Unique Limiting State Vector

- Previously, we discussed how to classify the states of a Markov chain.

- If a finite-state, homogeneous, discrete-time Markov chain is irreducible and aperiodic, then it has a unique limiting probability state vector $\underline{\pi} = \lim_{t \to \infty} \underline{p}_t$.

→ Normalization: $\sum_{j=1}^{K} \pi_j = 1$

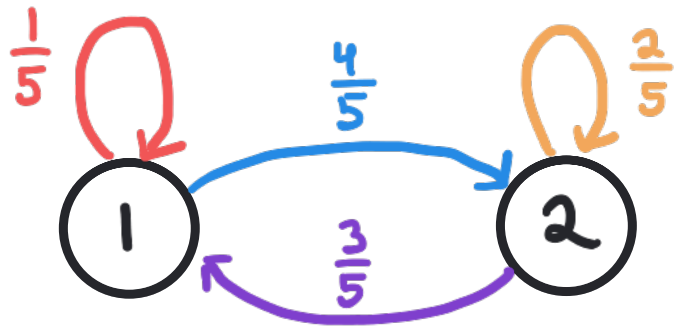
→ All states have positive probability: $\pi_j > 0$ for $j = 1, \ldots, K$.

→ Any initial probability state vector $\underline{p}_0$ will converge to $\underline{\pi}$.

→ $\underline{\pi}$ is an eigenvector of $\mathbf{P}^T$ with eigenvalue 1, $\mathbf{P}^T \underline{\pi} = \underline{\pi}$.

→ $\mathbf{P}^T \underline{\pi} = \underline{\pi}$ along with $\sum_{j=1}^{K} \pi_j = 1$ gives a system of K equations that we can solve for the K variables $\pi_1, \ldots, \pi_K$ (and avoid solving for the eigenvectors directly).

- **Example:**



$$P = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

→ Is there a unique limiting state probability vector $\underline{\pi} = \lim_{t \to \infty} \underline{p}_t$? If so, solve for it.

Communicating Classes: $C_1 = \{1, 2\}$

⇒ Since there is just one communicating class, the Markov chain is irreducible.

Period of $C_1 = 1$ since there is a cycle of length 1.

⇒ The Markov chain is aperiodic.

Since the chain is irreducible and aperiodic, there is a unique limit $\underline{\pi}$.

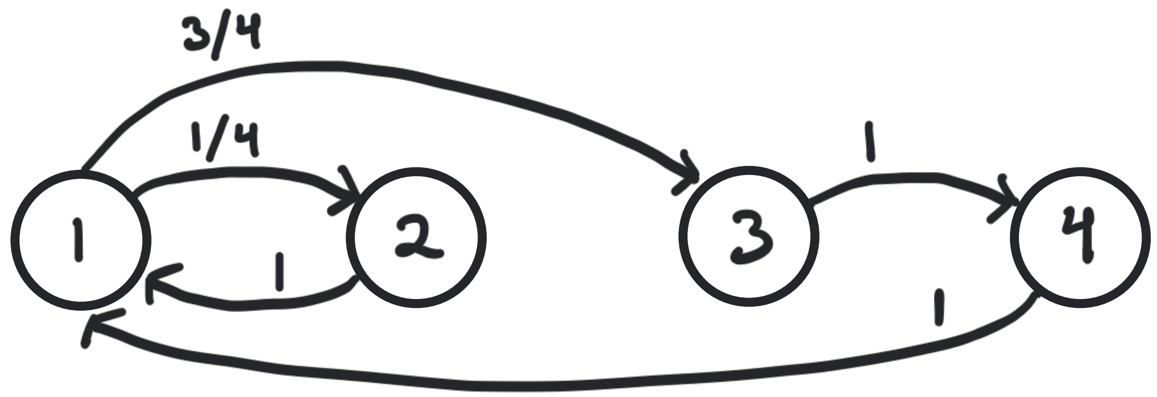
$$P^T \underline{\pi} = \underline{\pi} \Rightarrow \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \Rightarrow \begin{array}{l} \frac{1}{5}\pi_1 + \frac{3}{5}\pi_2 = \pi_1 \\ \frac{4}{5}\pi_1 + \frac{2}{5}\pi_2 = \pi_2 \end{array} \Rightarrow \frac{3}{5}\pi_2 = \frac{4}{5}\pi_1$$

$$\Downarrow$$

$$\boxed{\pi_1 = \frac{3}{4}\pi_2}$$

$$\sum_{j=1}^{K} \pi_j = 1 \Rightarrow \pi_1 + \pi_2 = 1 \Rightarrow \boxed{\frac{3}{4}\pi_2} + \pi_2 = 1 \Rightarrow \pi_2 = \frac{4}{7} \quad \pi_1 = \frac{3}{7}$$

• <u>Example</u>:



$$P = \begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

→ Is there a unique limiting state probability vector $\underline{\pi} = \lim_{t \to \infty} p_t$?
If so, solve for it.

Communicating Classes: $C_1 = \{1, 2, 3, 4\} \Rightarrow$ Irreducible

Period of C_1: $\gcd(2,3) = 1 \Rightarrow$ Aperiodic

⇒ The Markov chain is irreducible and aperiodic so $\underline{\pi}$ exists.

$$P^T \underline{\pi} = \underline{\pi} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ \frac{1}{4} & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$$

$$\Rightarrow \boxed{\tfrac{1}{4}\pi_1 = \pi_2} \Rightarrow \pi_2 = \tfrac{1}{11}$$

$$\Rightarrow \boxed{\tfrac{3}{4}\pi_1 = \pi_3} \Rightarrow \pi_3 = \tfrac{3}{11}$$

$$\Rightarrow \pi_3 = \pi_4 \Rightarrow \boxed{\tfrac{3}{4}\pi_1 = \pi_4} \Rightarrow \pi_4 = \tfrac{3}{11}$$

$$\sum_{j=1}^{K} \pi_j = 1 \Rightarrow \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

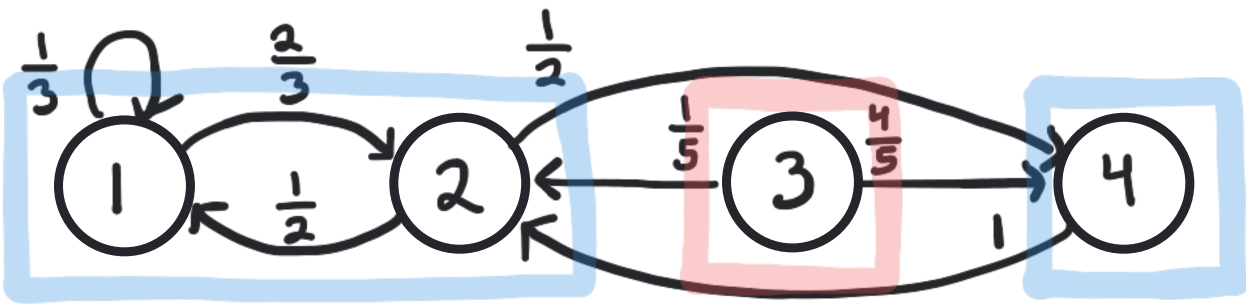
$$\Rightarrow \pi_1 + \boxed{\tfrac{1}{4}\pi_1} + \boxed{\tfrac{3}{4}\pi_1} + \boxed{\tfrac{3}{4}\pi_1} = 1$$

$$\Rightarrow \tfrac{11}{4}\pi_1 = 1 \Rightarrow \pi_1 = \tfrac{4}{11}$$

$$\underline{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 4/11 \\ 1/11 \\ 3/11 \\ 3/11 \end{bmatrix}$$

- We can also handle Markov chains with a single recurrent and aperiodic communicating class along with additional transient states.
  - → There is a unique limit $\underline{\pi} = \lim_{t \to \infty} p_t$.
  - → To solve for $\underline{\pi}$, first set $\pi_j = 0$ for all transient states, then solve for the remaining values as before.

- <u>Example</u>:



Communicating Classes:

$C_1 = \{1, 2, 4\}$
recurrent, aperiodic

$C_2 = \{3\}$
transient, aperiodic

  - → There is a unique limit $\underline{\pi} = \lim_{t \to \infty} p_t$.
  - → Set transient states to limiting probability 0, $\pi_3 = 0$.
  - → Solve for π_j for $j \in C_1$.

$$P^T \underline{\pi} = \underline{\pi} \Rightarrow \begin{bmatrix} 1/3 & 1/2 & 0 & 0 \\ 2/3 & 0 & 1/5 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 4/5 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ 0 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ 0 \\ \pi_4 \end{bmatrix} \Rightarrow \begin{aligned} \tfrac{1}{3}\pi_1 + \tfrac{1}{2}\pi_2 &= \pi_1 \Rightarrow \boxed{\tfrac{3}{4}\pi_2 = \pi_1} \Rightarrow \pi_1 = \tfrac{3}{9} \\ \tfrac{2}{3}\pi_1 + \pi_4 &= \pi_2 \\ \boxed{\tfrac{1}{2}\pi_2 = \pi_4} &\Rightarrow \pi_4 = \tfrac{2}{9} \end{aligned}$$

$$\sum_{j=1}^{K} \pi_j = 1 \Rightarrow \pi_1 + \pi_2 + \pi_4 = 1 \Rightarrow \boxed{\tfrac{3}{4}\pi_2} + \pi_2 + \boxed{\tfrac{1}{2}\pi_2} = 1 \Rightarrow \tfrac{9}{4}\pi_2 = 1 \Rightarrow \pi_2 = \tfrac{4}{9}$$