Counting

- We will now discuss a few basic counting techniques to help us develop more complex probability models.
  - → Especially useful in scenarios where all outcomes are equally likely since $\mathbb{P}[A] = \dfrac{\#\text{ outcomes in } A}{\#\text{ outcomes in } \Omega}$

- Basic Idea: If an experiment can be broken into m subexperiments and the i^{th} subexperiment consists of n_i sub-outcomes that can be chosen freely, then the total number of outcomes is $n_1 \cdot n_2 \cdot \dots \cdot n_m$.

- **Example**: Roll a six-sided die ($n_1 = 6$), then flip a coin ($n_2 = 2$), and finally roll a twelve-sided die ($n_3 = 12$). There are $6 \cdot 2 \cdot 12 = 144$ total outcomes.

- **Example:** How many ways are there to draw three face cards (J, Q, K) from a standard 52-card deck, without replacement between draws?

  1st card: (# faces) $\cdot$ (# suits) = $3 \cdot 4 = 12$ options

  2nd card: 11 options ♡ ♢ ♣ ♠

  3rd card: 10 options

  $12 \cdot 11 \cdot 10 = 1320$ possible draws

  What is the probability of such a draw?

  1st card: 52 options

  2nd card: 51 options

  3rd card: 50 options

  $52 \cdot 51 \cdot 50 = 132600$ possible draws

$$\mathbb{P}[\{\text{draw 3 face cards}\}] = \frac{\#\text{ ways to draw 3 face cards}}{\#\text{ ways to draw 3 cards}}$$

$$= \frac{1320}{132600} = \frac{11}{1105}$$

- Many counting problems can be decomposed into a sequence of sampling problems.
- A sampling problem consists of:
  - → n distinguishable elements
  - → k selections to be made
  - → selections made either with or without replacement
    - * <u>Ex</u>: Draw consecutive cards from a deck. Without replacement.
    - * <u>Ex</u>: Roll a die twice. With replacement.
  - → final outcome is either order dependent or independent
    - * <u>Ex</u>: Roll two dice, only care about sum. Order independent.
    - * <u>Ex</u>: Guess a locker combination. Order dependent.
- We now develop formulas for each of these four configurations.

- **Sampling with replacement, order dependent:**

$$\boxed{\#\text{ of possibilities } = n^k}$$

- **Example:** Roll a six-sided die three times. What is the probability of seeing three 2's?

  $\#$ ways to roll three 2's $= 1$

  $\#$ ways to roll $= 6^3 = 216$

  $$\mathbb{P}[\{⚁, ⚁, ⚁\}] = \frac{1}{216}$$

- <u>Sampling without replacement, order dependent:</u>

$$\boxed{\#\text{ of possibilities } = n\cdot(n-1)\cdot\cdots\cdot(n-k+1) = \frac{n!}{(n-k)!}}$$

→ Sometimes called a k-permutation.

→ Recall that $n! = n\cdot(n-1)\cdot\cdots\cdot 1$ ("n factorial") is the number of ways we can order n elements.

→ Define $0! = 1$ for convenience.

- <u>Example</u>: Draw 3 cards from a 52-card deck, without reinserting cards between draws. What is the probability of drawing $J\diamondsuit$ then $Q\diamondsuit$ then $K\diamondsuit$?

$\#$ ways to draw $J\diamondsuit, Q\diamondsuit, K\diamondsuit = 1$

$\#$ ways to draw $= \dfrac{52!}{49!} = 52\cdot 51\cdot 50$
$= 132600$

$$\mathbb{P}[\{J\diamondsuit, Q\diamondsuit, K\diamondsuit\}] = \frac{1}{132600}$$

- <u>Sampling without replacement, order independent:</u>

$$\boxed{\# \text{ of possibilities} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}}$$

→ Divide by $k!$ since we don't care about order of selections.

→ Sometimes called a k-combination.

- <u>Example:</u> Draw 3 cards from a standard deck without replacement. What is the probability of getting $J\diamondsuit$, $Q\diamondsuit$, $K\diamondsuit$ in any order?

$\#$ ways to draw $J\diamondsuit, Q\diamondsuit, K\diamondsuit = 1$

$$\# \text{ ways to draw} = \binom{52}{3} = \frac{52!}{3!\,49!} = \frac{52 \cdot 51 \cdot 50}{3 \cdot 2 \cdot 1} = 22100$$

$$\mathbb{P}[\{J\diamondsuit, Q\diamondsuit, K\diamondsuit\}] = \frac{1}{22100}$$

- <u>Sampling with replacement, order independent:</u>

$$\boxed{\#\text{ of possibilities} = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}}$$

- <u>Example</u>: How many order-independent configurations of three balls that can be red, blue, or green?

  $n = 3$ colors $\quad k = 3$ choices

  $$\binom{3+3-1}{3} = \binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2 \cdot 1} = 10 \text{ configurations}$$

→ Double Check: 

- Caution: This formula does not play a major role in our probability calculations since the configurations are usually not equally likely.

- **Example:** What is the probability of drawing 3 kings in a hand of 5 cards (without replacement)?

Two subexperiments: ① Draw 3 kings.
② Draw other 2 cards.

$$\#\text{ 5-card hands with 3 kings} = (\#\text{ ways to draw 3 kings}) \times (\#\text{ ways to draw 2 non-kings}) = \binom{4}{3} \times \binom{48}{2}$$

$$\#\text{ 5-card hands} = \binom{52}{5}$$

$$\mathbb{P}[\{3\text{ kings in 5-card hand}\}] = \frac{\#\text{ 5-card hands with 3 kings}}{\#\text{ 5-card hands}} = \frac{\binom{4}{3} \times \binom{48}{2}}{\binom{52}{5}} = \frac{4512}{2598960} \approx 0.0017$$