Expectation

- Recall that, for a discrete random variable, the expected value is $\mathbb{E}[X] = \sum_{x \in R_X} x P_X(x)$.

- The expected value $\mathbb{E}[X]$ of a continuous random variable X is

$$\boxed{\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx}$$

- The expected value of a function $\mathbb{E}[g(X)]$ of a continuous random variable X is

$$\boxed{\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx}$$

→ Note that this formula holds even if $g(X)$ is not a continuous random variable.

- **Linearity of Expectation:** For any constants a and b,

$$\boxed{\mathbb{E}[aX+b] = a\,\mathbb{E}[X] + b}$$

→ Why? $\mathbb{E}[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\,dx = a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{\text{Normalization}}$

$= a\,\mathbb{E}[X] + b \cdot 1$

- The variance and standard deviation are defined as before:

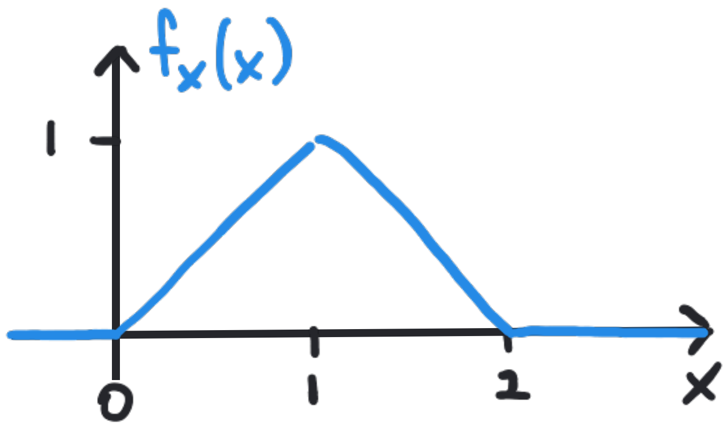
$$\boxed{\begin{aligned} \mathrm{Var}[X] &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \end{aligned}}$$

$$\boxed{\sigma_X = \sqrt{\mathrm{Var}[X]}}$$

- **Variance of a Linear Function:** For any constants a and b,

$$\boxed{\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X]}$$

- Example:



$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & 2 \leq x \end{cases}$$

→ Determine the CDF.

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$ integration variable

Evaluate case-by-case.

$x < 0$: $F_X(x) = \int_{-\infty}^{x} 0\,du = 0$

$0 \leq x < 1$: $F_X(x) = \int_{-\infty}^{0} 0\,du + \int_{0}^{x} u\,du = 0 + \left(\frac{1}{2}u^2\right)\Big|_0^x = \frac{1}{2}x^2$

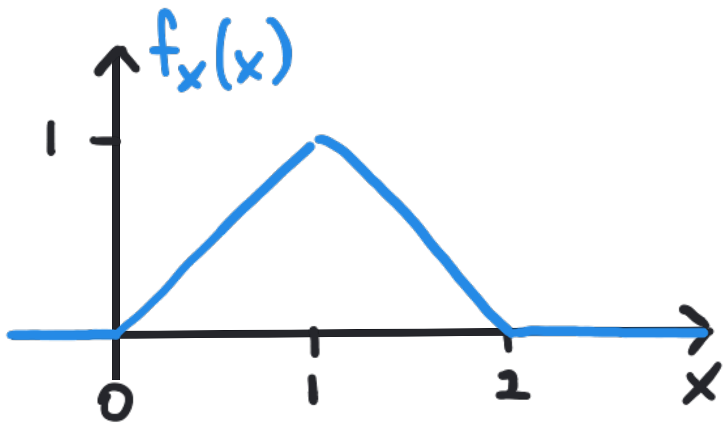
$1 \leq x < 2$: $F_X(x) = \int_{-\infty}^{0} 0\,du + \int_{0}^{1} u\,du + \int_{1}^{x} (2-u)\,du$

$$= 0 + \left(\frac{1}{2}u^2\right)\Big|_0^1 + \left(2u - \frac{1}{2}u^2\right)\Big|_1^x$$

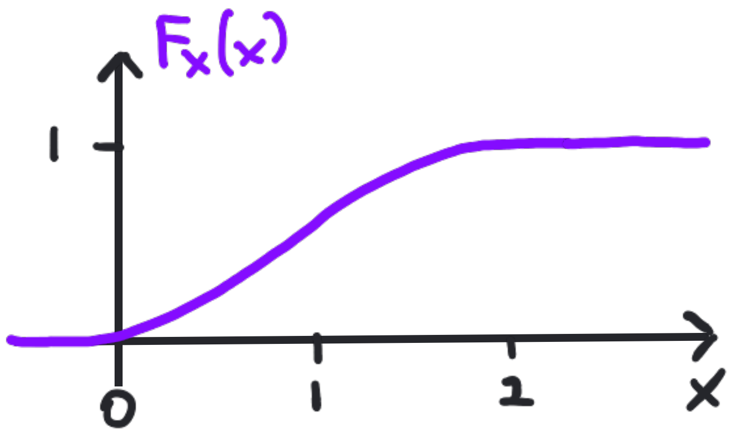
$$= 0 + \frac{1}{2} + 2x - \frac{1}{2}x^2 - 2 + \frac{1}{2} = -\frac{1}{2}x^2 + 2x - 1$$

$2 < x$: $F_X(x) = 1$ ← Since we've already integrated the non-zero values.

- **Example:**



$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 2 - x & 1 \le x < 2 \\ 0 & 2 \le x \end{cases}$$



$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & 0 \le x < 1 \\ -\frac{1}{2}x^2 + 2x - 1 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

→ Determine $\mathbb{P}\left[\left\{ \frac{3}{4} \le X \le \frac{5}{4} \right\}\right]$

Probability of an Interval

$$= F_X\left(\tfrac{5}{4}\right) - F_X\left(\tfrac{3}{4}\right) = -\frac{1}{2}\left(\frac{5}{4}\right)^2 + 2\cdot\frac{5}{4} - 1 - \frac{1}{2}\left(\frac{3}{4}\right)^2$$

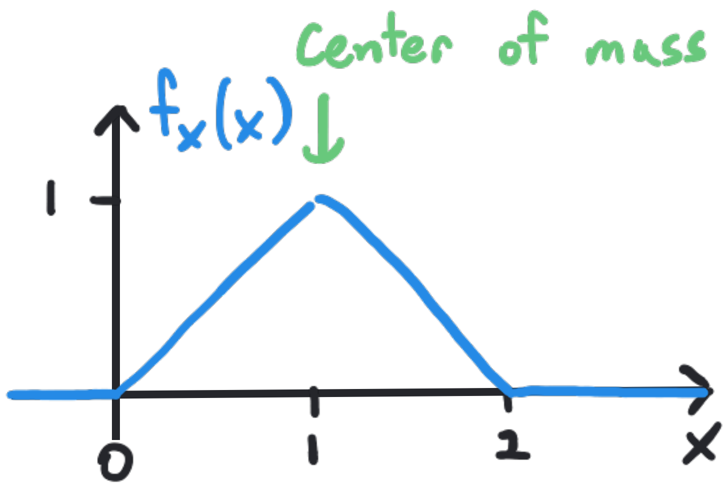
$$= -\frac{25}{32} + \frac{5}{2} - 1 - \frac{9}{32}$$

$$= \frac{3}{2} - \frac{34}{32}$$

$$= \frac{14}{32}$$

$$= \frac{7}{16}$$

- **Example:**



$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & 2 \le x \end{cases}$$

→ Determine $\mathbb{E}[X]$. Intuitively, seems like this should be 1 from the plot.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{0} x\cdot 0\,dx + \int_{0}^{1} x\cdot x\,dx + \int_{1}^{2} x\cdot(2-x)\,dx + \int_{2}^{\infty} x\cdot 0\,dx$$

$$= 0 + \left(\tfrac{1}{3}x^3\right)\Big|_0^1 + \left(x^2 - \tfrac{1}{3}x^3\right)\Big|_1^2 + 0$$

$$= \tfrac{1}{3} + 2^2 - \tfrac{1}{3}\cdot 2^3 - \left(1^2 - \tfrac{1}{3}\cdot 1^3\right) = 4 - 1 + \tfrac{1}{3} - \tfrac{8}{3} + \tfrac{1}{3} = 1$$

→ Determine $\mathbb{E}[X^2]$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \overset{0}{\cancel{\int_{-\infty}^{0} x^2\cdot 0\,dx}} + \int_{0}^{1} x^2\cdot x\,dx + \int_{1}^{2} x^2\cdot(2-x)\,dx + \overset{0}{\cancel{\int_{2}^{\infty} x^2\cdot 0\,dx}}$$

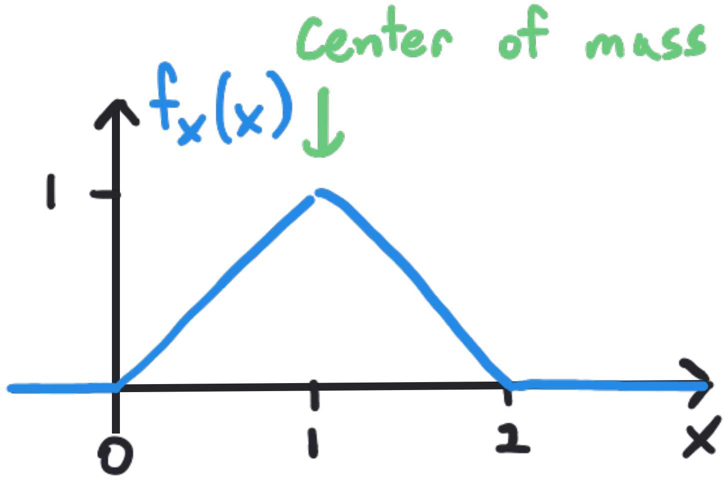
$$= \left(\tfrac{1}{4}x^4\right)\Big|_0^1 + \left(\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_1^2 = \tfrac{1}{4} + \left(\tfrac{16}{3} - 4\right) - \left(\tfrac{2}{3} - \tfrac{1}{4}\right)$$

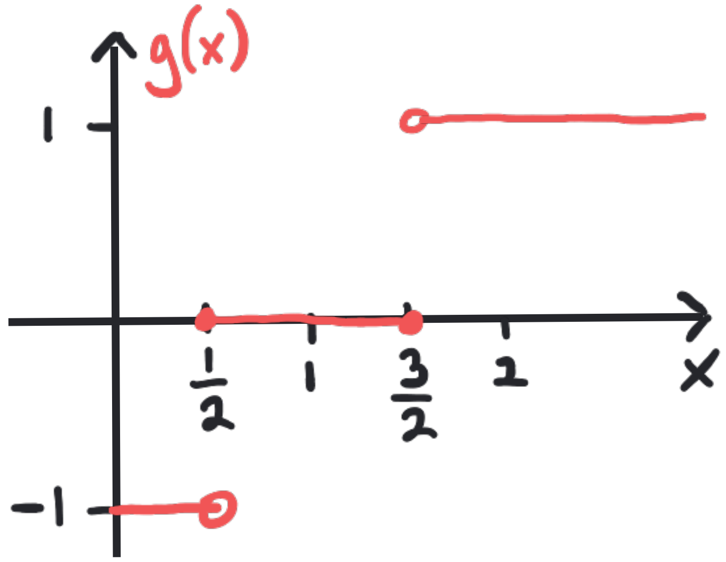
$$= \tfrac{1}{2} + \frac{16 - 12 - 2}{3} = \tfrac{7}{6}$$

→ Determine $\mathrm{Var}[X]$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2 = \tfrac{7}{6} - (1)^2 = \tfrac{1}{6}$$

- **Example:**





$$f_x(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & 2 \le x \end{cases}$$

$$g(x) = \begin{cases} -1 & x < \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le \frac{3}{2} \\ +1 & \frac{3}{2} < x \end{cases}$$

→ Determine $\mathbb{E}[g(X)]$.

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_x(x)\,dx = \cancelto{0}{\int_{-\infty}^{0} (-1)\cdot 0\,dx} + \int_{0}^{1/2} (-1)\cdot x\,dx + \cancelto{0}{\int_{1/2}^{1} 0\cdot x\,dx}$$

$$+ \cancelto{0}{\int_{1}^{3/2} 0\cdot(2-x)\,dx} + \int_{3/2}^{2} (+1)\cdot(2-x)\,dx + \cancelto{0}{\int_{2}^{\infty} (+1)\cdot 0\,dx}$$

$$= (-1)\left(\tfrac{1}{2}x^2\right)\Big|_0^{1/2} + (+1)\left(2x - \tfrac{1}{2}x^2\right)\Big|_{3/2}^{2} = -\frac{1}{8} + (4-2) - \left(3 - \frac{9}{8}\right) = 0$$