Cumulative Distribution Function (CDF)

- The cumulative distribution function (CDF) takes as an input a real number x and returns the probability that X is less than or equal to x.

$$\boxed{\begin{aligned} F_X(x) &= \mathbb{P}[\{\omega \in \Omega : X(\omega) \le x\}] \\ &= \mathbb{P}[\{X \le x\}] \quad \text{shorthand notation} \\ &= \mathbb{P}[X \le x] \quad \text{shorthand notation} \end{aligned}}$$

→ This concept is not very useful for discrete random variables.

→ However, it will later serve as a connection between discrete and continuous random variables.

- **Basic CDF Properties:**

  - $\rightarrow F_X(-\infty) = \lim_{x \to -\infty} F_X(x) = 0$

  - $\rightarrow F_X(\infty) = \lim_{x \to \infty} F_X(x) = 1$ (Normalization)

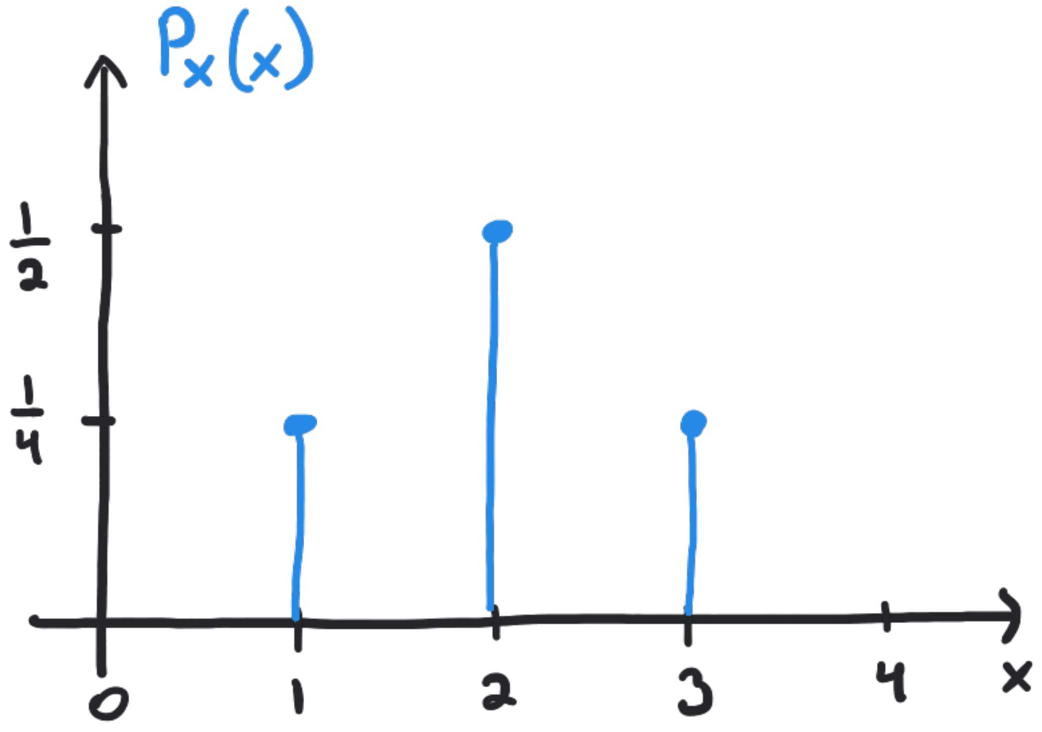
  - $\rightarrow$ $F_X(x)$ is a non-decreasing function of x. (Non-negativity)

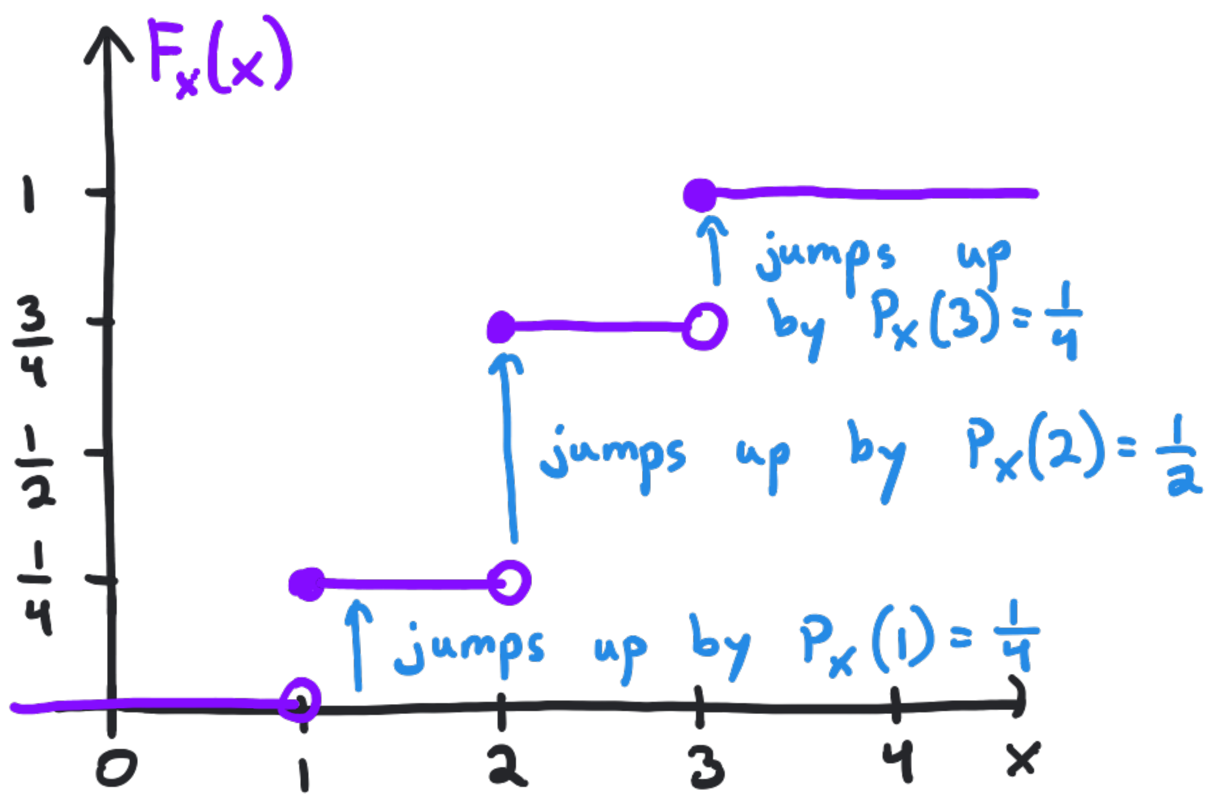
  - $\rightarrow$ For $a \leq b$, $F_X(b) - F_X(a) = \mathbb{P}[a < X \leq b]$.

  - $\rightarrow \lim_{\epsilon \downarrow 0} F_X(x + \epsilon) = F_X(x)$

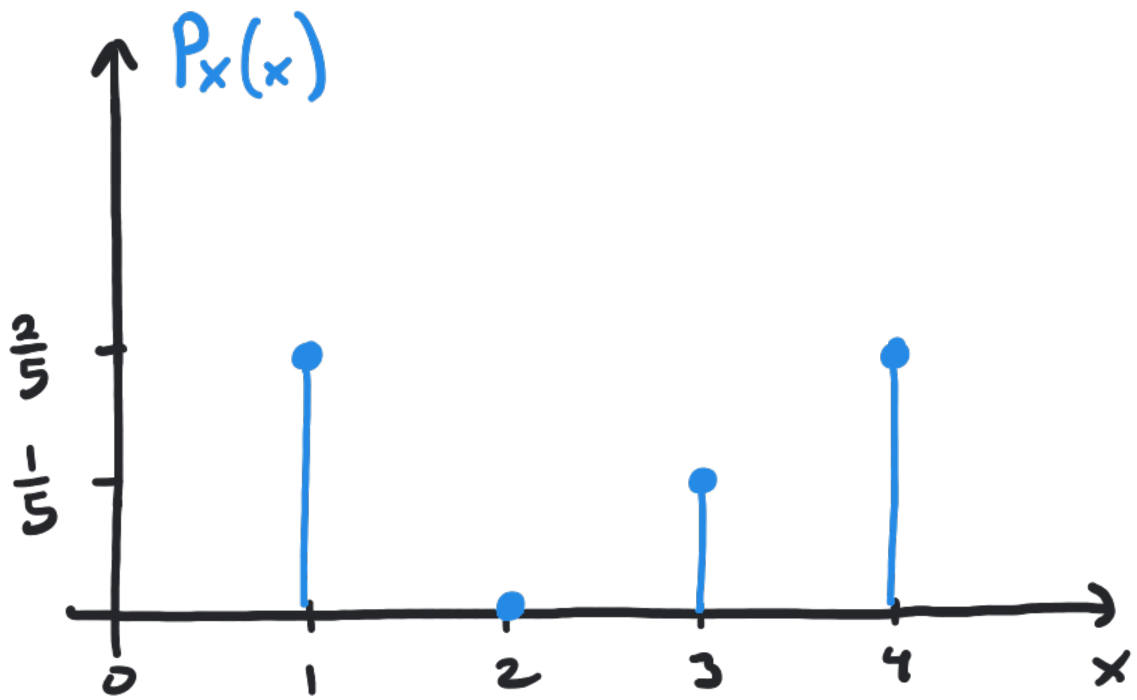
  - $\rightarrow$ For discrete random variables: $F_X(x)$ is a piecewise constant function that jumps up by $P_X(x)$ at each point x in the range R_X.

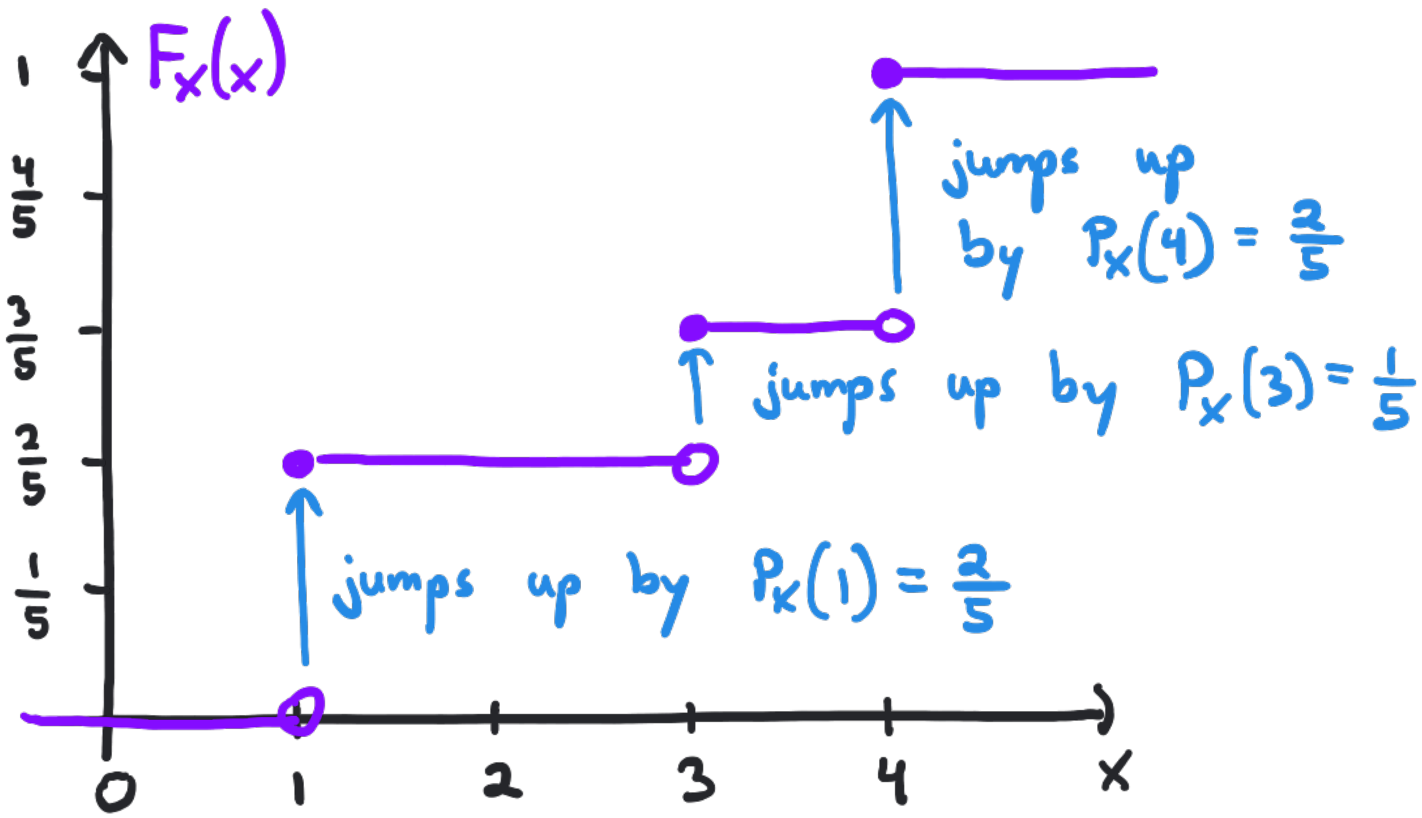
• **Example:**



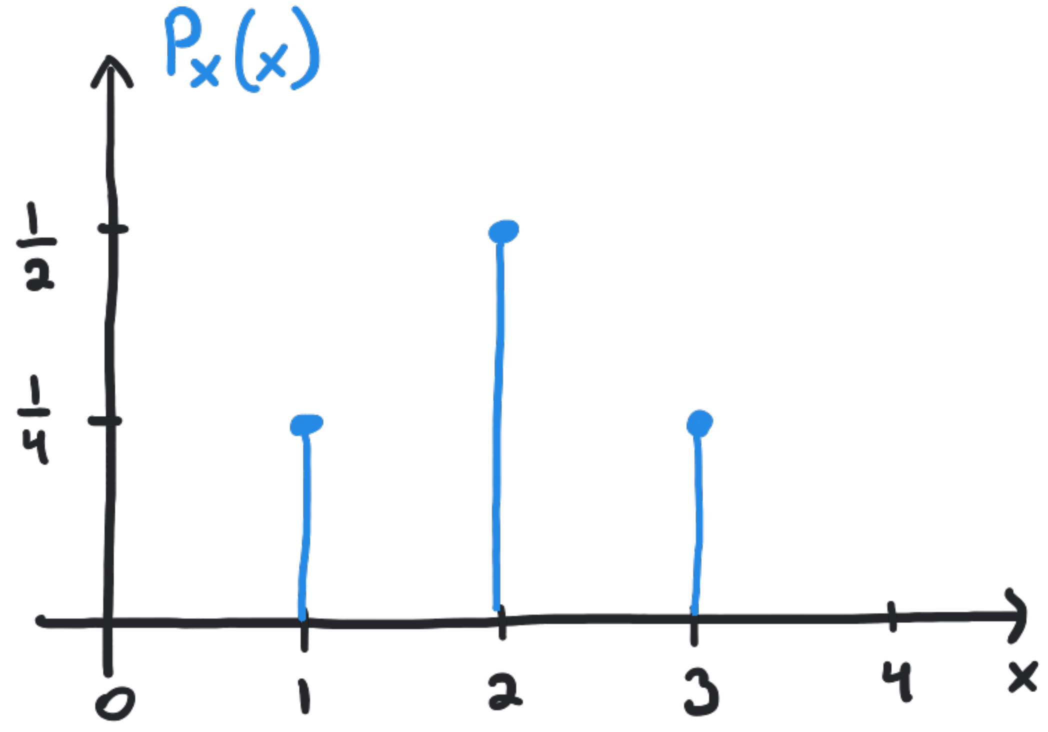


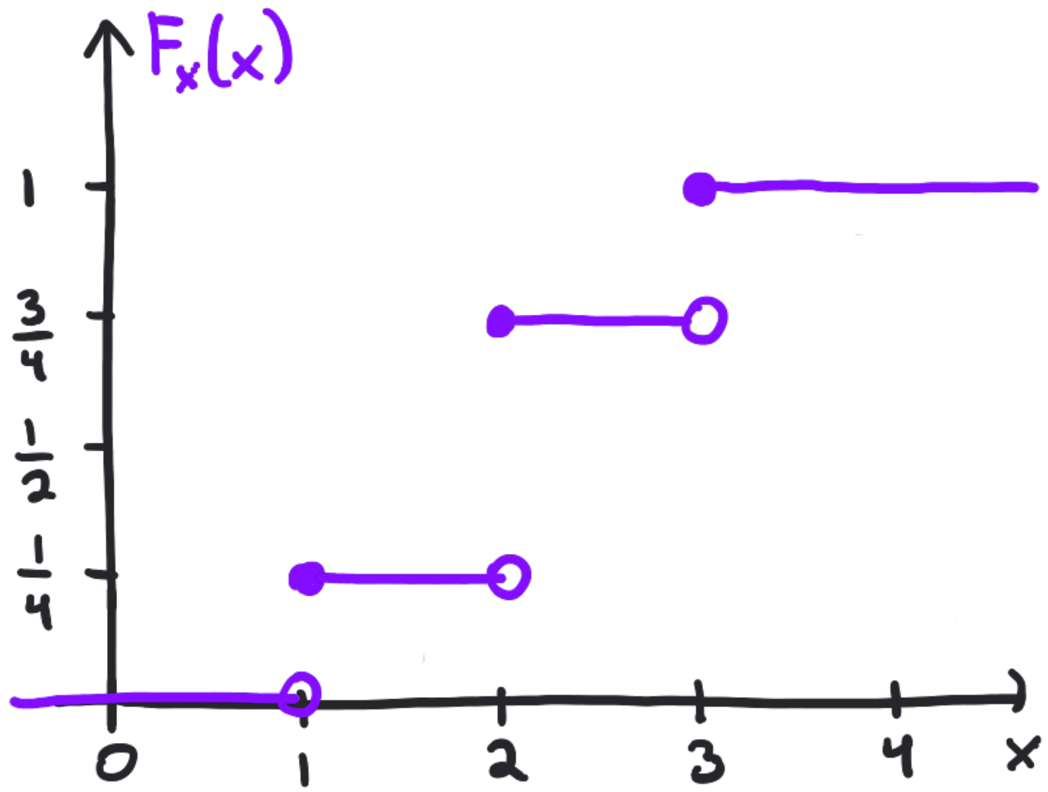
• **Example:**





- **Example:**





$$P_X(x) = \begin{cases} \frac{1}{4} & x = 1, 3 \\ \frac{1}{2} & x = 2 \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4} & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

What is the probability X is less than or equal to 2?

PMF: $\mathbb{P}[X \le 2] = P_X(1) + P_X(2) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

CDF: $\mathbb{P}[X \le 2] = F_X(2) = \frac{3}{4}$

What is the probability X is greater than 1 and less than or equal to 3?

PMF: $\mathbb{P}[1 < X \le 3] = P_X(2) + P_X(3) = \frac{1}{2} + \frac{1}{4}$

$= \frac{3}{4}$

CDF: $\mathbb{P}[1 < X \le 3] = F_X(3) - F_X(1) = 1 - \frac{1}{4}$

$= \frac{3}{4}$

Property from previous slide