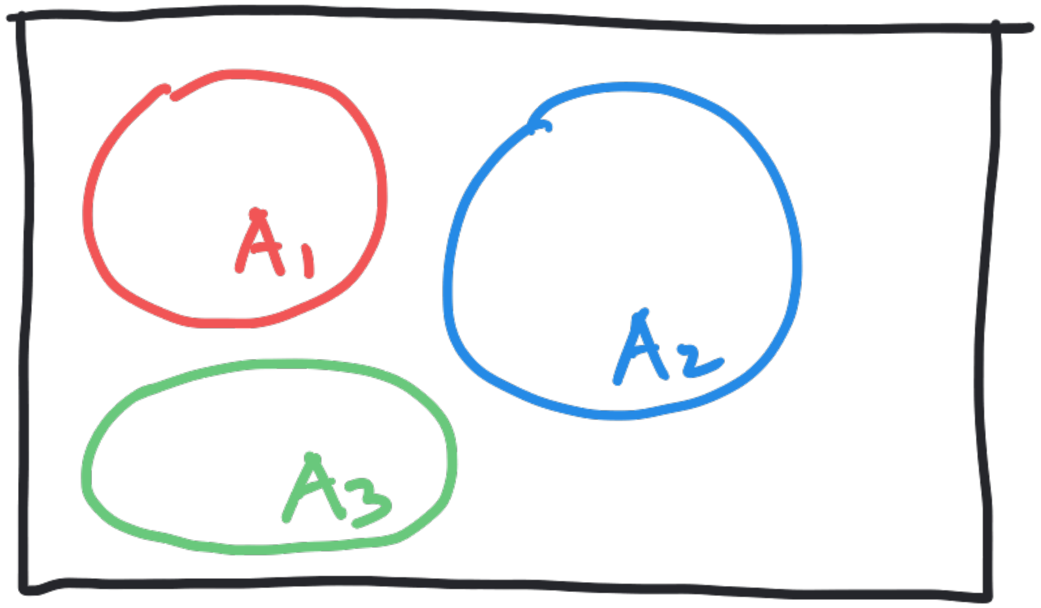


Determine $A^c = (3,5] = C$ $A \cap B = (2,3]$ $A \cap C = \emptyset$

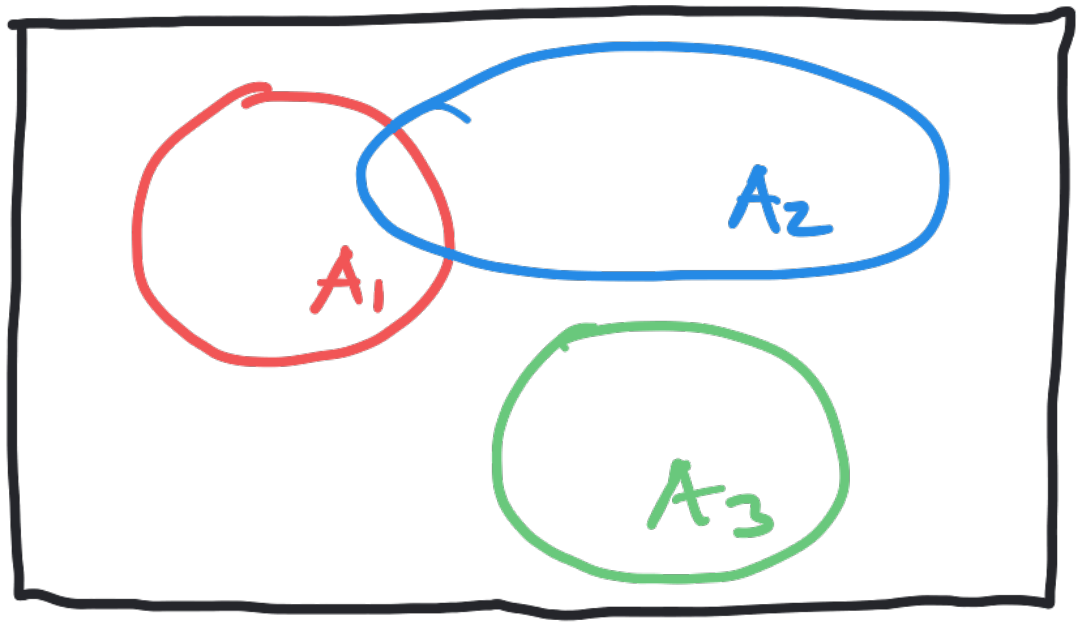
$B \cup C = (2,5]$ $B^c = [1,2] \cup [4,5]$

Other Set Concepts

- A and B are disjoint or mutually exclusive if $A \cap B = \emptyset$.
- A collection of sets $A_1, A_2, \ldots$ is mutually exclusive (or disjoint) if $A_i \cap A_j = \emptyset$ for all $i \neq j$.



A_1, A_2, A_3 mutually exclusive

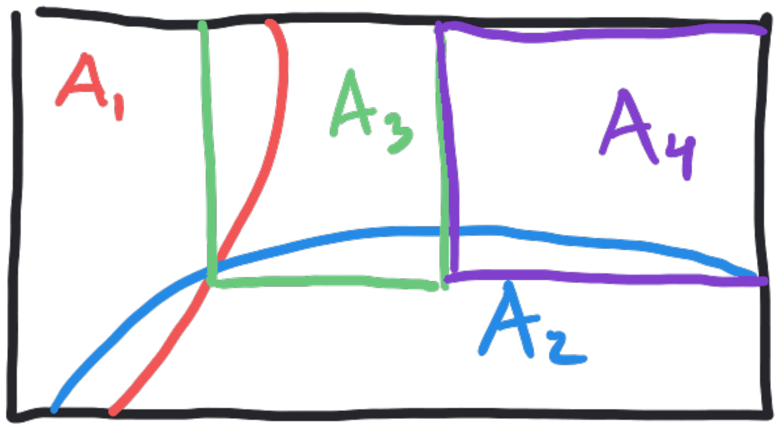


A_1, A_2, A_3 not mutually exclusive

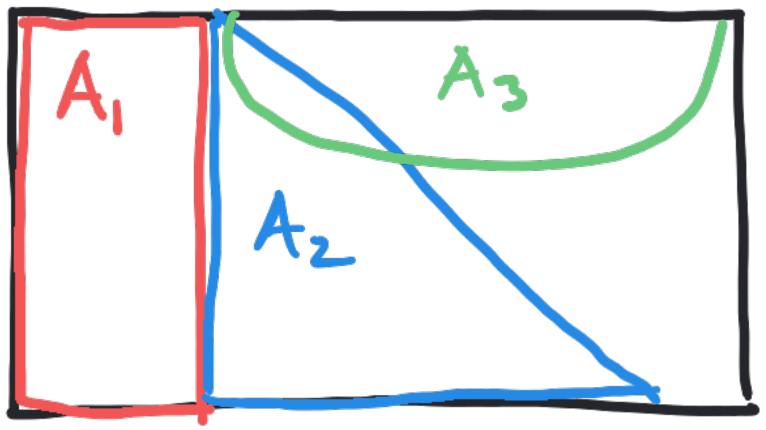
A_1, A_3 mutually exclusive

A_2, A_3 mutually exclusive

- a collection of sets $A_1, A_2, \ldots$ is collectively exhaustive if $A_1 \cup A_2 \cup \cdots = \Omega$.

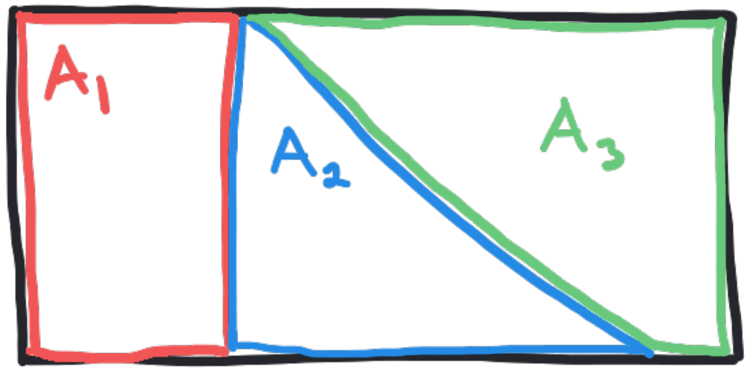


A_1, A_2, A_3, A_4 are collectively exhaustive



A_1, A_2, A_3 are not collectively exhaustive

- a collection of sets $A_1, A_2, \ldots$ is a partition if they are both mutually exclusive and collectively exhaustive.



A_1, A_2, A_3 are a partition.