Conditioning for Continuous Random Variables

- Recall that the conditional probability of event A given event B is $\mathbb{P}[A \mid B] = \dfrac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$ (assuming $\mathbb{P}[B] > 0$).

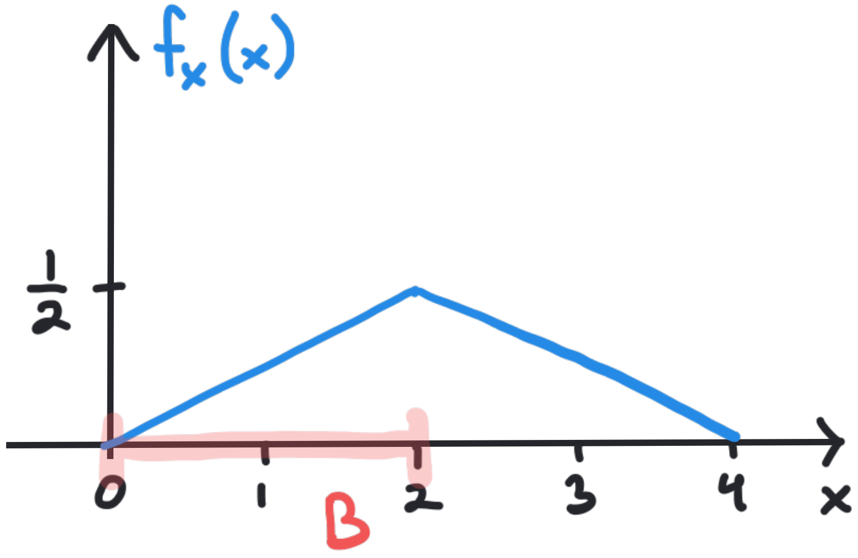
- Recall also that, for a continuous random variable X, we can calculate the probability that X lands in A via its PDF, $\mathbb{P}[\{X \in A\}] = \int_A f_X(x)\,dx$.

- Say we want to calculate the probability that X lands in A given that X lands in B. Can we define a conditional PDF so that $\mathbb{P}[\{X \in A\} \mid \{X \in B\}] = \int_A f_{X|B}(x)\,dx$? Yes!

- The conditional PDF $f_{X|B}(x)$ of X given the event $\{X \in B\}$ is

$$\boxed{f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[\{X \in B\}]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \dfrac{f_X(x)}{\int_B f_X(x)\,dx} & x \in B \\ 0 & x \notin B \end{cases}}$$

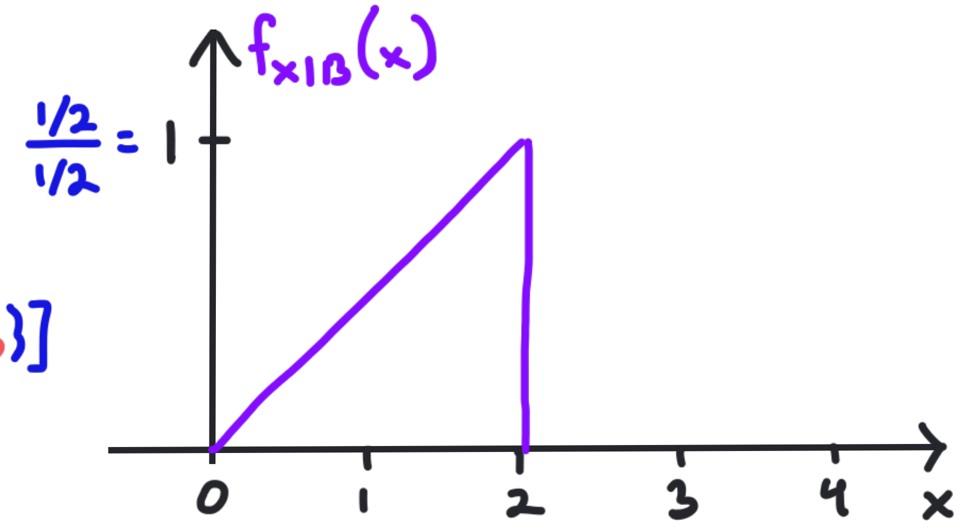
- **Example:**



Let $B = [0,2]$

What is $f_{X|B}(x)$?

Restrict to B



Rescale by dividing by $\mathbb{P}[\{X \in B\}]$

$$f_X(x) = \begin{cases} \frac{1}{4}x & 0 \le x \le 2 \\ 1 - \frac{1}{4}x & 2 < x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}[\{X \in B\}] = \mathbb{P}[0 \le X \le 2]$$
$$= \int_0^2 f_X(x)\,dx$$
$$= \int_0^2 \frac{1}{4}x\,dx$$
$$= \frac{1}{8}x^2 \Big|_0^2 = \frac{1}{2}$$

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[\{X \in B\}]} & x \in B \\ 0 & x \notin B \end{cases}$$
$$= \begin{cases} \dfrac{f_X(x)}{1/2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \dfrac{x}{2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

- **Intuition:** $f_{X|B}(x)$ is $f_X(x)$ restricted to the values in B and rescaled so it integrates to 1.

- <u>Conditional PDF Properties</u>:
  - → $f_{X|B}(x) \geq 0$ (Non-negativity)
  - → $\int_B f_{X|B}(x)\,dx = 1$ (Normalization)
  - → For any event A, the conditional probability that X falls into A given that X falls into B is

$$\mathbb{P}[\{X \in A\} \mid \{X \in B\}] = \int_A f_{X|B}(x)\,dx \quad \text{(additivity)}$$

- Given that X falls into B, what is the average value of X? We need to define conditional expectation for continuous random variables.

- The conditional expected value $\mathbb{E}[X \mid B]$ given the event $\{X \in B\}$ is

$$\boxed{\mathbb{E}[X \mid B] = \int_B x f_{X|B}(x)\,dx = \frac{\int_B x f_X(x)\,dx}{\int_B f_X(x)\,dx}}$$

- The conditional expected value $\mathbb{E}[g(X) \mid B]$ of a function $g(X)$ given the event $\{X \in B\}$ is

$$\boxed{\mathbb{E}[g(X) \mid B] = \int_B g(x) f_{X|B}(x)\,dx = \frac{\int_B g(x) f_X(x)\,dx}{\int_B f_X(x)\,dx}}$$

- The conditional variance $\mathrm{Var}[X \mid B]$ of X given the event $\{X \in B\}$ is

$$\boxed{\mathrm{Var}[X \mid B] = \mathbb{E}[(X - \mathbb{E}[X \mid B])^2 \mid B] = \mathbb{E}[X^2 \mid B] - (\mathbb{E}[X \mid B])^2}$$

- **Example:** X is Uniform$(-1,1)$. Condition on the event $\{|X| \geq \frac{1}{2}\}$

→ Determine $f_{X|B}(x)$.

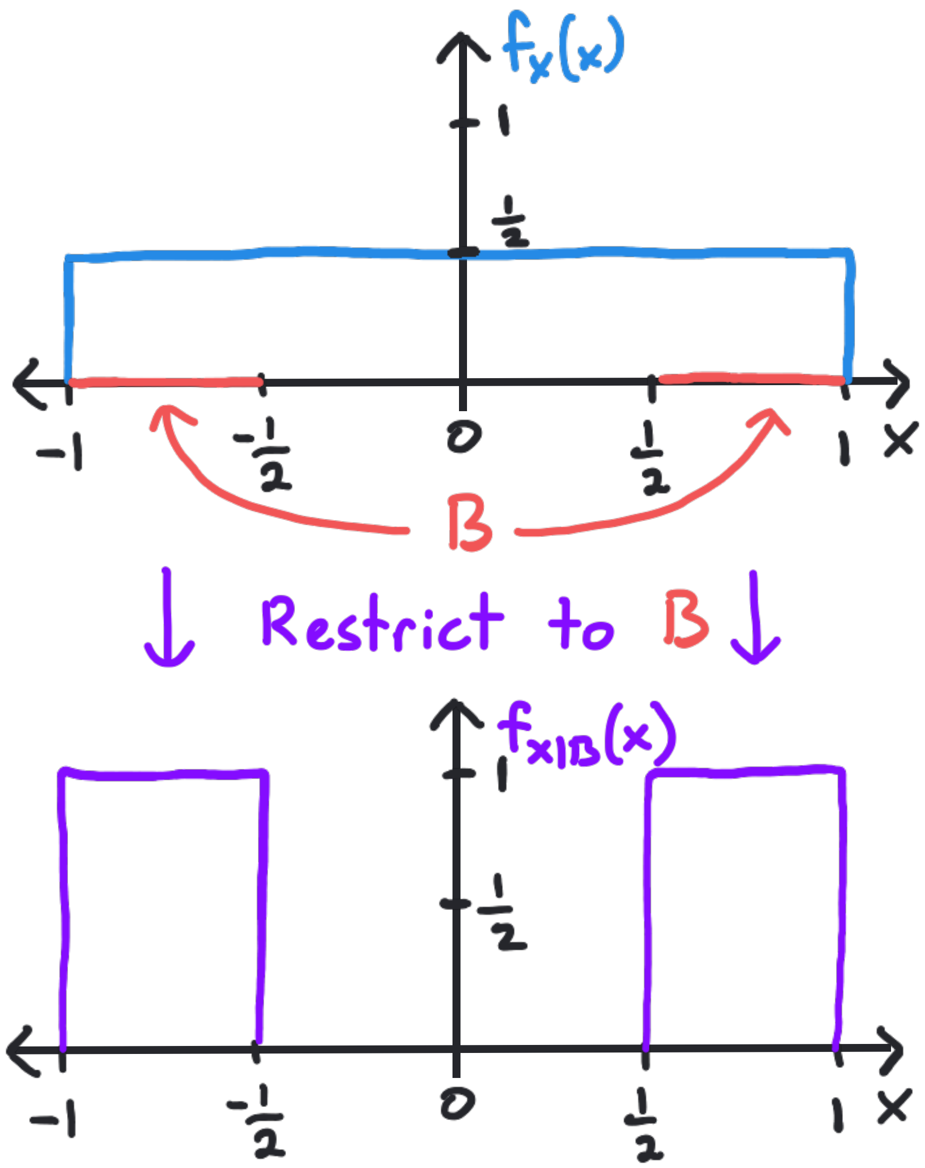
$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\int_B f_X(x)\,dx} & x \in B \\ 0 & x \notin B \end{cases}$$

$$\int_B f_X(x)\,dx = \int_{-1}^{-1/2} \frac{1}{2}\,dx + \int_{1/2}^{1} \frac{1}{2}\,dx$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$f_{X|B}(x) = \begin{cases} \dfrac{\frac{1}{2}}{\frac{1}{2}} & \frac{1}{2} \leq |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

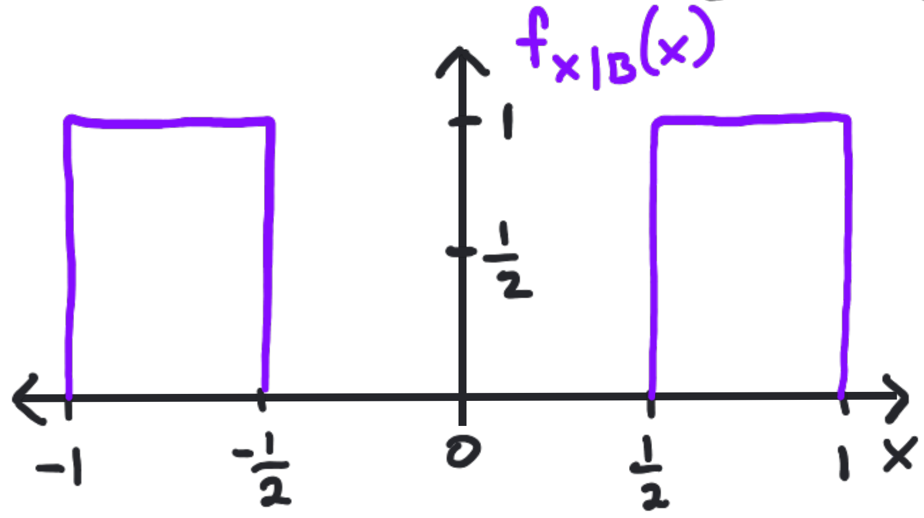
$$= \begin{cases} 1 & \frac{1}{2} \leq |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Rescale to height 1
so the total area is 1.

- **Example:** X is Uniform$(-1,1)$. Condition on the event $\{|X| \geq \frac{1}{2}\}$

$$f_{X|B}(x) = \begin{cases} 1 & \frac{1}{2} \leq |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



→ Determine $\mathbb{E}[X|B]$

$$\mathbb{E}[X|B] = \int_B x f_{X|B}(x)\,dx = \int_{-1}^{-1/2} x\,dx + \int_{1/2}^{1} x\,dx$$

$$= \left(\tfrac{1}{2}x^2\right)\Big|_{-1}^{-1/2} + \left(\tfrac{1}{2}x^2\right)\Big|_{1/2}^{1} = \frac{1}{4} - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = 0$$

→ Determine $\mathrm{Var}[X|B]$. First compute $\mathbb{E}[X^2|B]$.

$$\mathbb{E}[X^2|B] = \int_B x^2 f_{X|B}(x)\,dx = \int_{-1}^{-1/2} x^2\,dx + \int_{1/2}^{1} x^2\,dx = \left(\tfrac{1}{3}x^3\right)\Big|_{-1}^{-1/2} + \left(\tfrac{1}{3}x^3\right)\Big|_{1/2}^{1}$$

$$= -\frac{1}{24} - \left(-\frac{1}{3}\right) + \frac{1}{3} - \frac{1}{24} = \frac{-1+8+8-1}{24} = \frac{7}{12}$$

$$\mathrm{Var}[X|B] = \mathbb{E}[X^2|B] - (\mathbb{E}[X|B])^2$$
$$= \frac{7}{12} - 0^2 = \frac{7}{12}$$

→ Determine $\mathbb{E}[(3X-1)^2|B] = \mathbb{E}[9X^2 - 6X + 1 | B]$

Linearity of Expectation
$$= 9\,\mathbb{E}[X^2|B] - 6\,\mathbb{E}[X|B] + 1$$
$$= 9 \cdot \frac{7}{12} - 6 \cdot 0 + 1 = \frac{63+12}{12} = \frac{75}{12} = \frac{25}{4}$$