Axiomatic Theory of Probability

- The basic model for probability starts with:
  - → An experiment which is a procedure for generating observable outcomes.
  - → An outcome is a possible observation from an experiment. We use the notation ω for an outcome.
  - → a sample space Ω, which is the set of all possible outcomes
- An event is a subset of Ω: it is a collection of possible outcomes.

- <u>Example 1</u>:

Experiment: Roll a six-sided die once.

Outcome: a number $\omega = 1, 2, 3, 4, 5, 6$

Sample Space: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Possible Events: $E_1 = \{\text{roll is odd}\} = \{1, 3, 5\}$

$E_2 = \{\text{roll is less than } 3\} = \{1, 2\}$

- **Example 2:**

Experiment: Two consecutive rolls of a four-sided die (order matters).

Outcomes: a pair $\omega = (i, j)$ where $i, j \in \{1, 2, 3, 4\}$

Sample Space: 16 distinct pairs

$$\Omega = \{(i, j) : i, j \in \{1, 2, 3, 4\}\}$$
$$= \{(1,1), \dots (1,4), \dots, (4,1), \dots, (4,4)\}$$

Possible Events: $E_1 = \{\text{sum of rolls is } 4\} = \{(1,3), (2,2), (3,1)\}$

$E_2 = \{\text{rolls equal}\} = \{(1,1), (2,2), (3,3), (4,4)\}$

- <u>Example 3:</u>

Experiment: Go to St. Mary's Green Line Station and wait for an inbound train. Record the time you wait in minutes (including decimals).

Outcome: a non-negative number ω.

Sample Space: $\Omega = [0, \infty)$

Possible Events: $E_1 = \{$train arrives in under two and a half minutes$\}$

$= [0, 2.5)$

$E_2 = \{$train takes more than twenty minutes$\}$

$= (20, \infty)$

- The event space $\mathcal{E}$ is a collection of subsets of the sample space Ω, chosen so that
  - → $\mathcal{E}$ contains the sample space, $\Omega \in \mathcal{E}$
  - → $\mathcal{E}$ is closed under complements, $A \in \mathcal{E} \Rightarrow A^c \in \mathcal{E}$
  - → $\mathcal{E}$ is closed under countable unions,

    $$A_1, A_2, \ldots \in \mathcal{E} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$$

- We will only work with events from the event space.
  - → This will be constructed for us, no need to worry about it.

Probability Axioms

- The probability measure $\mathbb{P}$ assigns probability values from $[0,1]$ to every event in the event space $\mathcal{E}$. It satisfies the following axioms:

① For any event $A \in \mathcal{E}$, $\mathbb{P}[A] \geq 0$. (Non-negativity)

② $\mathbb{P}[\Omega] = 1$ (Normalization)

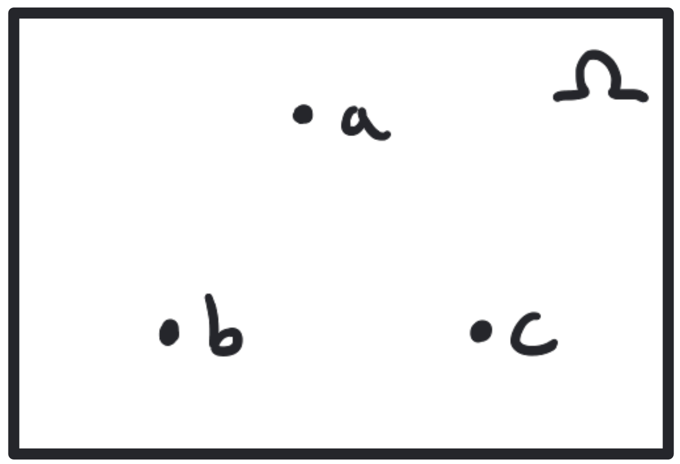
③ For any countable collection $A_1, A_2, \ldots$ of mutually exclusive events, $\mathbb{P}[A_1 \cup A_2 \cup \cdots] = \mathbb{P}[A_1] + \mathbb{P}[A_2] + \cdots$ (Countable Additivity)

(Can also write this as $\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$)

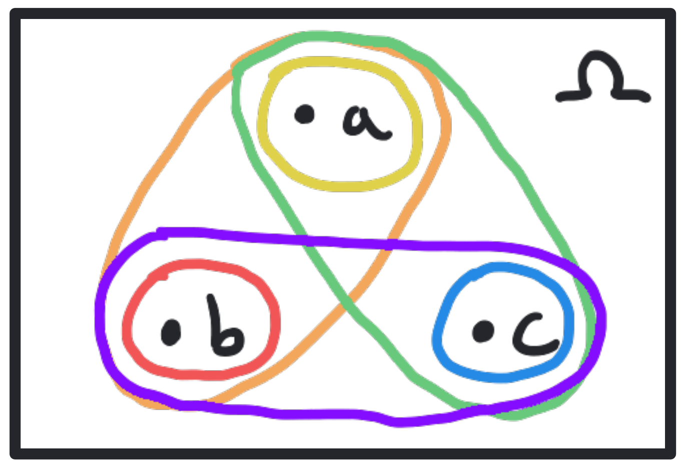
- Basic intuition for probabilities:
  - $\rightarrow$ $\mathbb{P}[A] = 0$ means the event A never occurs.
  - $\rightarrow$ $\mathbb{P}[A] = 1$ means the event A always occurs.
  - $\rightarrow$ $0 < \mathbb{P}[A] < 1$ means the event A sometimes occurs.

    Higher values of $\mathbb{P}[A]$ correspond proportionally to higher likelihood of occurrence.

- a **probability space** $(\Omega, \mathcal{E}, \mathbb{P})$ consists of a
  - → sample space Ω, which is the set of possible outcomes
  - → event space $\mathcal{E}$, which is the set of possible events, each of which is a subset of Ω
  - → probability measure $\mathbb{P}$, which assigns a probability between 0 and 1 to each event in $\mathcal{E}$

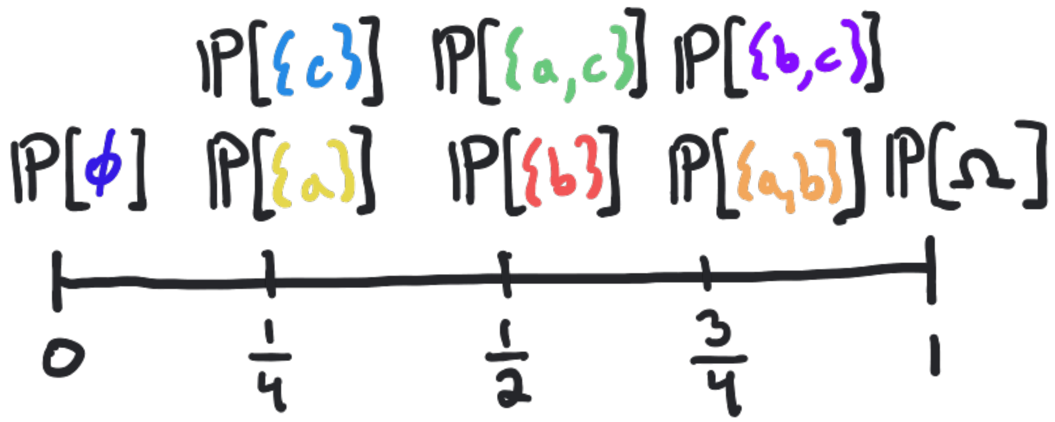


Sample Space



Event Space

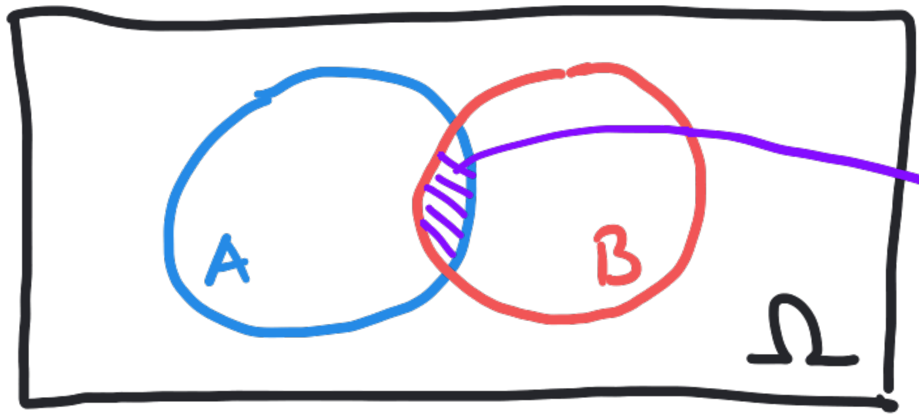
ϕ



Probability Measure

Basic Probability Properties

- For any finite collection $A_1, A_2, \ldots, A_n$ of mutually exclusive events, $\mathbb{P}[\bigcup_{i=1}^{n} A_i] = \sum_{i=1}^{n} \mathbb{P}[A_i]$.
  - → For two mutually exclusive events A, B, $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$.
- $\mathbb{P}[\phi] = 0$.
- For any event A, $\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$. (Complement)
- For any events A, B, $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$ (Inclusion - Exclusion)



$\mathbb{P}[A] + \mathbb{P}[B]$ counts the intersection probability $\mathbb{P}[A \cap B]$ twice.

- <u>Back to Example 1</u>: Roll a six-sided die. $\Omega = \{1,2,3,4,5,6\}$

  Assume outcomes are equally likely: $\mathbb{P}[\{1\}] = \cdots = \mathbb{P}[\{6\}] = \alpha$

→ Solve for α. We know that $\sum_{\omega=1}^{6} \mathbb{P}[\{\omega\}] \underset{\text{additivity}}{=} \mathbb{P}[\Omega] \underset{\text{Normalization}}{=} 1$

Therefore, $\sum_{\omega=1}^{6} \mathbb{P}[\{\omega\}] = \sum_{\omega=1}^{6} \alpha = 1 \Rightarrow 6\alpha = 1 \Rightarrow \alpha = \frac{1}{6}$

→ What is the probability the roll is odd?

$$\mathbb{P}[\{\text{roll is odd}\}] = \mathbb{P}[E_1] = \mathbb{P}[\{1,3,5\}] \overset{\text{additivity}}{=} \mathbb{P}[\{1\}] + \mathbb{P}[\{3\}] + \mathbb{P}[\{5\}]$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

→ What is the probability the roll is less than 3?

$$\mathbb{P}[\{\text{less than } 3\}] = \mathbb{P}[E_2] = \mathbb{P}[\{1,2\}] \overset{\text{additivity}}{=} \mathbb{P}[\{1\}] + \mathbb{P}[\{2\}] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

→ What is the probability the roll is odd and/or less than 3?

$$\mathbb{P}[E_1 \cap E_2] = \mathbb{P}[\{1\}] = \frac{1}{6} \qquad \mathbb{P}[E_1 \cup E_2] = \mathbb{P}[\{1,2,3,5\}] = \frac{4}{6} = \frac{2}{3}$$

Alternatively, by Inclusion - Exclusion, $\mathbb{P}[E_1 \cup E_2] = \mathbb{P}[E_1] + \mathbb{P}[E_2] - \mathbb{P}[E_1 \cap E_2]$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$$

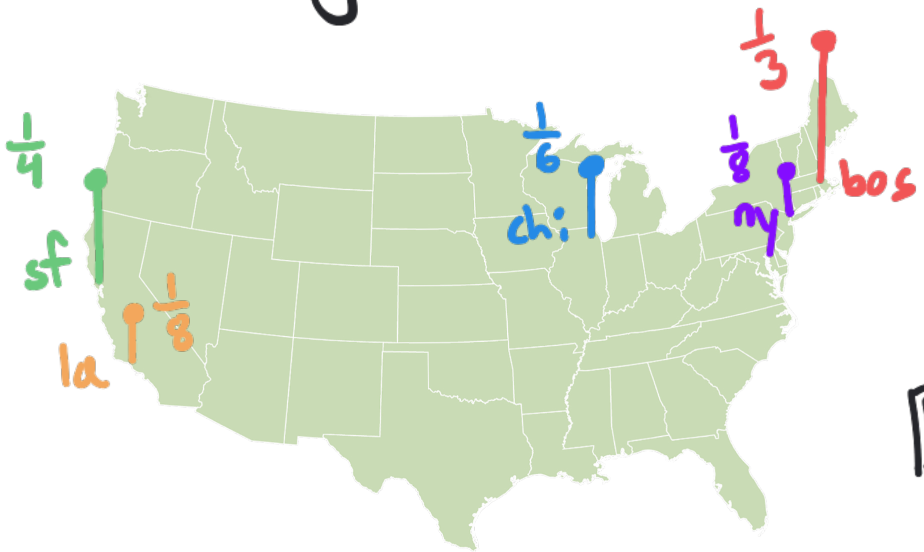
- If a sample space Ω consists of a finite number of outcomes, we can simply take the event space $\mathcal{E}$ to be all possible subsets of Ω and construct the probability measure $\mathbb{P}$ by assigning probabilities to outcomes.

- Ex: Experiment: Ask a stranger which city they prefer: Boston, Chicago, Los Angeles, New York, San Francisco

Sample Space: $\Omega = \{\text{bos}, \text{chi}, \text{la}, \text{ny}, \text{sf}\}$

Assign Probabilities: $\mathbb{P}[\{\text{bos}\}] = \frac{1}{3}$ $\quad \mathbb{P}[\{\text{chi}\}] = \frac{1}{6}$ $\quad \mathbb{P}[\{\text{la}\}] = \frac{1}{8}$

$\mathbb{P}[\{\text{ny}\}] = \frac{1}{8}$ $\quad \mathbb{P}[\{\text{sf}\}] = \frac{1}{4}$



$W = \{\text{prefer west-coast city}\} = \{\text{la}, \text{sf}\}$

$$\mathbb{P}[W] = \mathbb{P}[\{\text{la}, \text{sf}\}] = \mathbb{P}[\{\text{la}\}] + \mathbb{P}[\{\text{sf}\}]$$

$$= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$