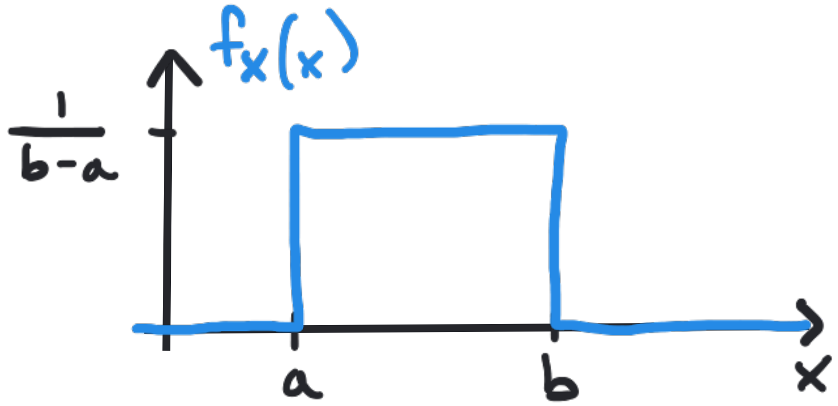
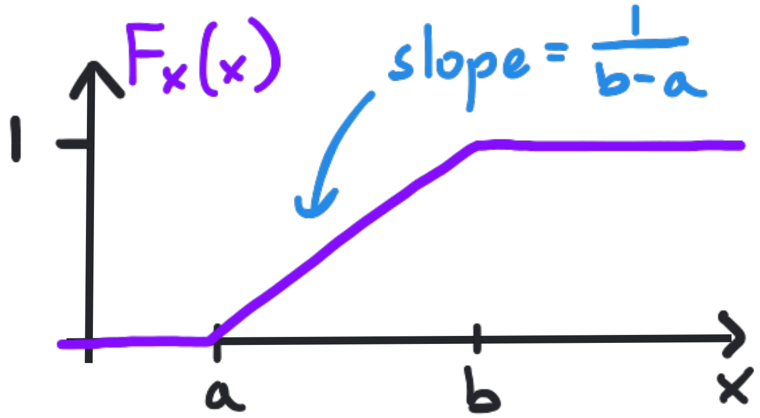
Important Families of Continuous Random Variables

- **Uniform:** X is a Uniform(a,b) random variable if it has PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



→ CDF: $F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & b < x \end{cases}$



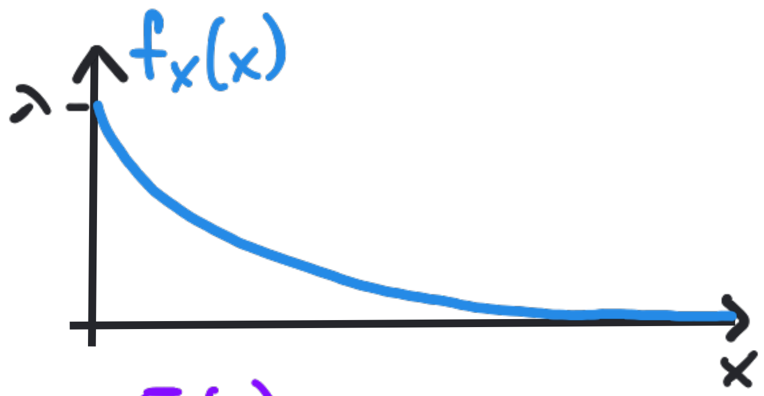
→ Mean: $\mathbb{E}[X] = \frac{a+b}{2}$ → Variance: $\mathrm{Var}[X] = \frac{(b-a)^2}{12}$

→ Interpretation: Equally likely to take any value between a and b.

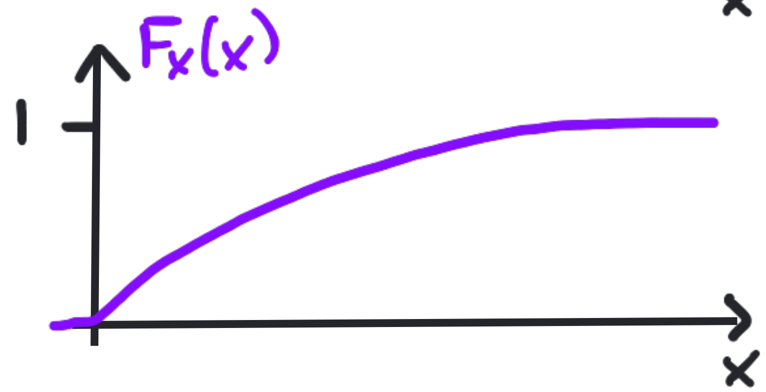
→ Application: Measurement noise / uncertainty in a bounded range.

- Exponential: X is an Exponential(λ) random variable if it has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



→ CDF: $F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$



→ Mean: $\mathbb{E}[X] = \frac{1}{\lambda}$

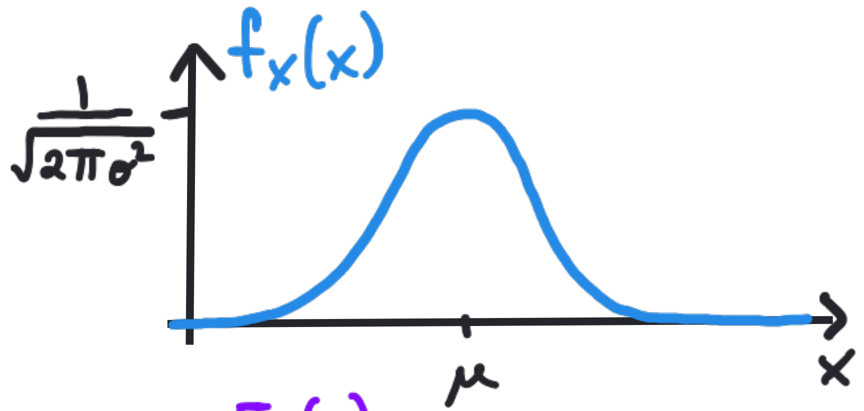
→ Variance: $\mathrm{Var}[X] = \frac{1}{\lambda^2}$

→ Interpretation: "Continuous version" of a geometric random variable.

→ Applications:
* Hard drive lifetimes
* Simple model of infectious period
* Time until component failure

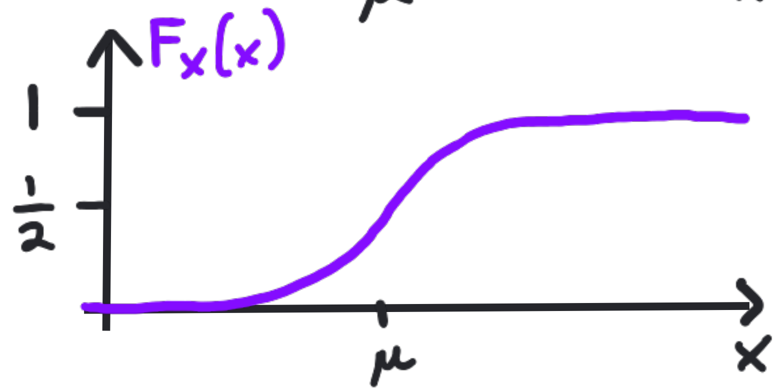
- **Gaussian:** X is a Gaussian(μ, σ^2) (or $\mathcal{N}(\mu, \sigma^2)$) random variable if it has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



standard normal CDF (defined on next page)

→ CDF: $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$



→ Mean: $\mathbb{E}[X] = \mu$ → Variance: $\text{Var}[X] = \sigma^2$

→ Interpretation: Sum or average of many small random quantities.

→ Applications:
- * Noise modeling
- * Linear systems
- * High-dimensional data

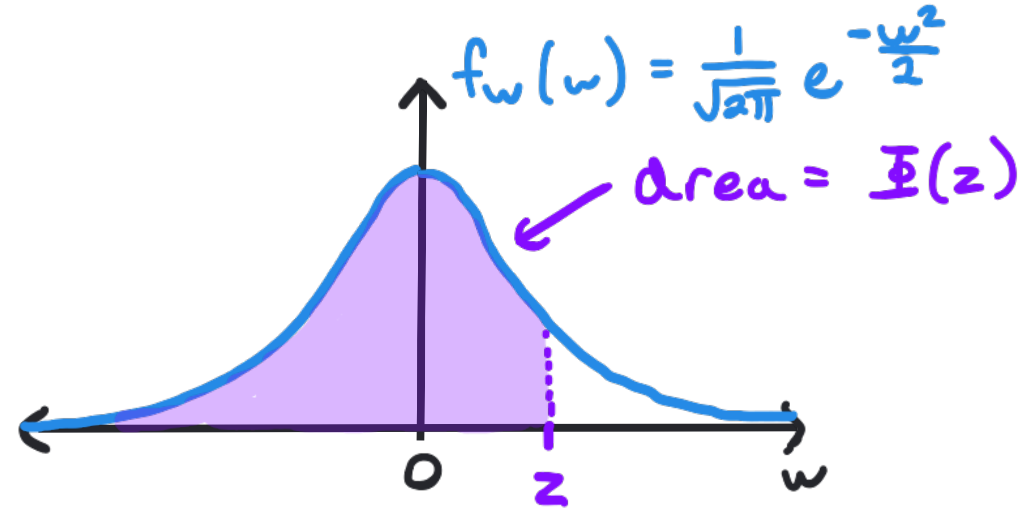
- The standard normal CDF $\Phi(z)$ is the CDF of a Gaussian$(0,1)$ random variable,

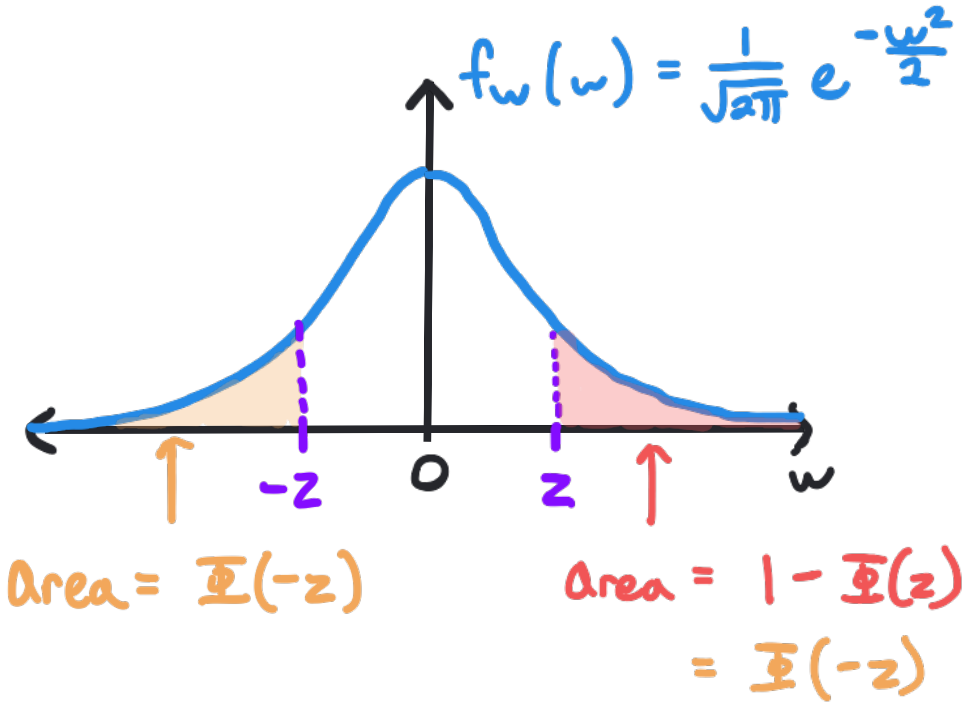
$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\, dw$$

  → Can evaluate using a lookup table, MATLAB, Wolfram alpha, etc.

  → $\Phi(0) = \frac{1}{2}$ by symmetry.

  → $\Phi(-z) = 1 - \Phi(z)$ by symmetry.





- For large values of z, it is usually easier to work with the standard normal complementary CDF $Q(z) = 1 - \Phi(z) = \Phi(-z)$

- Probability of an Interval: $\mathbb{P}[\{a \le X \le b\}] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

• **Example:** X is Gaussian$(2, 9)$. Calculate $\mathbb{P}[X > 5]$.

$$\text{Gaussian}(\underset{\mu}{2}, \underset{\sigma^2}{9})$$

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \text{ for a Gaussian.}$$

$$\mathbb{P}[X > 5] = 1 - \mathbb{P}[X \le 5]$$

$$= 1 - F_X(5)$$

$$= 1 - \Phi\left(\frac{5-2}{\sqrt{9}}\right) = 1 - \Phi\left(\frac{3}{3}\right) = 1 - \Phi(1) \underset{\text{lookup table}}{\approx} 1 - 0.8413$$

$$= 0.1587$$

→ Calculate $\mathbb{P}[X < 8 \mid X > 5] \overset{\text{shorthand for}}{=} \mathbb{P}[\underbrace{\{X < 8\}}_{\text{event } A} \mid \underbrace{\{X > 5\}}_{\text{event } B}] = \mathbb{P}[A \mid B] \underset{\text{Definition of Conditional Probability}}{=} \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$

$$= \frac{\mathbb{P}[5 < X < 8]}{\mathbb{P}[X > 5]}$$

$$= \frac{\Phi\left(\frac{8-2}{\sqrt{9}}\right) - \Phi\left(\frac{5-2}{\sqrt{9}}\right)}{\mathbb{P}[X > 5]}$$

$$= \frac{\Phi\left(\frac{6}{3}\right) - \Phi\left(\frac{3}{3}\right)}{\mathbb{P}[X > 5]}$$

work above

$$= \frac{\Phi(2) - \Phi(1)}{1 - \Phi(1)} \underset{\text{lookup table}}{\approx} \frac{0.9772 - 0.8413}{0.1587} = 0.8563$$

- <u>Important Property</u>:

$\boxed{\text{If } X \text{ is Gaussian}(\mu, \sigma^2) \text{ and } Y = aX + b \text{ is a linear function of } X, \text{ then } Y \text{ is Gaussian}(a\mu + b, a^2\sigma^2).}$

- In general, determining the PDF of a function $Y = g(X)$ of a continuous random variable X is quite involved. However, for a linear function of a Gaussian, we just need to update the mean and variance.

- <u>Example</u>: X is Gaussian$(-1, 3)$ and $Y = 2X - 1$.
  What kind of a random variable is Y?

Y is a linear function of a Gaussian so it is Gaussian.

$$\mathbb{E}[Y] = \mathbb{E}[2X - 1] = 2\,\mathbb{E}[X] - 1 = 2 \cdot (-1) - 1 = -3$$

$$\text{Var}[Y] = \text{Var}[2X - 1] = 2^2\,\text{Var}[X] = 4 \cdot 3 = 12$$

Y is Gaussian$(-3, 12)$.