Conditional Probability Models

- Recall that, for a general probability space $(\Omega, \mathcal{E}, \mathbb{P})$, we defined the conditional probability of event $A \in \mathcal{E}$ occurring given that event $B \in \mathcal{E}$ occurs as

sample space — event space — probability measure

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \qquad \text{(assuming } \mathbb{P}[B] > 0\text{)}$$

- Let X be a discrete random variable with PMF

$$P_X(x) = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}] = \mathbb{P}[X = x]$$

- The conditional PMF $P_{X|B}(x)$ of X given event $\{X \in B\}$ is

$$P_{X|B}(x) = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\} \mid \{X \in B\}]$$

$$\Rightarrow \boxed{P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[\{X \in B\}]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \dfrac{P_X(x)}{\sum_{x \in B} P_X(x)} & x \in B \\ 0 & x \notin B \end{cases}} \qquad \text{(assuming } \mathbb{P}[\{X \in B\}] > 0\text{)}$$

- Why?

$$P_{X|B}(x) = \mathbb{P}[\overbrace{\{\omega \in \Omega : X(\omega) = x\}}^{\text{event } A} \mid \overbrace{\{X \in B\}}^{\text{event } C}]$$

$$= \mathbb{P}[A \mid C]$$

$$= \frac{\mathbb{P}[A \cap C]}{\mathbb{P}[C]}$$

(assuming $\mathbb{P}[C] > 0$, otherwise undefined)

$$= \frac{\mathbb{P}[\{\omega \in \Omega : X(\omega) = x\} \cap \{X \in B\}]}{\mathbb{P}[\{X \in B\}]}$$

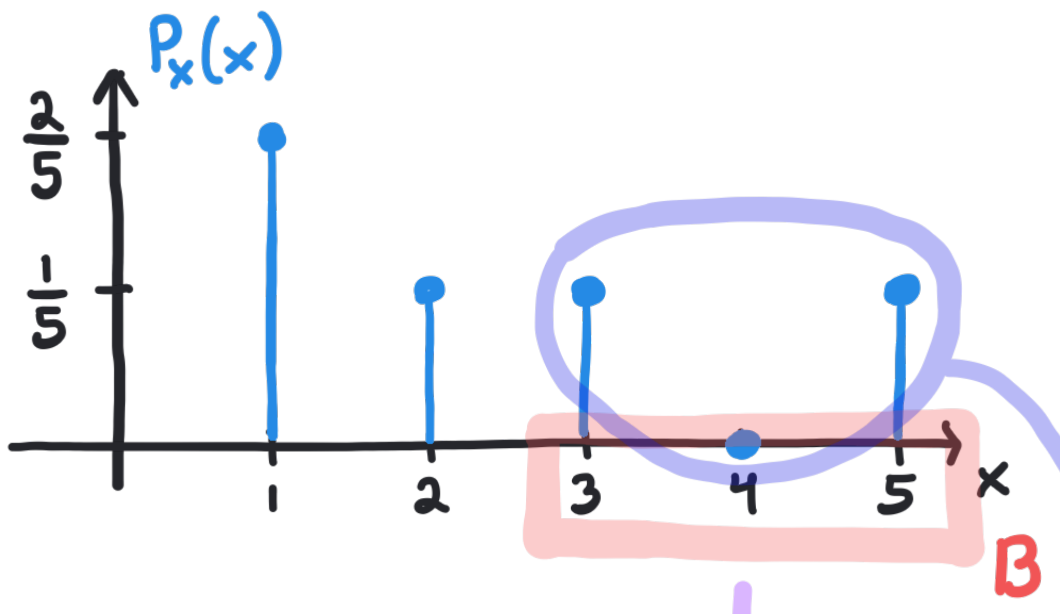
only outcomes in B occur

$$= \begin{cases} \dfrac{\mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}]}{\mathbb{P}[\{X \in B\}]} & x \in B \\ 0 & x \notin B \end{cases}$$

Definition of PMF

$$= \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[\{X \in B\}]} & x \in B \\ 0 & x \notin B \end{cases} \overset{\text{additivity}}{=} \begin{cases} \dfrac{P_X(x)}{\sum_{x \in B} P_X(x)} & x \in B \\ 0 & x \notin B \end{cases}$$

• **Example:**



$$P_X(x) = \begin{cases} \frac{2}{5} & x = 1 \\ \frac{1}{5} & x = 2, 3, 5 \end{cases}$$

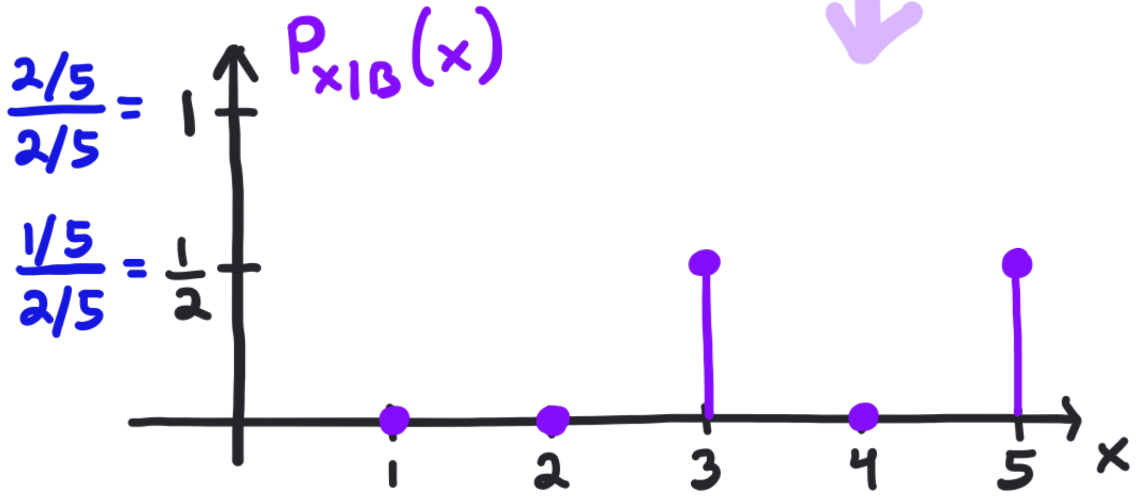
$$\mathbb{P}[\{X \in B\}] = \sum_{x \in B} P_X(x)$$
$$= P_X(3) + P_X(4) + P_X(5)$$
$$= \frac{1}{5} + 0 + \frac{1}{5}$$
$$= \frac{2}{5}$$

Let $B = \{3, 4, 5\}$.

Find $P_{X|B}(x)$.

Restrict to B

Rescale by dividing by $\mathbb{P}[\{X \in B\}]$



$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[\{X \in B\}]} & x \in B \\ 0 & x \notin B \end{cases}$$
$$= \begin{cases} \dfrac{1/5}{2/5} & x = 3, 5 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{1}{2} & x = 3, 5 \\ 0 & \text{otherwise} \end{cases}$$

→ **Intuition:** $P_{X|B}(x)$ is just $P_X(x)$ restricted to the values in B and rescaled so these remaining values sum to 1.

- <u>Conditional PMF Properties:</u>

  → $P_{X|B}(x) \geq 0$ (Non-negativity)

  → $\sum_{x \in B} P_{X|B}(x) = 1$ (Normalization)

  → For any event $C \subset R_X$, the conditional probability that X falls into C given that X falls into event B is

$$\mathbb{P}_{X|B}[C] = \mathbb{P}[\{X \in C\} | \{X \in B\}] = \sum_{x \in C} P_{X|B}(x)$$

(additivity)

- We can also develop versions of the multiplication rule, the law of total probability, and Bayes' rule. (See lecture notes for details.)

- Given that the event B occurs, what is the average value of X? We need to generalize our notion of expectation to allow for conditioning.

- The conditional expected value $\mathbb{E}[X \mid B]$ of X given event B is

$$\boxed{\mathbb{E}[X \mid B] = \sum_{x \in B} x \, P_{X \mid B}(x)}$$

- The conditional expected value $\mathbb{E}[g(X) \mid B]$ of a function $g(X)$ given event B is

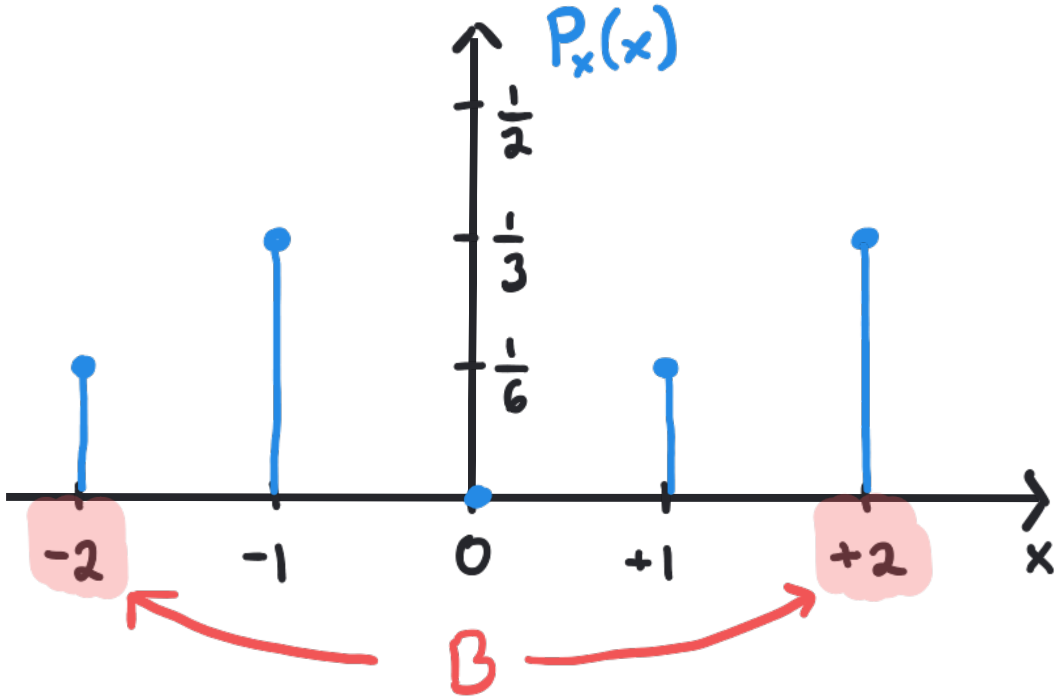
$$\boxed{\mathbb{E}[g(X) \mid B] = \sum_{x \in B} g(x) \, P_{X \mid B}(x)}$$

→ Linearity of Expectation: $\mathbb{E}[aX + b \mid B] = a\,\mathbb{E}[X \mid B] + b$

- The conditional variance $\mathrm{Var}[X \mid B]$ of X given event B is

$$\boxed{\mathrm{Var}[X \mid B] = \mathbb{E}\left[\left(X - \mathbb{E}[X \mid B]\right)^2 \mid B\right] = \mathbb{E}[X^2 \mid B] - \left(\mathbb{E}[X \mid B]\right)^2}$$

• <u>Example</u>:



$$P_X(x) = \begin{cases} \frac{1}{3} & x = -1, +2 \\ \frac{1}{6} & x = -2, +1 \\ 0 & \text{otherwise} \end{cases}$$

→ Condition on the event that $|X| > 1$.

$$\{|X| > 1\} = \{X \in B\}$$

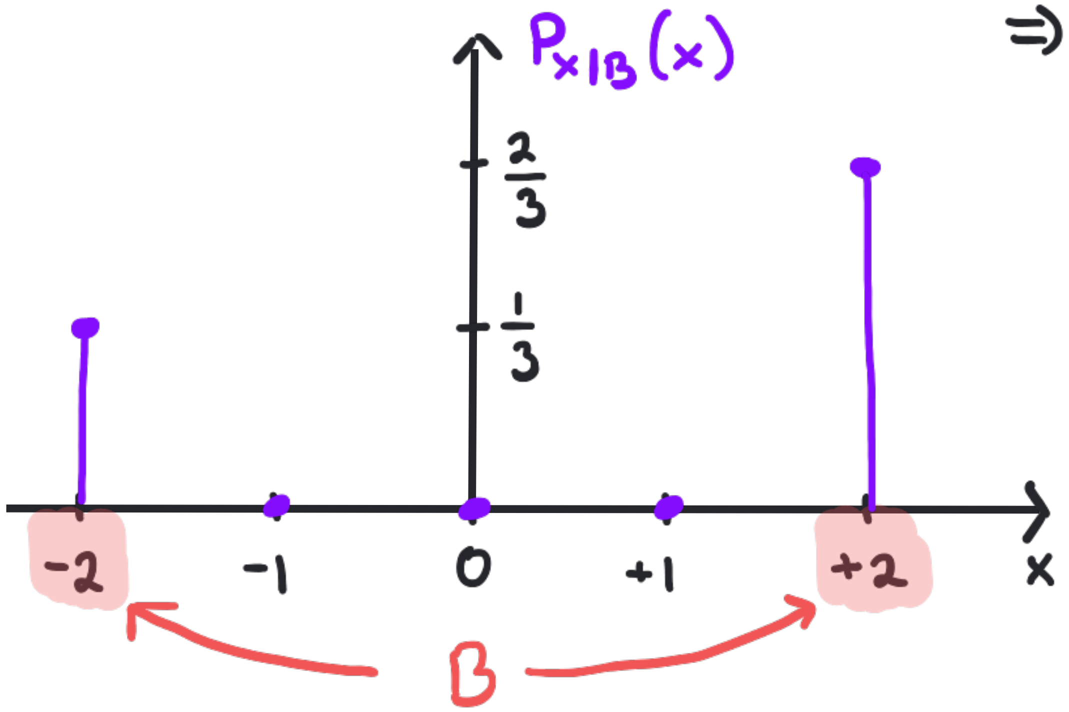
$$B = \{-2, +2\}$$

→ Determine the conditional PMF.

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[\{X \in B\}]} & x \in B \\ 0 & x \notin B \end{cases}$$

$$\mathbb{P}[\{X \in B\}] = \sum_{x \in B} P_X(x)$$

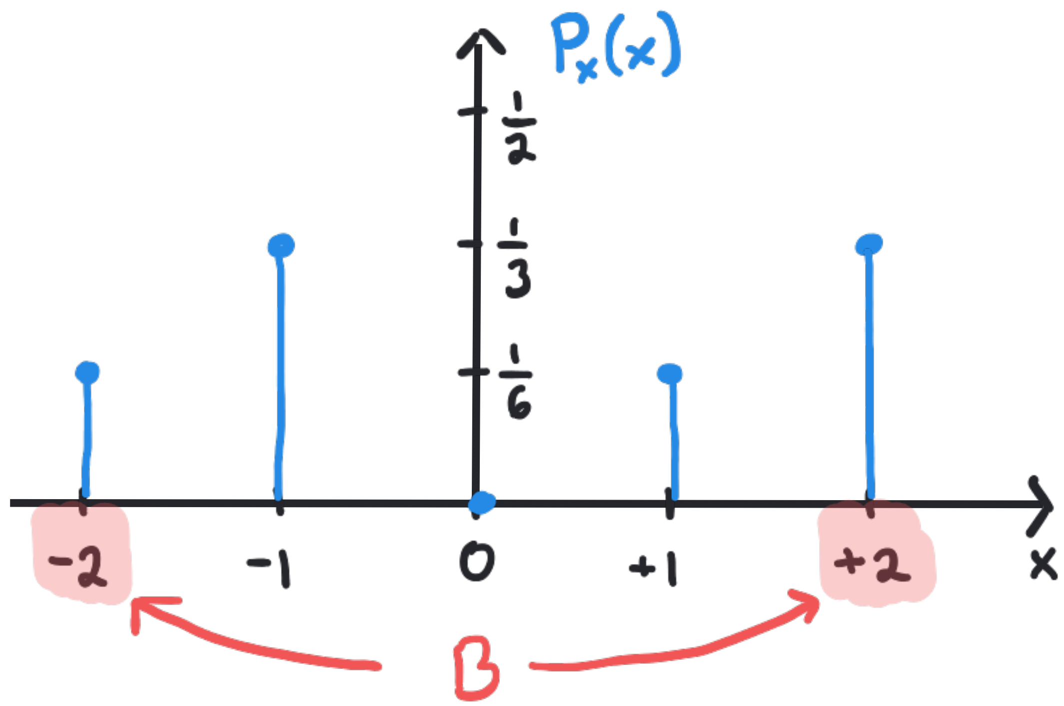
$$= P_X(-2) + P_X(+2) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$



$$\Rightarrow P_{X|B}(x) = \begin{cases} \frac{2}{3} & x = +2 \\ \frac{1}{3} & x = -2 \\ 0 & \text{otherwise} \end{cases}$$

PMF is restricted to B and rescaled so it sums to 1.

- **Example:**

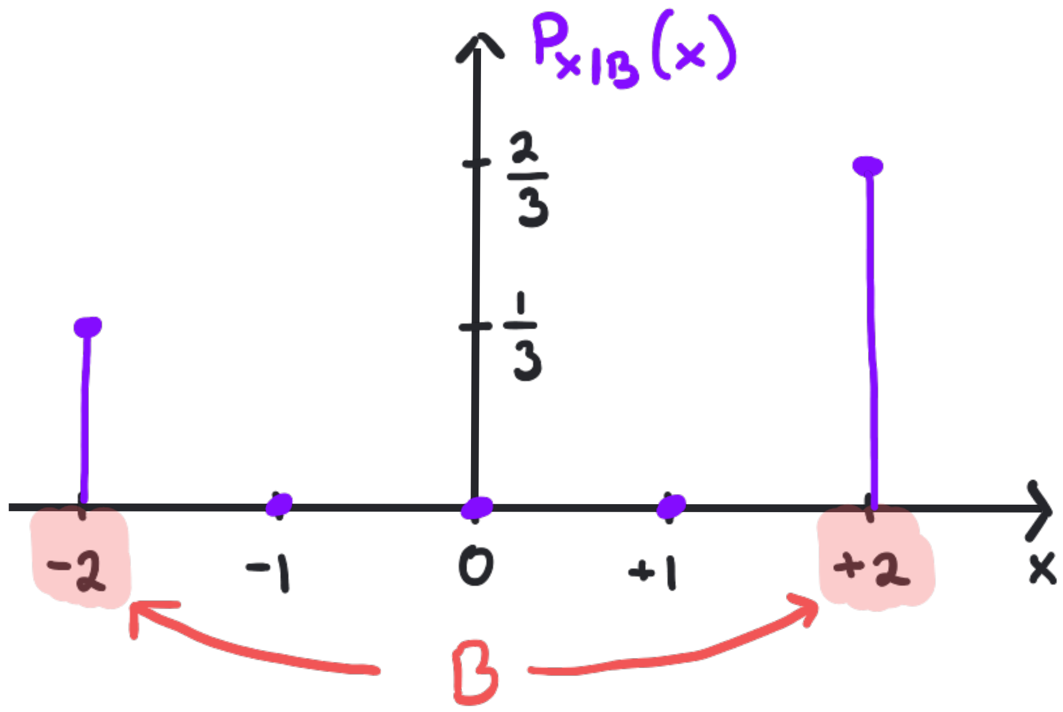


$$P_X(x) = \begin{cases} \frac{1}{3} & x = -1, +2 \\ \frac{1}{6} & x = -2, +1 \\ 0 & \text{otherwise} \end{cases}$$

→ Condition on the event that $|X| > 1$.

$$\{|X| > 1\} = \{X \in B\}$$

$$B = \{-2, +2\}$$



$$P_{X|B}(x) = \begin{cases} \frac{2}{3} & x = +2 \\ \frac{1}{3} & x = -2 \\ 0 & \text{otherwise} \end{cases}$$

→ Calculate $\mathbb{E}[X]$, $\mathbb{E}[X|B]$.

$$\mathbb{E}[X] = \sum_{x \in R_X} x\, P_X(x) = (-2)\cdot\frac{1}{6} + (-1)\cdot\frac{1}{3} + (+1)\cdot\frac{1}{6} + (+2)\cdot\frac{1}{3} = \frac{1}{6}$$

$$\mathbb{E}[X|B] = \sum_{x \in B} x\, P_{X|B}(x) = (-2)\cdot\frac{1}{3} + (+2)\cdot\frac{2}{3} = \frac{2}{3}$$

- **Example:** Wait for delivery of a missing part. Arrival time X is uniformly distributed over $\{1, 2, \dots, 20\}$ days.

→ Assume part has not yet arrived after 6 days. Determine the conditional PMF, expected value, and variance.

$$B = \{7, 8, \dots, 20\} \qquad \sum_{x \in B} P_X(x) = \sum_{x=7}^{20} P_X(x) = \sum_{x=7}^{20} \frac{1}{20} = \frac{14}{20}$$

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\sum_{x \in B} P_X(x)} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \dfrac{1/20}{14/20} = \dfrac{1}{14} & x = 7, 8, \dots, 20 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X \mid B] = \sum_{x \in B} x \, P_{X|B}(x) = \sum_{x=7}^{20} x \cdot \frac{1}{14} = \frac{7 + 20}{2} = \frac{27}{2}$$

→ Simplifying observation: $P_{X|B}(x)$ is PMF for Discrete Uniform(7, 20)

$$\Rightarrow \operatorname{Var}[X \mid B] = \frac{(20 - 7 + 2)(20 - 7)}{12} = \frac{15 \cdot 13}{12} = \frac{65}{4}$$

- **Example:** X is Binomial$(5, \frac{1}{3})$. Let $B = \{0, 1, 2\}$.
  Determine the conditional PMF of X given $\{X \in B\}$ and the conditional expected value.

$$P_X(x) = \begin{cases} \binom{5}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{5-x} & x = 0, 1, \ldots, 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{x \in B} P_X(x) = P_X(0) + P_X(1) + P_X(2)$$

$$= \binom{5}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^5 + \binom{5}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^4 + \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3$$

$$= \underbrace{\frac{32}{243}}_{P_X(0)} + \underbrace{\frac{80}{243}}_{P_X(1)} + \underbrace{\frac{80}{243}}_{P_X(2)} = \frac{192}{243}$$

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{\sum_{x \in B} P_X(x)} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \dfrac{32/243}{192/243} & x = 0 \\ \dfrac{80/243}{192/243} & x = 1, 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{6} & x = 0 \\ \frac{5}{12} & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X|B] = \sum_{x \in B} x\, P_{X|B}(x) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{5}{12} + 2 \cdot \frac{5}{12} = \frac{15}{12} = \frac{5}{4}$$