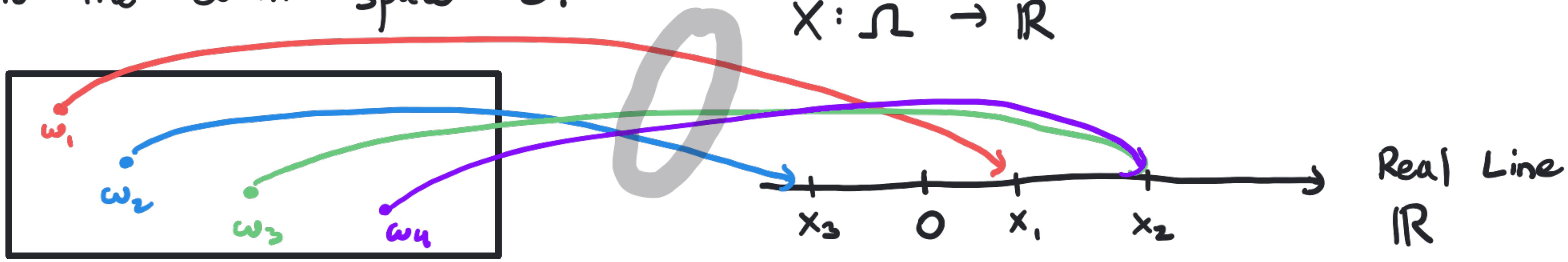
Discrete Random Variables

- For the rest of the course, we will focus on modeling scenarios using random numbers (rather than random cards, letters, etc.).
  - → Opens door to more powerful techniques.
  - → Will allow us to model dependencies between random numbers.

- A **random variable** X is a function that maps outcomes to real numbers, $X : \Omega \to \mathbb{R}$, such that for any interval (a,b), the set of outcomes $\{\omega \in \Omega : a < X(\omega) < b\}$ belongs to the event space $\mathcal{E}$.



- We usually denote random variables using uppercase letters such as X, Y, and Z.
- We usually denote the specific values a random variable can take by lowercase letters, such as x, y, and z.
- The range R_X is just the range of X.
  - → a random variable X is discrete if its range R_X is countable.

- Formally, the experiment is the source of randomness and X is a function of the outcome.
  - → However, it can help to build intuition by simply thinking of X as a random number.

- **Experiment 1:** Count # of photons that hit a CCD pixel in 1 ms.

  $\Omega = \{0, 1, 2, \ldots\}$

  $X(\omega) = \omega$ X is the same as the outcome.

  $R_X = \{0, 1, 2, \ldots\}$ X is a discrete random variable.

- **Experiment 2:** Measure exact time (in milliseconds) between 1st and 2nd photon arrival.

  $\Omega = [0, \infty)$

  $X(\omega) = \omega$ X is the same as the outcome.

  $R_X = [0, \infty)$ X is not a discrete random variable.

- **Experiment 3:** Roll two 4-sided dice.

$$\Omega = \{ \omega = (i,j) : i,j \in \{1,2,3,4\} \}$$

X is the sum of the rolls

$X(\omega) = i + j$ X is a function of the outcome.

$R_X = \{2, 3, 4, 5, 6, 7, 8\}$ X is a discrete random variable.

- **Experiment 4:** Same setup as above, including X.

$$Y = X^2$$

$$R_Y = \{4, 9, 16, 25, 36, 49, 64\}$$

Y is a function of the random variable X and itself a random variable. Y is a discrete random variable.

- <u>Experiment 5:</u> Ask a student about their letter grade in a class.

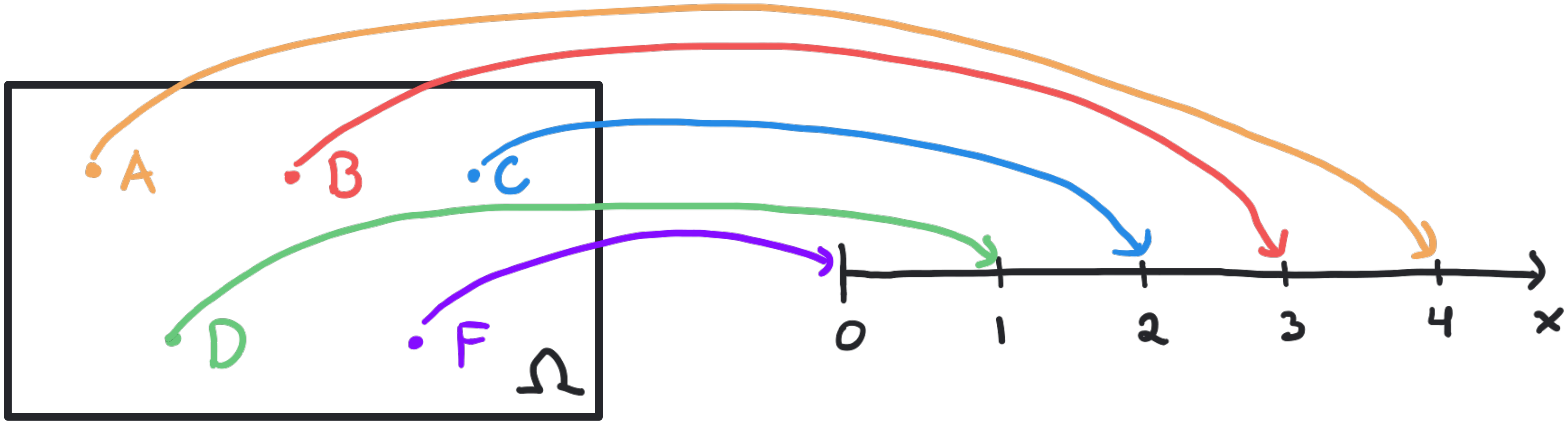
$$\Omega = \{A, B, C, D, F\}$$

$$X(\omega) = \begin{cases} 4 & \omega = A \\ 3 & \omega = B \\ 2 & \omega = C \\ 1 & \omega = D \\ 0 & \omega = F \end{cases}$$

X maps letters to real numbers, which is useful for computing things like grade point averages.

$$R_X = \{0, 1, 2, 3, 4\}$$

X is a discrete random variable.



Probability Mass Function (PMF)

- The probability mass function (PMF) $P_X(x)$ is a function whose input is a possible value $x \in R_X$ of the random variable X and whose output is the probability that X assumes this value.

$$\boxed{\begin{aligned} P_X(x) &= \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}] \\ &= \mathbb{P}[\{X = x\}] \\ &= \mathbb{P}[X = x] \end{aligned}}$$

shorthand notation

shorthand notation

→ We will start using the notation $\mathbb{P}[X = x]$ more frequently but it is important to remember that probabilities are assigned to event, which are subsets of Ω.

- **Back to Experiment 3:** Roll two 4-sided dice, all outcomes equally likely. X is the sum of the rolls.

$\Omega = \{\omega = (i,j) : i,j \in \{1,2,3,4\}\}$ $\quad X(\omega) = i+j$ $\quad R_X = \{2,3,4,5,6,7,8\}$

$P_X(2) = \mathbb{P}[\{(1,1)\}] = \frac{1}{16}$

$P_X(3) = \mathbb{P}[\{(1,2),(2,1)\}] = \frac{1}{8}$

$P_X(4) = \mathbb{P}[\{(1,3),(2,2),(3,1)\}] = \frac{3}{16}$

$P_X(5) = \mathbb{P}[\{(1,4),(2,3),(3,2),(4,1)\}] = \frac{1}{4}$

$P_X(6) = \mathbb{P}[\{(2,4),(3,3),(4,2)\}] = \frac{3}{16}$

$P_X(7) = \mathbb{P}[\{(3,4),(4,3)\}] = \frac{1}{8}$

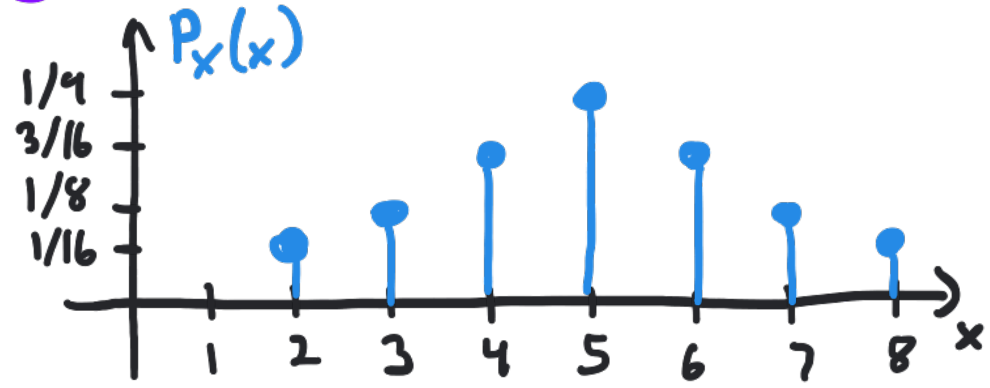
$P_X(8) = \mathbb{P}[\{(4,4)\}] = \frac{1}{16}$

Three Useful Ways to Write a PMF:

① Case-by-Case

$$P_X(x) = \begin{cases} \frac{1}{16} & x = 2, 8 \\ \frac{1}{8} & x = 3, 7 \\ \frac{3}{16} & x = 4, 6 \\ \frac{1}{4} & x = 5 \end{cases}$$

② Plot



③ Table

x	2	3	4	5	6	7	8
$P_X(x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

- <u>Basic PMF Properties:</u>

  → $P_X(x) \geq 0$ (Non-negativity)

  → $\sum_{x \in R_X} P_X(x) = 1$ (Normalization)

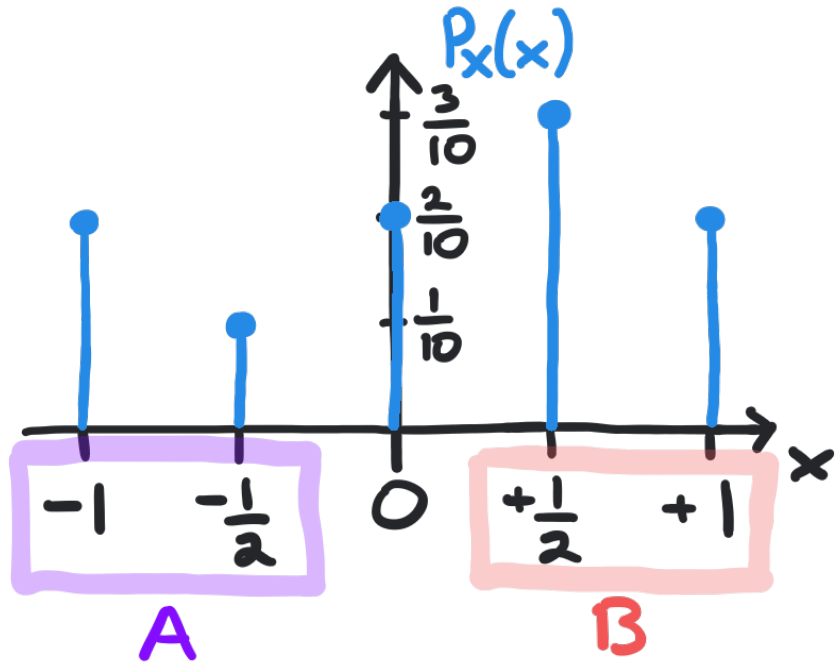
  → For any event $B \subset R_X$, the probability that X falls into B is $\mathbb{P}_X[B] = \mathbb{P}[\{X \in B\}] = \mathbb{P}[\{\omega \in \Omega : X(\omega) \in B\}]$

  $$= \sum_{x \in B} P_X(x) \quad \text{(additivity)}$$

- Notation: $\sum_{x \in B} P_X(x)$ means add up $P_X(x)$ for every x that belongs to the set B.

  → Example: $B = \{1, 5, 8\}$ $\sum_{x \in B} P_X(x) = P_X(1) + P_X(5) + P_X(8)$

- <u>Example</u>: $P_X(x) = \begin{cases} \frac{1}{10} & x = -\frac{1}{2} \\ \frac{2}{10} & x = -1, 0, +1 \\ \frac{3}{10} & x = +\frac{1}{2} \end{cases}$



$$\mathbb{P}[X = +\tfrac{1}{2}] = P_X(+\tfrac{1}{2}) = \frac{3}{10}$$

$$\mathbb{P}[X > 0] = \mathbb{P}[X \in B] = \sum_{x \in B} P_X(x) = P_X(+\tfrac{1}{2}) + P_X(+1) = \frac{3}{10} + \frac{2}{10} = \frac{1}{2}$$

$$B = \{ x \in R_X : x > 0 \}$$
$$= \{ +\tfrac{1}{2}, +1 \} \quad \text{since} \quad R_X = \{-1, -\tfrac{1}{2}, 0, +\tfrac{1}{2}, +1\}$$

- Any probability question can be translated into "membership in a set."

$$\mathbb{P}[\{X \text{ is negative}\}] = \mathbb{P}[X < 0] = \mathbb{P}[X \in A]$$
$$A = \{ x \in R_X : x < 0 \} = \{-1, -\tfrac{1}{2}\}$$