Independence

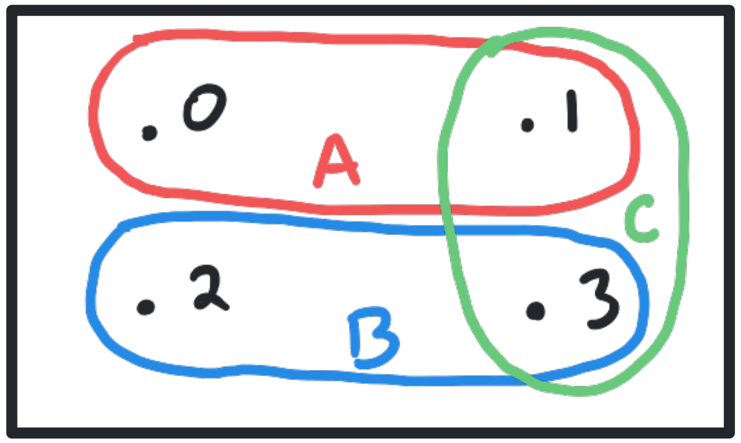
- Two event A and B are independent if

$$\boxed{\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]}$$

- **Example:** Sample Space: $\Omega = \{0, 1, 2, 3\}$

Probabilities: $\mathbb{P}[\{0\}] = \frac{2}{9}$ $\quad \mathbb{P}[\{1\}] = \frac{1}{9}$

$\mathbb{P}[\{2\}] = \frac{4}{9}$ $\quad \mathbb{P}[\{3\}] = \frac{2}{9}$



Events: $A = \{0, 1\}$ $\quad B = \{2, 3\}$ $\quad C = \{1, 3\}$

Are A and B independent?

No. $\mathbb{P}[A] = \frac{1}{3}$ $\quad \mathbb{P}[B] = \frac{2}{3}$ $\quad \mathbb{P}[A \cap B] = 0$

$\mathbb{P}[A \cap B] \neq \mathbb{P}[A] \cdot \mathbb{P}[B]$

A and B are mutually exclusive.

Are A and C independent? Yes. $\mathbb{P}[C] = \frac{1}{3}$ $\quad \mathbb{P}[A \cap C] = \frac{1}{9}$

$\mathbb{P}[A \cap C] = \mathbb{P}[A] \cdot \mathbb{P}[C]$

A and C are not mutually exclusive.

- Independence is not the same as mutually exclusive.
  - → If A and B are mutually exclusive, $\mathbb{P}[A \cap B] = \mathbb{P}[\phi] = 0$. Therefore, they are only independent if $\mathbb{P}[A] = 0$ or $\mathbb{P}[B] = 0$. (This is not an interesting scenario.)

- Conditional probability perspective:
  - → If $\mathbb{P}[B] > 0$, then "A and B are independent" is equivalent to "$\mathbb{P}[A \mid B] = \mathbb{P}[A]$."
  - → If $\mathbb{P}[A] > 0$, then "A and B are independent" is equivalent to "$\mathbb{P}[B \mid A] = \mathbb{P}[B]$."
  - → Intuitively, if A and B are independent, then knowing that A occurs does not help us predict whether B occurs and vice versa.

- Three or more events $A_1, A_2, \ldots, A_n$ are **(mutually) independent** if:
  - → all collections of $n-1$ events chosen from $A_1, A_2, \ldots, A_n$ are (mutually) independent.
  - → $\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2] \cdot \ldots \cdot \mathbb{P}[A_n]$.

- Let's open up this recursive definition for $n = 3$ events:
  - → A_1, A_2, A_3 are independent if
    - * A_1, A_2 are independent: $\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2]$
    - * A_1, A_3 are independent: $\mathbb{P}[A_1 \cap A_3] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_3]$
    - * A_2, A_3 are independent: $\mathbb{P}[A_2 \cap A_3] = \mathbb{P}[A_2] \cdot \mathbb{P}[A_3]$
  - → $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2] \cdot \mathbb{P}[A_3]$

- <u>Example</u>: Experiment: Flip a coin twice. $\Omega = \{HH, HT, TH, TT\}$

Events: $A = \{1^{st} \text{ flip heads}\} = \{HH, HT\}$ all outcomes equally likely.

$B = \{2^{nd} \text{ flip heads}\} = \{HH, TH\}$

$C = \{1^{st} \text{ and } 2^{nd} \text{ flips different}\} = \{HT, TH\}$

$\mathbb{P}[A] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $\quad \mathbb{P}[B] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $\quad \mathbb{P}[C] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\mathbb{P}[A \cap B] = \frac{1}{4}$ $\quad \mathbb{P}[A \cap C] = \frac{1}{4}$ $\quad \mathbb{P}[B \cap C] = \frac{1}{4}$

$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B] \Rightarrow$ A and B independent

$\mathbb{P}[A \cap C] = \mathbb{P}[A] \cdot \mathbb{P}[C] \Rightarrow$ A and C independent

$\mathbb{P}[B \cap C] = \mathbb{P}[B] \cdot \mathbb{P}[C] \Rightarrow$ B and C independent

$\mathbb{P}[A \cap B \cap C] = \mathbb{P}[\emptyset] = 0 \neq \mathbb{P}[A] \cdot \mathbb{P}[B] \cdot \mathbb{P}[C]$

$\Rightarrow$ A, B, C are not independant. Intuitively, any two events can be used to improve our prediction of the other event.

- Three or more events $A_1, A_2, \ldots, A_n$ are pairwise independent if $\mathbb{P}[A_i \cap A_j] = \mathbb{P}[A_i] \cdot \mathbb{P}[A_j]$ for all $i \neq j$.
  - → In the previous example, A, B, and C are pairwise independent.
  - → Intuition: Any one event does not help to predict any other event.
- In general, independence can be very tedious to check.
- More often, we will assume independence as part of our modeling assumptions.
  - → If a communication channel is corrupted by noise, it makes sense to assume the noise is independent of the message.
  - → Component failures may be highly dependent within a vehicle but independent across vehicles.

- Two events A and B are conditionally independent given C if $\mathbb{P}[A \cap B \mid C] = \mathbb{P}[A \mid C] \cdot \mathbb{P}[B \mid C]$.

→ Independence does not imply conditional independence.

→ Conditional independence does not imply independence.

→ Intuition: Given that C occurs, A does not tell us anything additional about whether B occurs (and B tells us nothing additional about A).

→ If $\mathbb{P}[B \cap C] > 0$, "A and B are conditionally independent given C" is equivalent to "$\mathbb{P}[A \mid B \cap C] = \mathbb{P}[A \mid C]$."

→ If $\mathbb{P}[A \cap C] > 0$, "A and B are conditionally independent given C" is equivalent to "$\mathbb{P}[B \mid A \cap C] = \mathbb{P}[B \mid C]$."

- <u>Example</u>: Experiment: Flip a coin twice. $\Omega = \{HH, HT, TH, TT\}$

  Events: $A = \{1^{st} \text{ flip heads}\} = \{HH, HT\}$ all outcomes equally likely.

  $B = \{2^{nd} \text{ flip heads}\} = \{HH, TH\}$

  $C = \{1^{st} \text{ and } 2^{nd} \text{ flips different}\} = \{HT, TH\}$

  $$\mathbb{P}[A \cap B \mid C] = \frac{\mathbb{P}[A \cap B \cap C]}{\mathbb{P}[C]} = \frac{\mathbb{P}[\emptyset]}{\mathbb{P}[C]} = 0$$

  $$\mathbb{P}[A \mid C] = \frac{\mathbb{P}[A \cap C]}{\mathbb{P}[C]} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

  $$\Rightarrow \mathbb{P}[A \cap B \mid C] \neq \mathbb{P}[A \mid C] \cdot \mathbb{P}[B \mid C]$$

  $$\mathbb{P}[B \mid C] = \frac{\mathbb{P}[B \cap C]}{\mathbb{P}[C]} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

  A and B are not conditionally independent given C.

- Even though A and B are independent, they become conditionally dependent given C. Intuitively, knowing whether the flips differ lets us predict one flip using the other.

- Three or more events $A_1, A_2, \ldots, A_n$ are conditionally independent given B if:
  - → Any collection of $n-1$ events from $A_1, A_2, \ldots, A_n$ is conditionally independent given B.
  - → $\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n \mid B] = \mathbb{P}[A_1 \mid B] \cdot \mathbb{P}[A_2 \mid B] \cdot \cdots \cdot \mathbb{P}[A_n \mid B]$

- Independence is preserved under complements.
  - → If A and B are independent, so are
    A and B^c,
    A^c and B, as well as
    A^c and B^c.