Conditional Probability: Examples

- Consider a quick, cheap test for a disease.
  - → Let $A = \{\text{subject has disease}\}$ and $B = \{\text{test positive}\}$.

- Study has shown that
  - → Given subject has the disease, test positive with probability $\frac{4}{5}$.

    $\Rightarrow \mathbb{P}[B \mid A] = \frac{4}{5} \xrightarrow{\text{Complement}} \mathbb{P}[B^c \mid A] = \frac{1}{5}$
  - → Given subject healthy, test negative with probability $\frac{3}{4}$.

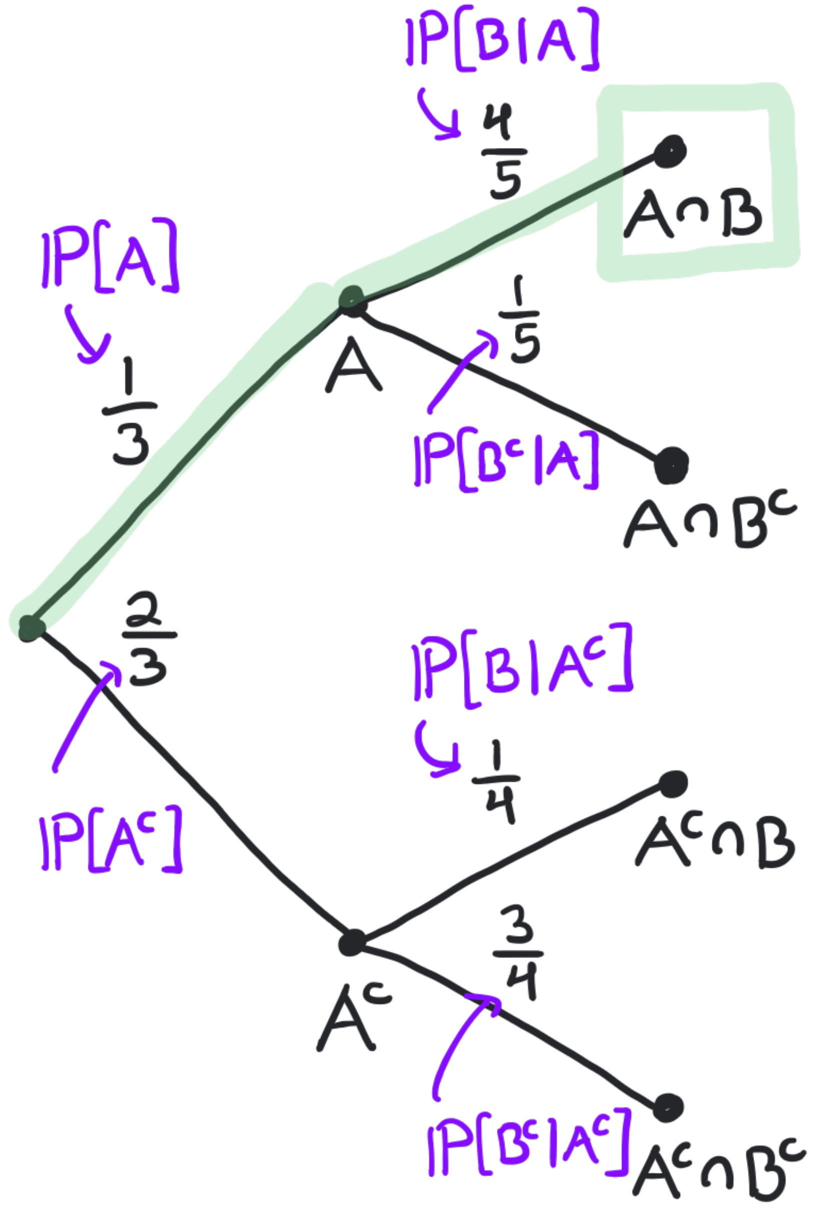
    $\Rightarrow \mathbb{P}[B^c \mid A^c] = \frac{3}{4} \xrightarrow{\text{Complement}} \mathbb{P}[B \mid A^c] = \frac{1}{4}$

- One out of three subjects currently have the disease.

  $\Rightarrow \mathbb{P}[A] = \frac{1}{3} \xrightarrow{\text{Complement}} \mathbb{P}[A^c] = \frac{2}{3}$

- What is the probability that a subject has the disease, given the test is positive?

• To build intuition, let's draw a conditional probability tree.



• What is the probability that the $\underbrace{\text{subject has the disease}}_{A}$ and the $\underbrace{\text{test is positive}}_{B}$?

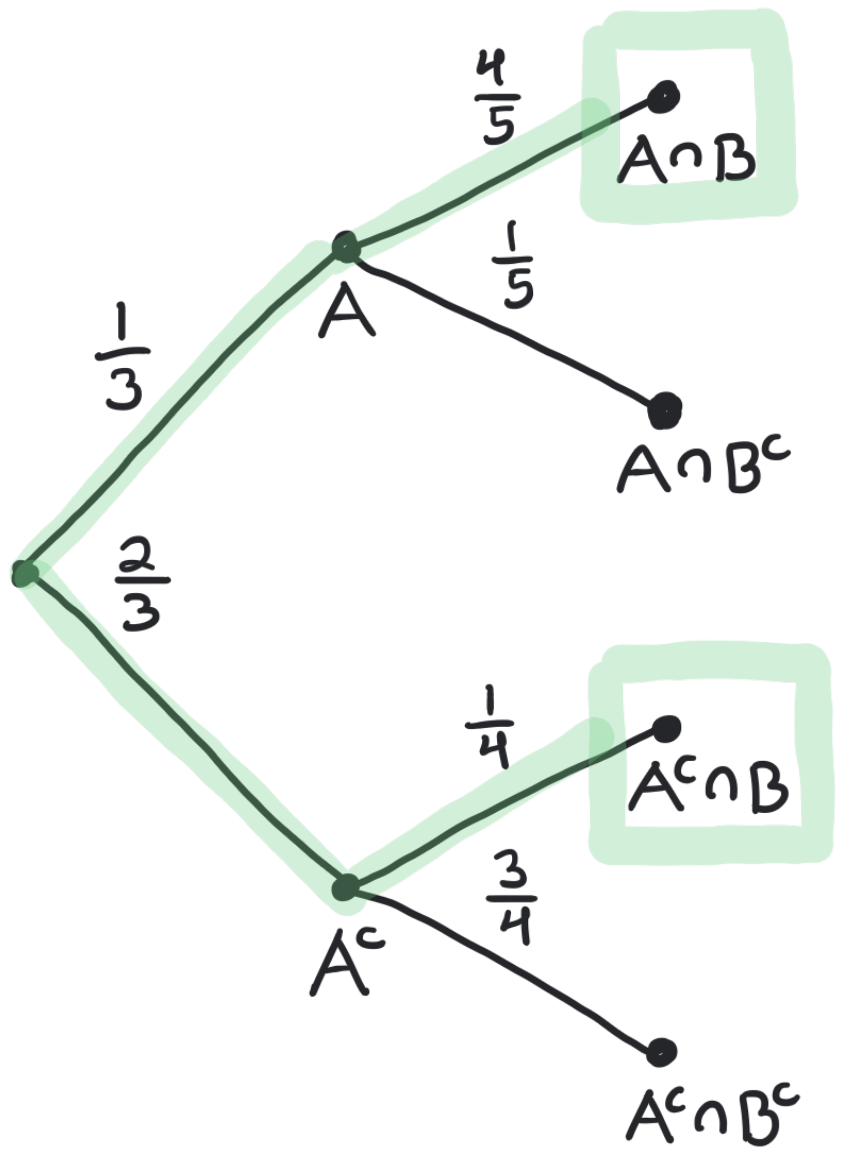
$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B \mid A]$$

Multiplication Rule

$$= \frac{1}{3} \cdot \frac{4}{5} = \frac{4}{15}$$

• For any event that appears directly on the tree, we can compute its probability by multiplying all of the values from the root to the event.

$$\mathbb{P}[A \cap B] = \frac{1}{3} \cdot \frac{4}{5} = \frac{4}{15}$$



- What is the probability that the test is positive? B

$$\mathbb{P}[B] = \mathbb{P}[A]\cdot\mathbb{P}[B|A] + \mathbb{P}[A^c]\cdot\mathbb{P}[B|A^c]$$

Law of Total Probability

$$= \frac{1}{3}\cdot\frac{4}{5} + \frac{2}{3}\cdot\frac{1}{4}$$

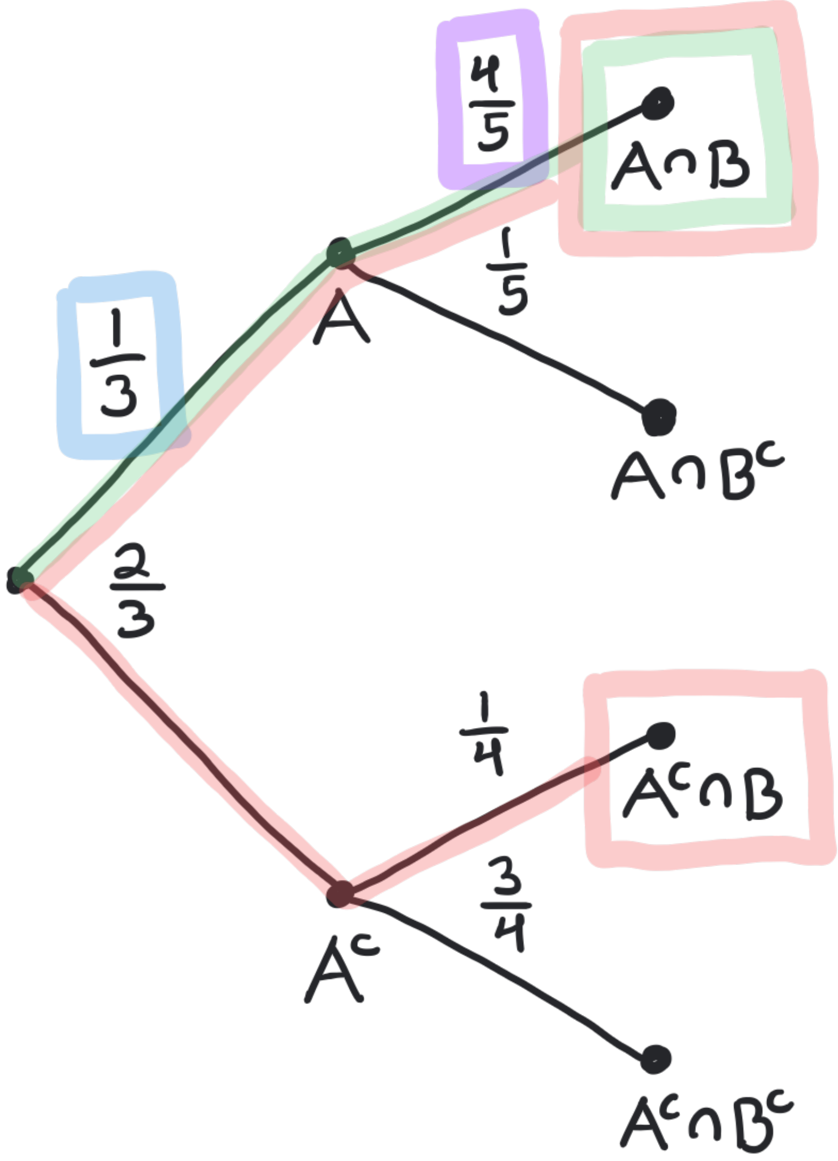
$$= \frac{16+10}{60} = \frac{26}{60} = \frac{13}{30}$$

- To find the probability of an event appearing on the tree as part of other events,

① Find all nodes (at the same depth) that include the event.

② Add up their probabilities.

$$\mathbb{P}[B] = \frac{1}{3}\cdot\frac{4}{5} + \frac{2}{3}\cdot\frac{1}{4} = \frac{13}{30}$$



- What is the probability that a subject has the disease, given that the test is positive?

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A] \cdot \mathbb{P}[A]}{\mathbb{P}[B]}$$

← previous problem

Bayes' Rule to "flip" conditioning

$$= \frac{\frac{4}{5} \cdot \frac{1}{3}}{\frac{13}{30}} = \frac{8}{13}$$

- We can also find this using the tree and the definition of conditional probability.

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

$$= \frac{\frac{1}{3} \cdot \frac{4}{5}}{\frac{1}{3} \cdot \frac{4}{5} + \frac{2}{3} \cdot \frac{1}{4}} = \frac{\frac{4}{15}}{\frac{13}{30}} = \frac{8}{13}$$