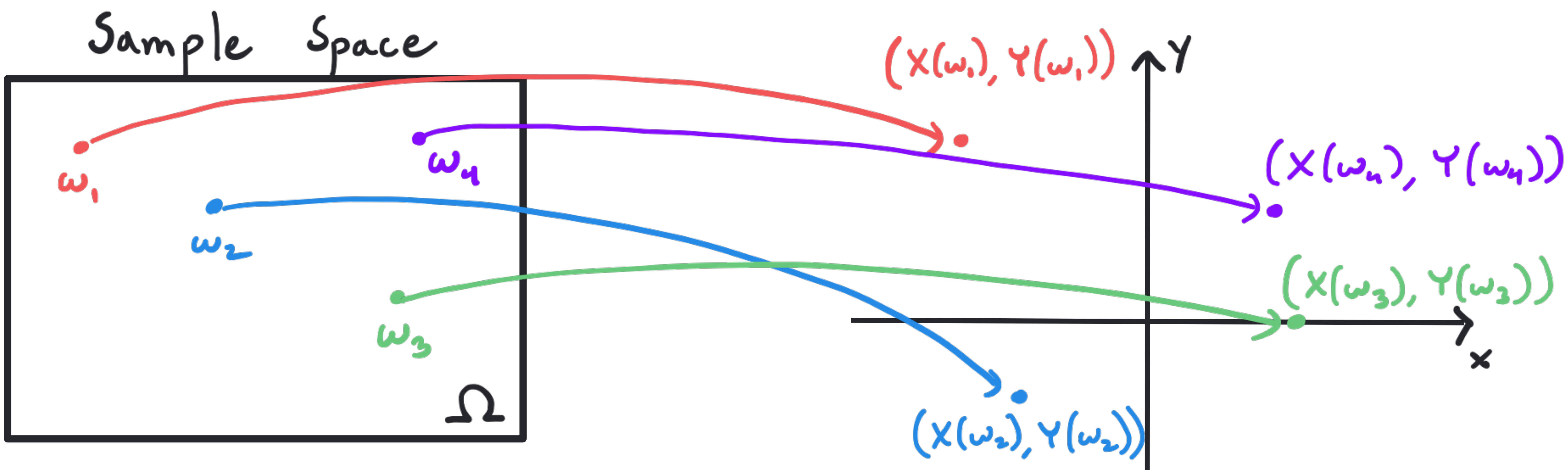
Pairs of Random Variables

- Now that we have a good understanding of a single random variable, we can study multiple random variables $X_1, X_2, \ldots, X_n$ and the relationships (e.g., dependencies) between them.

- Most of the intuition is captured by $n = 2$ random variables.
  → Notation: We usually use X and Y rather than X_1 and X_2.
  → In general, we cannot think of Y as a function of X (and vice versa).

- Examples: → X is the desired signal and Y is a noisy version.
  → X and Y are the temperatures in Boston on two consecutive days
  → X and Y are the ratings of a movie by two friends



Random Variables: $X: \Omega \to \mathbb{R}$
$Y: \Omega \to \mathbb{R}$

- We will use the CDF as a unifying framework (in the background) to describe discrete and continuous random variables.
  - → Discrete: Use the CDF to get the PMF, which is easier to work with.
  - → Continuous: Use the CDF to get the PDF, which is easier to work with.

Joint Cumulative Distribution Function

- The joint cumulative distribution function (CDF) $F_{X,Y}(x,y)$ of a pair of random variables X and Y is the probability that X is less than or equal to x and Y is less than or equal to y,

$$\boxed{\begin{aligned} F_{X,Y}(x,y) &= \mathbb{P}\left[\{\omega \in \Omega : X(\omega) \le x, \ Y(\omega) \le y\}\right] \\ &= \mathbb{P}\left[\{X \le x\} \cap \{Y \le y\}\right] \quad \text{shorthand notation} \\ &= \mathbb{P}\left[X \le x, \ Y \le y\right] \quad \text{Shorthand notation Comma means "and"} \end{aligned}}$$

- **Properties**: → $F_{X,Y}(x,y) \ge 0$ (Non-negativity) → $\lim_{x,y \to \infty} F_{X,Y}(x,y) = 1$ (Normalization)

→ $F_{X,Y}(x,y) \le F_{X,Y}(\tilde{x}, \tilde{y})$ for $x \le \tilde{x}$, $y \le \tilde{y}$ (Non-decreasing)

→ $\lim_{y \to \infty} F_{X,Y}(x,y) = F_X(x)$ and $\lim_{x \to \infty} F_{X,Y}(x,y) = F_Y(y)$ (Marginalization)

Joint Probability Mass Function

- The joint probability mass function (PMF) $P_{X,Y}(x,y)$ of a pair of discrete random variables X and Y is the probability that X equals x and Y equals y,

$$\boxed{\begin{aligned} P_{X,Y}(x,y) &= \mathbb{P}[\{\omega \in \Omega : X(\omega) = x,\ Y(\omega) = y\}] \\ &= \mathbb{P}[\{X = x\} \cap \{Y = y\}] \\ &= \mathbb{P}[X = x,\ Y = y] \end{aligned}}$$

shorthand notation

- The range $R_{X,Y}$ of a pair of discrete random variables is the set of possible values,

$$\boxed{R_{X,Y} = \{ (x,y) \in \mathbb{R}^2 : P_{X,Y}(x,y) > 0 \}}$$

- We can derive the joint PMF from the joint CDF but it will be easier to work with the joint PMF directly.

- Properties of the Joint PMF:
  - → $P_{X,Y}(x,y) \geq 0$ (Non-negativity)
  - → $\sum_{(x,y)\in R_{X,Y}} P_{X,Y}(x,y) = 1$ (Normalization)
  - → $\mathbb{P}[\{(X,Y) \in B\}] = \sum_{(x,y)\in B} P_{X,Y}(x,y)$ (additivity)

- The marginal PMFs $P_X(x)$ and $P_Y(y)$ are the PMFs for the individual random variables X and Y.
  - → Sum the joint PMF over the undesired variable to get a marginal PMF. $P_X(x) = \sum_{y\in R_Y} P_{X,Y}(x,y)$ $\quad P_Y(y) = \sum_{x\in R_X} P_{X,Y}(x,y)$
  - → The marginal PMFs alone are not enough to determine the joint PMF.

- Can visualize the joint PMF as a table:

Marginalize to get $P_Y(y)$

$P_{X,Y}(x,y)$		X				
		x_1	x_2	x_3	$\cdots$	x_m
Y	y_1	$P_{X,Y}(x_1,y_1)$	$P_{X,Y}(x_2,y_1)$	$P_{X,Y}(x_3,y_1)$	$\cdots$	$P_{X,Y}(x_m,y_1)$
	y_2	$P_{X,Y}(x_1,y_2)$	$P_{X,Y}(x_2,y_2)$	$P_{X,Y}(x_3,y_2)$	$\cdots$	$P_{X,Y}(x_m,y_2)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	y_n	$P_{X,Y}(x_1,y_n)$	$P_{X,Y}(x_2,y_n)$	$P_{X,Y}(x_3,y_n)$	$\cdots$	$P_{X,Y}(x_m,y_n)$

Sum over Columns

$$P_Y(y) = \begin{cases} P_Y(y_1) & y = y_1 \\ P_Y(y_2) & y = y_2 \\ \vdots & \vdots \\ P_Y(y_n) & y = y_n \end{cases}$$

Marginalize to get $P_X(x)$

Sum over Rows

$$P_X(x) = \begin{cases} P_X(x_1) & x = x_1 \\ P_X(x_2) & x = x_2 \\ P_X(x_3) & x = x_3 \\ \vdots & \vdots \\ P_X(x_m) & x = x_m \end{cases}$$

<u>Properties</u>

Non-negativity: No negative entries.

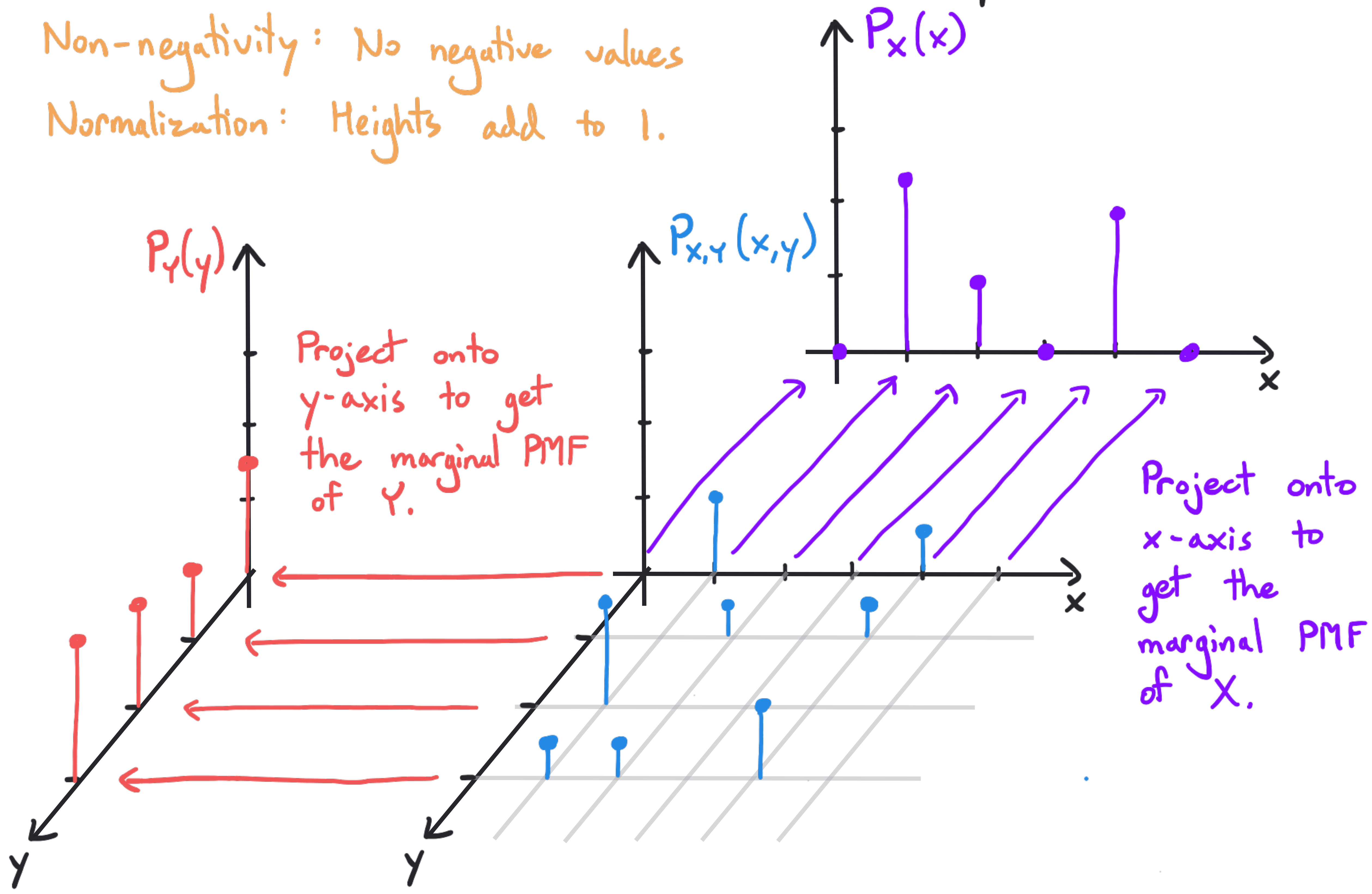
Normalization: Table entries sum to 1.

Additivity: Add up subset of entries.

• Can visualize the joint PMF as a 3-d plot:

Non-negativity: No negative values

Normalization: Heights add to 1.



Joint PMF

- **Example:** $p_{X,Y}(x,y)$

		$x=1$	$x=2$	$x=3$
y	1	$\frac{1}{3}$	0	0
	2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Sum over columns

Marginal PMF of Y

$$p_Y(y) = \begin{cases} \frac{1}{3} & y=1 \\ \frac{2}{3} & y=2 \end{cases}$$

Sum over rows

$$p_X(x) = \begin{cases} \frac{1}{2} & x=1 \\ \frac{1}{6} & x=2 \\ \frac{1}{3} & x=3 \end{cases}$$

Marginal PMF of X

Calculate $\mathbb{P}[X \le Y]$.

→ This is shorthand for the event $\{(X,Y) \in B\}$ with $B = \{(1,1), (1,2), (2,2)\}$

$$\mathbb{P}[X \le Y] = \mathbb{P}[\{(X,Y) \in B\}]$$
$$= \sum_{(x,y) \in B} p_{X,Y}(x,y)$$
$$= p_{X,Y}(1,1) + p_{X,Y}(1,2) + p_{X,Y}(2,2)$$
$$= \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$$

- Recall that all probability questions are implicitly about the probability of membership in a set.