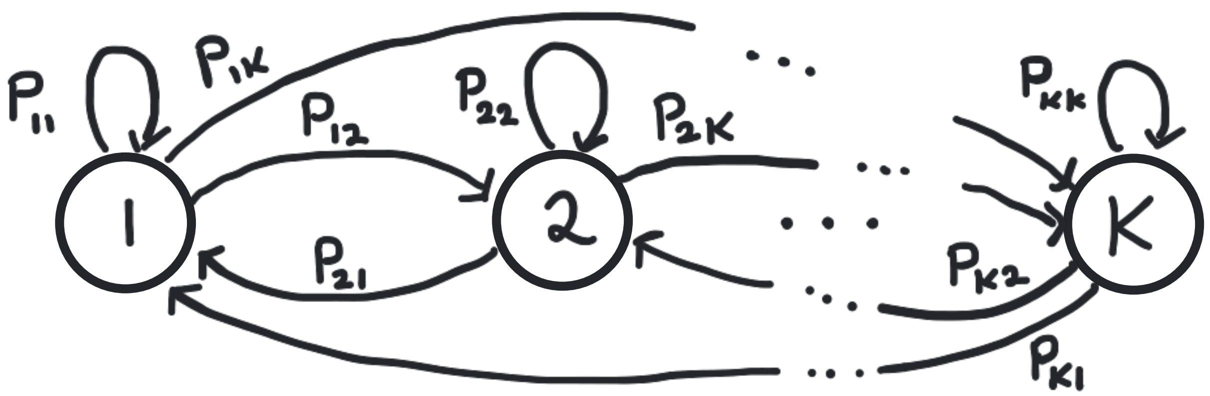
State Vector and Transition Matrix

- Consider a discrete-time Markov chain $X_0, X_1, X_2, \ldots$ with the following properties:
  - → Finite Range: $R_X = \{1, 2, \ldots, K\}$ (or any other labels)
  - → Homogeneous: $\mathbb{P}[X_{t+1} = k \mid X_t = j] = P_{jk}$ for all $t = 0, 1, 2, \ldots$
- Recall that the P_{jk} are the transition probabilities: the probability of going from state j to state k (in one step).
  - → Non-Negativity: $P_{jk} \geq 0$
  - → Normalization: $\sum_{k=1}^{K} P_{jk} = 1$
  - → n-step transition probabilities $\mathbb{P}[X_{t+n} = k \mid X_t = j] = P_{jk}(n)$ can be found using $\underbrace{P_{jk}(n+m) = \sum_{i=1}^{K} P_{ji}(n) P_{ik}(m)}_{\text{Chapman-Kolmogorov Equations}}$ with $P_{jk}(1) = P_{jk}$.



This illustration assumes that all $P_{jk} > 0$. If $P_{jk} = 0$, we do not draw an arc between j and k.

- It is often more convenient to write the transition probabilities as a matrix.

- The **state transition matrix** (or **transition probability matrix**) $\mathbf{P}$ is a $K \times K$ matrix whose $(j,k)^{th}$ entry is P_{jk}.

$$\boxed{\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1K} \\ P_{21} & P_{22} & \cdots & P_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ P_{K1} & P_{K2} & \cdots & P_{KK} \end{bmatrix}}$$

→ Row index refers to the current state.

→ Column index refers to the next state.

→ Normalization: Each row sums up to 1.

- The **n-step state transition matrix** (or **n-step transition probability matrix**) $\mathbf{P}(n)$ is a $K \times K$ matrix whose $(j,k)^{th}$ entry is $P_{jk}(n)$.

$$\boxed{\mathbf{P}(n) = \begin{bmatrix} P_{11}(n) & P_{12}(n) & \cdots & P_{1K}(n) \\ P_{21}(n) & P_{22}(n) & \cdots & P_{2K}(n) \\ \vdots & \vdots & \ddots & \vdots \\ P_{K1}(n) & P_{K2}(n) & \cdots & P_{KK}(n) \end{bmatrix}}$$

→ Chapman-Kolmogorov Equations:

$$\mathbf{P}(n+m) = \mathbf{P}(n)\,\mathbf{P}(m)$$

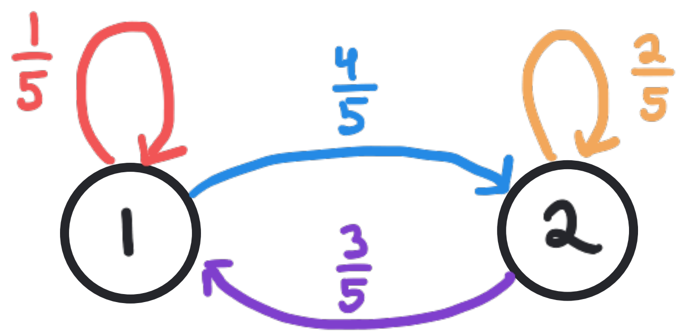
→ $\mathbf{P}(n) = \mathbf{P}^n$

→ Plus all the properties above.

- <u>Example</u>: Determine the two-step transition probability matrix.

$P_{11} = \frac{1}{5}$ $\quad P_{12} = \frac{4}{5}$ $\quad P_{21} = \frac{3}{5}$ $\quad P_{22} = \frac{2}{5}$

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$



$$\mathbf{P}(2) = \mathbf{P}^2 = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{3}{5} & \frac{1}{5} \cdot \frac{4}{5} + \frac{4}{5} \cdot \frac{2}{5} \\ \frac{3}{5} \cdot \frac{1}{5} + \frac{2}{5} \cdot \frac{3}{5} & \frac{3}{5} \cdot \frac{4}{5} + \frac{2}{5} \cdot \frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{13}{25} & \frac{12}{25} \\ \frac{9}{25} & \frac{16}{25} \end{bmatrix}$$

- This is the same answer we obtained before when we used the Chapman-Kolmogorov equations.

- Intuitively, we are just tracking how probabilities "flow" from current states through all possible paths to the next state.

- The state probability vector at time t $\underline{p}_t$ is a length-K column vector whose j^{th} entry is the probability of occupying state j at time t, $P_{X_t}(j) = \mathbb{P}[X_t = j]$.

$$\boxed{\underline{p}_t = \begin{bmatrix} P_{X_t}(1) \\ \vdots \\ P_{X_t}(K) \end{bmatrix}}$$

  → $\underline{p}_0$ is called the initial probability state vector.

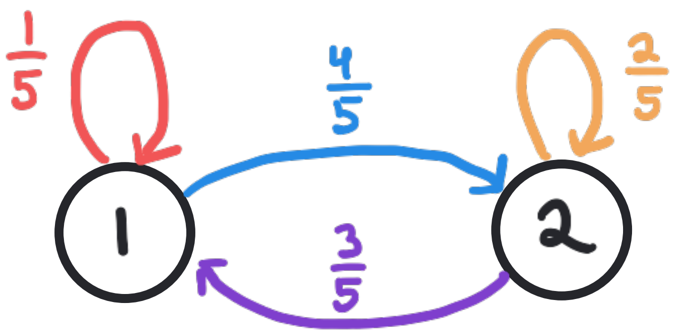
- We can determine how $\underline{p}_t$ changes in one step using the
  → transition probabilities $P_{X_{t+1}}(k) = \sum_{j=1}^{K} P_{X_t}(j) P_{jk}$, or

  → transition probability matrix $\underline{p}_{t+1} = \mathbf{P}^T \underline{p}_t$

- We can determine how $\underline{p}_t$ changes in n steps using the
  → n-step transition probabilities $P_{X_{t+n}}(k) = \sum_{j=1}^{K} P_{X_t}(j) P_{jk}(n)$

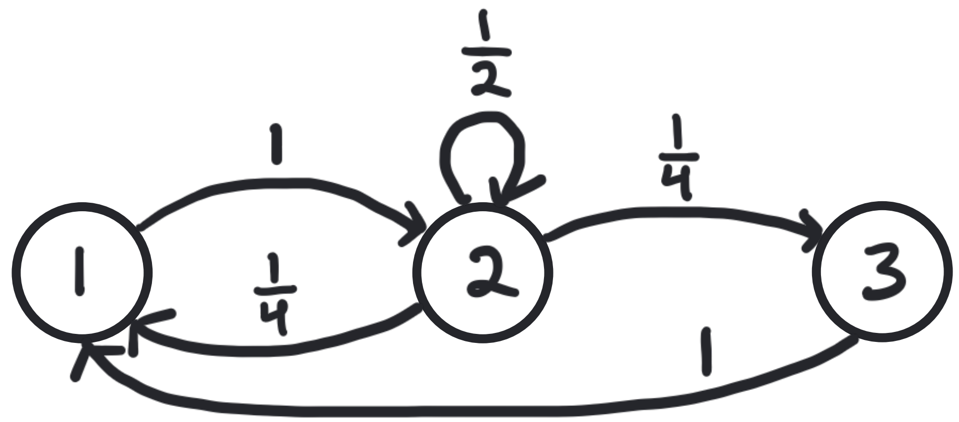
  → n-step transition probability matrix $\underline{p}_{t+n} = (\mathbf{P}(n))^T \underline{p}_t$

- <u>Example</u>: Say $\underline{p}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, determine $\underline{p}_1, \underline{p}_2$.



$$\underline{p}_1 = \mathbf{P}^T \underline{p}_0 = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} \qquad \underline{p}_2 = \mathbf{P}^T \underline{p}_1 = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{13}{25} \\ \frac{12}{25} \end{bmatrix}$$

• **Example:** $\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 \end{bmatrix} \qquad p_0 = \begin{bmatrix} \frac{1}{3} \\ \boxed{\frac{2}{3}} \\ 0 \end{bmatrix}$



→ Calculate p_2.

$$p_1 = P^T p_0 = \begin{bmatrix} 0 & \frac{1}{4} & 1 \\ 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{1}{6} \end{bmatrix}$$

$$p_2 = P^T p_1 = \begin{bmatrix} 0 & \frac{1}{4} & 1 \\ 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \boxed{\frac{1}{6}} \end{bmatrix}$$

→ Calculate $\mathbb{P}[X_0 = 2 \mid X_2 = 3] = \dfrac{\mathbb{P}[X_2 = 3 \mid X_0 = 2]\ \boxed{\mathbb{P}[X_0 = 2]}}{\boxed{\mathbb{P}[X_2 = 3]}}$

Bayes' Rule

$$\mathbb{P}[X_2 = 3 \mid X_0 = 2] = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \qquad = \frac{\frac{1}{8} \cdot \frac{2}{3}}{\frac{1}{6}} = \frac{1}{2}$$