# Steady-State Behavior

- Recall that a homogeneous, finite-state, discrete-time Markov chain with range $R_X = \{1, \dots, K\}$ is fully specified by its initial distribution $P_{X_0}(x_0)$ and transition probabilities $P_{jk}$ for $j, k \in \{1, \dots, K\}$.

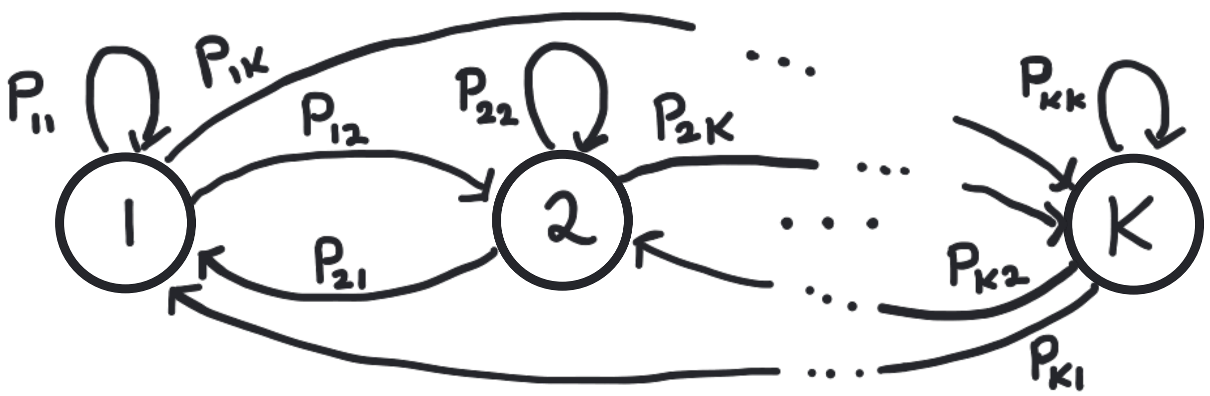
  Probability of going to state $k$ from state $j$.

- The state probability vector at time $t$ is $\underline{p}_t = \begin{bmatrix} P_{X_t}(1) \\ \vdots \\ P_{X_t}(K) \end{bmatrix}$ and evolves according to $\underline{p}_{t+1} = \mathbf{P}^T \underline{p}_t$ where

  $\mathbf{P} = \begin{bmatrix} P_{11} & \cdots & P_{1K} \\ \vdots & \ddots & \vdots \\ P_{K1} & \cdots & P_{KK} \end{bmatrix}$ is the transition probability matrix.

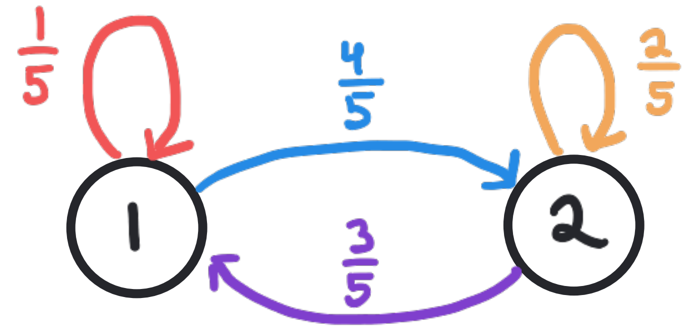
  Row index $j$ = current state. Column index $k$ = next state.

- How does $\underline{p}_t$ behave in the long run? Does it settle into a limit $\lim_{t \to \infty} \underline{p}_t$? Does it oscillate forever? The answer depends on the structure of the Markov chain.




This illustration assumes that all $P_{jk} > 0$. If $P_{jk} = 0$, we do not draw an arc between $j$ and $k$.

• <u>Example:</u> Recall the following Markov chain.

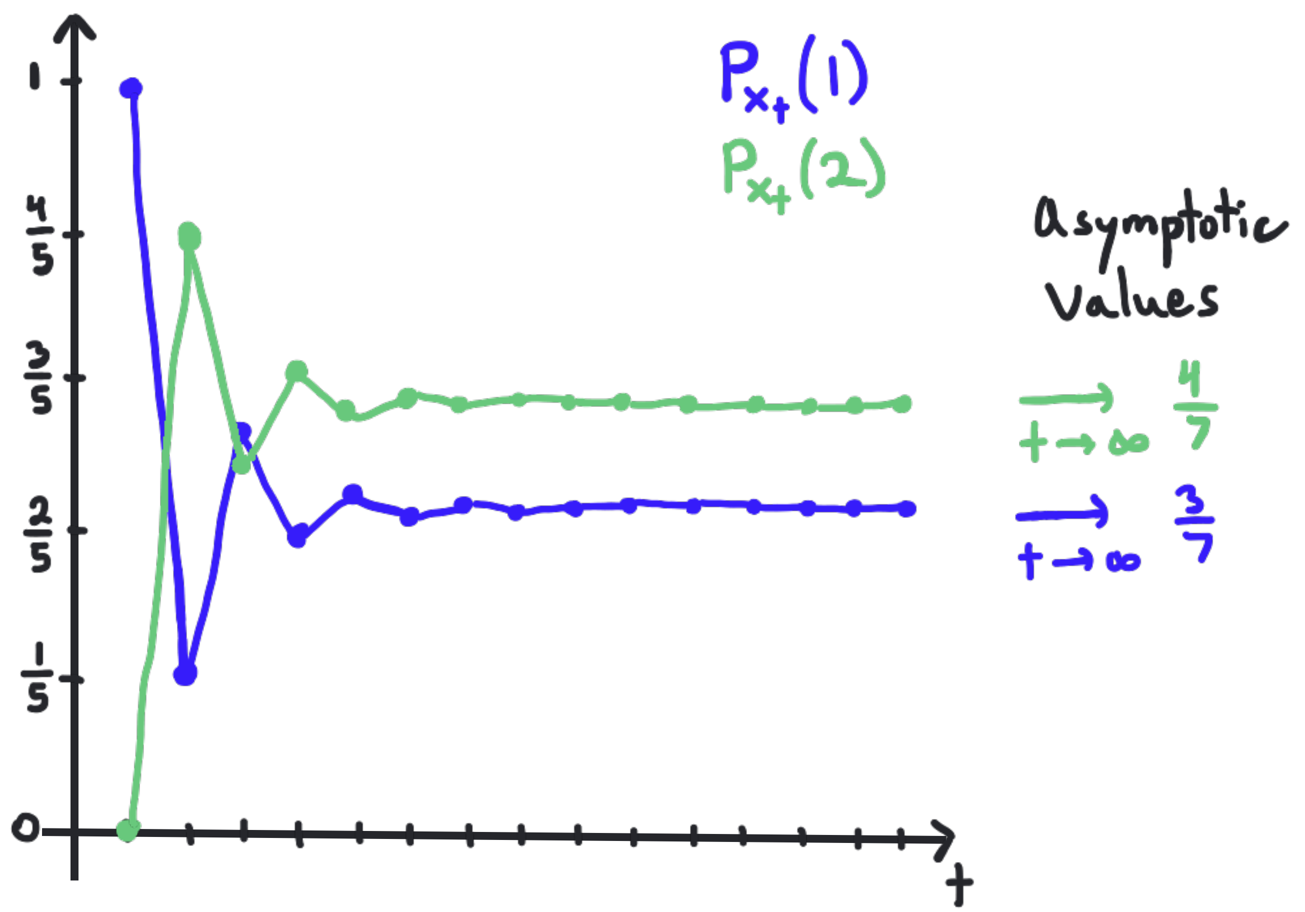



$$P = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

We previously showed that for $p_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have

$$p_1 = P^T p_0 = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} \qquad p_2 = P^T p_1 = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{13}{25} \\ \frac{12}{25} \end{bmatrix}$$

→ Using the update equation $p_{t+1} = P^T p_t$, we will plot $p_t = \begin{bmatrix} P_{x_t}(1) \\ P_{x_t}(2) \end{bmatrix}$ for $t = 1, \dots, 15$.




→ In the long run, the state 1 is occupied $\frac{3}{7}$ of the time and the state 2 is occupied $\frac{4}{7}$ of the time (regardless of how $p_0$ is chosen).

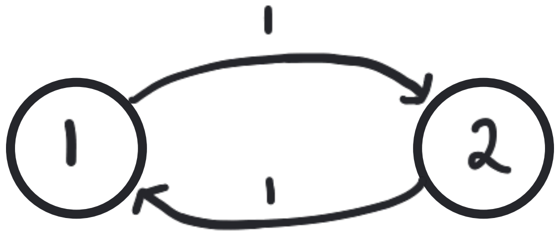
→ Do all Markov chains exhibit this limiting behavior?

→ For those that do, how do we determine the limit $\lim\limits_{t \to \infty} p_t$?

- Formally, we are interested in determining the limiting probability state vector $\underline{\pi} = \lim_{t\to\infty} \underline{p}_t$.
  - $\rightarrow$ $\underline{\pi} = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_K \end{bmatrix}$ where $\pi_j = \lim_{t\to\infty} P_{X_t}(j)$ and, by Normalization, $\sum_{j=1}^{K} \pi_j = 1$.
  - $\rightarrow$ But first, we need to be sure the limit exists!

- <u>Problematic Case 1</u>: If there are periodic oscillations, the Markov chain may never converge.

$\rightarrow$ <u>Ex</u>:




$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

* Notice that if $X_0 = 1$, then $X_i = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$.

The resulting sequence of $\underline{p}_t$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ldots$ No limit!

* Same issue for any initial distribution.

For $\underline{p}_0 = \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}$, we have $\underline{p}_1 = \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix}$, $\underline{p}_2 = \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}, \ldots$ No limit!

- <u>Problematic Case 2:</u> Some states are unreachable from other states.

→ <u>Ex</u>:

$$\begin{tikzcd} \textcircled{1} \arrow[loop above, "1"] & \textcircled{2} \arrow[l, bend right, "\frac{1}{2}"'] \arrow[r, bend left, "\frac{1}{2}"] & \textcircled{3} \arrow[loop above, "1"] \end{tikzcd}$$

Initial State = 2

$$p_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

* With probability $\frac{1}{2}$, we jump from $X_0 = 2$ to $X_1 = 1$.
  In this case, $p_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for $i \geq 1$.

* With probability $\frac{1}{2}$, we jump from $X_0 = 2$ to $X_1 = 3$.
  In this case, $p_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for $i \geq 1$.

* There is no unique limiting probability state vector.

- To avoid all problematic cases, we need a systematic way to classify Markov chains.

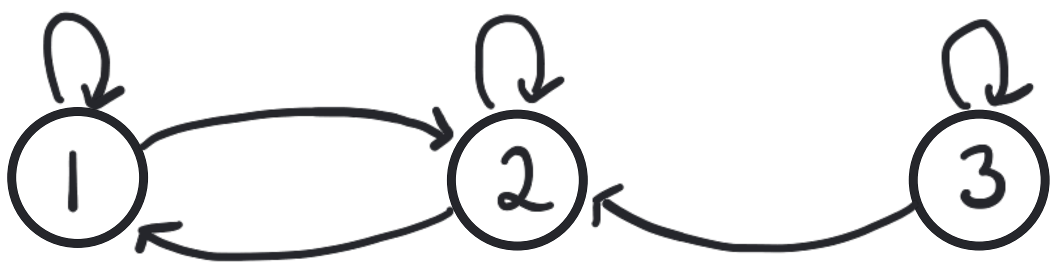
- **State Classification:**

→ State $k$ is accessible from state $j$, $j \to k$, if it is possible to reach state $k$ starting from state $j$ in one or more time steps, $P_{jk}(n) > 0$ for some $n \geq 0$.

* $P_{jk}(0)$ is defined to be the probability of going from state $j$ to state $k$ in $0$ time steps.

  Thus, $P_{jk}(0) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ and we always have $j \to j$.

* Ex:




For state classification, the actual $P_{jk}$ values do not matter, just that they are positive.

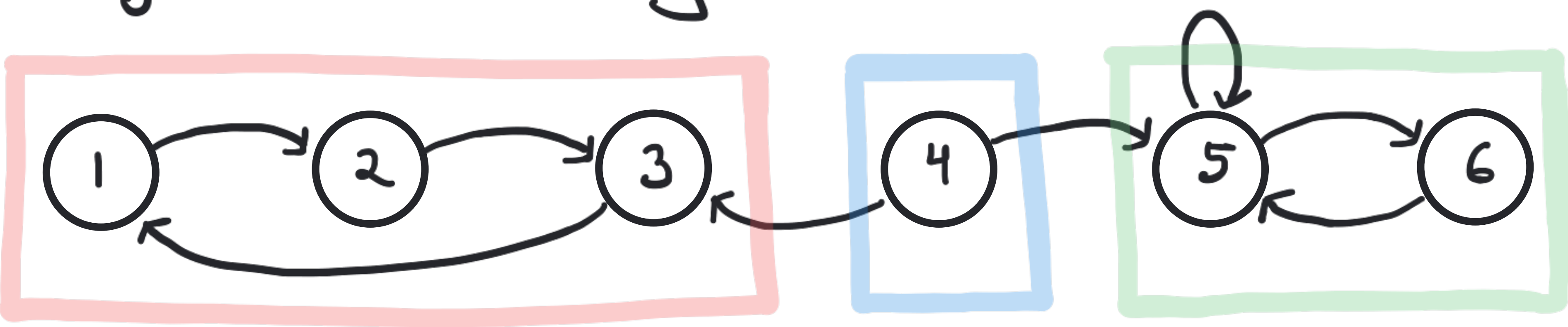
Which states are accessible from each other?

| | | |
|---|---|---|
| $1 \to 1$ | $1 \to 2$ | |
| $2 \to 2$ | $2 \to 1$ | |
| $3 \to 3$ | $3 \to 2$ | $3 \to 1$ |

By default

Not possible to reach 3 from 1 or 2.

- **State Classification:**
  - → States $j$ and $k$ communicate $j \leftrightarrow k$ if $j \to k$ and $k \to j$.
    - * Since $j \to j$ by default, we also have $j \leftrightarrow j$ by default.
    - * Intuitively, $j \leftrightarrow k$ means that it is possible to go back and forth between $j$ and $k$ (maybe with many steps).
  - → A communicating class $C$ is a subset of the states, $C \subset \{1, \dots, K\}$, such that all states that belong to $C$ communicate with each other. That is, if $j \in C$, then $k \in C$ if and only if $j \leftrightarrow k$.
    - * A finite-state Markov chain can always be partitioned into disjoint communicating classes.
    - * Ex:

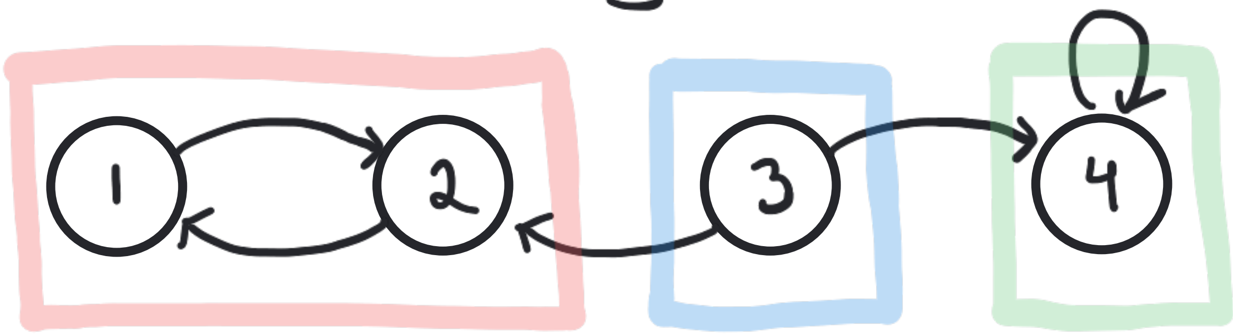



Communicating Classes: $C_1 = \{1, 2, 3\}$ $C_2 = \{4\}$ $C_3 = \{5, 6\}$

- **State Classification:**

→ We say that a Markov chain is irreducible if it only has one communicating class.

* Ex:




Communicating Classes:

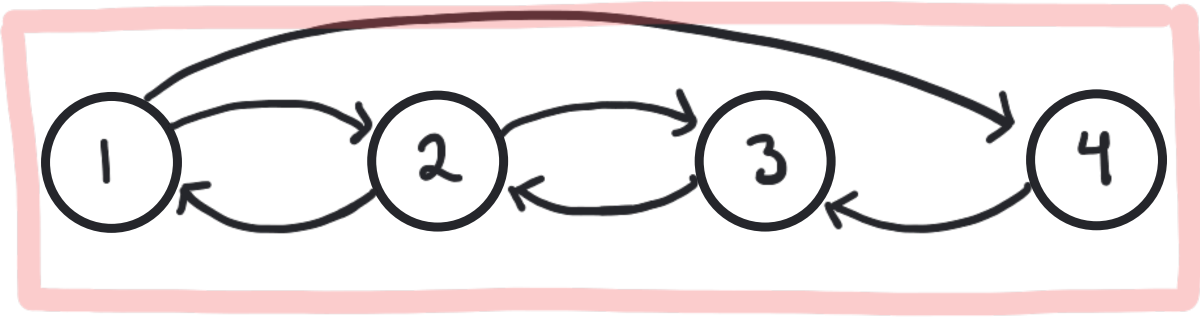
$C_1 = \{1, 2\}$ recurrent

$C_2 = \{3\}$ transient

$C_3 = \{4\}$ recurrent

Not Irreducible

* Ex:




Communicating Classes:

$C_1 = \{1, 2, 3, 4\}$ recurrent

Irreducible

→ The communicating class $C$ is transient if there are states $j \in C$, $k \notin C$ such that $j \to k$ and $k \not\to j$.

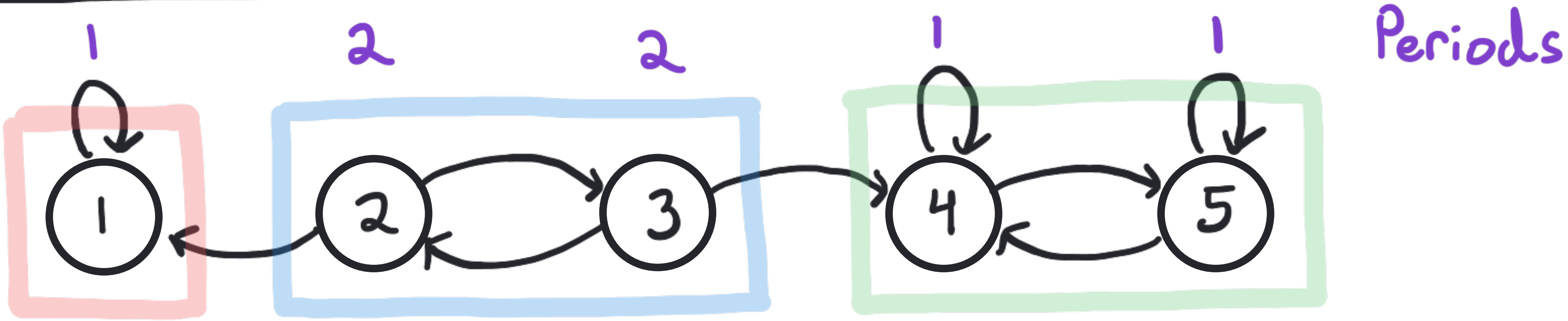
* Intuition: There is a path to leave $C$ from which there is no return.

* If $C$ is not transient, we say $C$ is recurrent.

* At least one communicating class is recurrent.

- **State Classification:**

→ Ex:




Communicating Classes: $C_1 = \{1\}$ $C_2 = \{2,3\}$ $C_3 = \{4,5\}$

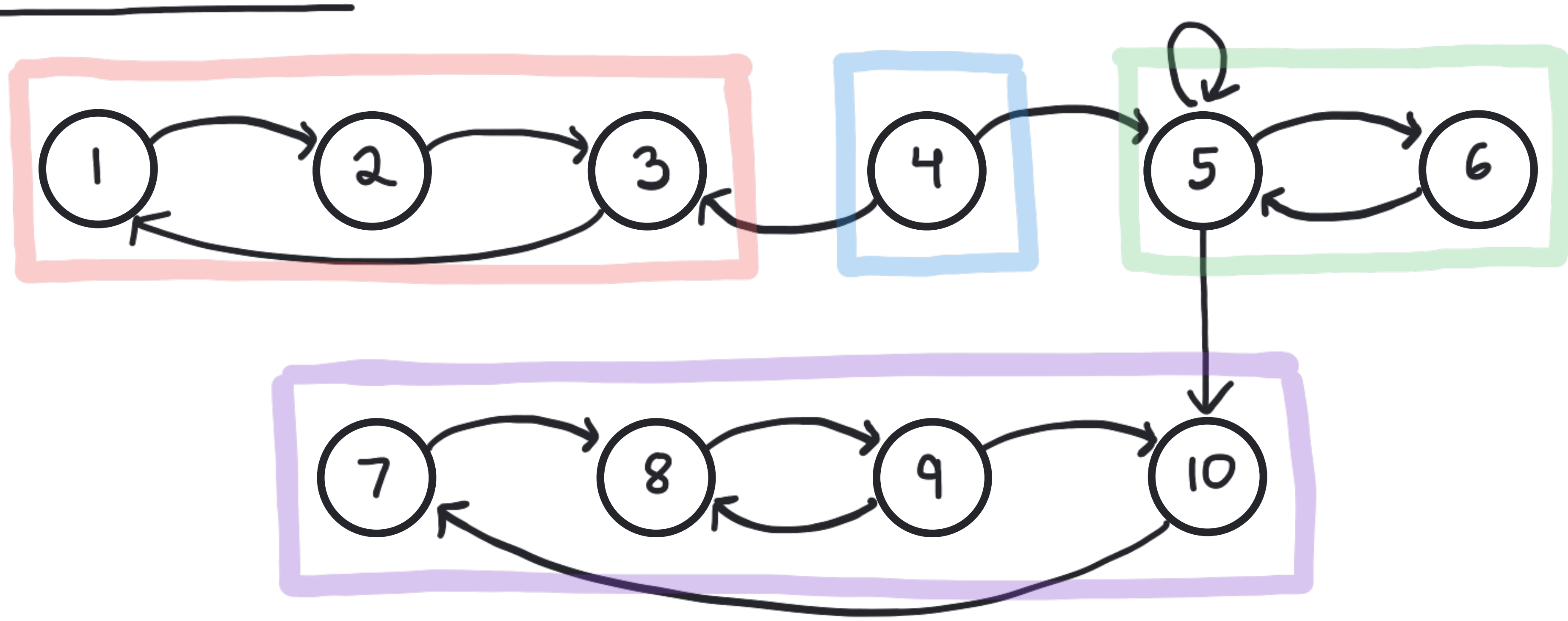
$C_1$ and $C_3$ are recurrent. $C_2$ is transient ($2 \to 1$ and $1 \not\to 2$)

→ The period of a state $j$ is the greatest common divisor of the lengths of all cycles from $j$ back to itself.

* All states in a communicating class have the same period.
* We say a state is aperiodic if it has period 1.
* If there no cycles from a state back to itself, we set its period to 1 by default.
* Shortcut: If a communicating class contains a cycle of length 1, the class is aperiodic.
* A Markov chain is aperiodic if all states are aperiodic.

- State Classification:

→ Ex:




| | Communicating Classes | | | |
|---|---|---|---|---|
| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
| States | 1, 2, 3 | 4 | 5, 6 | 7, 8, 9, 10 |
| Cycle Lengths | 3 | No cycles | 1, 2, 3, ... | 2, 4, 6, ... |
| Period | $\gcd(3) = 3$ | 1 | $\gcd(1,2,\ldots) = 1$ | $\gcd(2,4,\ldots) = 2$ |
| Transient or Recurrent | Recurrent | Transient | Transient | Recurrent |