Conditional PMFs

- Say we have a pair of discrete random variables X and Y described by joint PMF $P_{X,Y}(x,y)$.
  - → Observe that $Y=y$.
  - → How can we update the joint PMF to include this?
- By conditioning on $\{Y=y\}$, we restrict the joint PMF to pairs where $Y=y$ and rescale by dividing by $P_Y(y)$.
- The conditional PMF $P_{X|Y}(x|y)$ of X given Y is

$$\boxed{P_{X|Y}(x|y) = \begin{cases} \dfrac{P_{X,Y}(x,y)}{P_Y(y)} & (x,y) \in R_{X,Y} \\ 0 & \text{otherwise} \end{cases}}$$

- Similarly, the conditional PMF $P_{Y|X}(y|x)$ of Y given X is

$$\boxed{P_{Y|X}(y|x) = \begin{cases} \dfrac{P_{X,Y}(x,y)}{P_X(x)} & (x,y) \in R_{X,Y} \\ 0 & \text{otherwise} \end{cases}}$$

- Why? Let $B = \{x\}$.

$$P_{Y|X}(y|x) = P_{Y|B}(y) = \mathbb{P}[\{Y=y\} \mid \{X \in B\}]$$

$$= \mathbb{P}[\{Y=y\} \mid \{X=x\}]$$

$$= \begin{cases} \dfrac{\mathbb{P}[\{Y=y\} \cap \{X=x\}]}{\mathbb{P}[\{X=x\}]} & \mathbb{P}[\{X=x\}] > 0 \\ 0 & \text{otherwise} \end{cases}$$

Positive if and only if $(x,y) \in R_{X,Y}$, which also guarantees $x \in R_X$.

$$= \begin{cases} \dfrac{\boxed{P_{X,Y}(x,y)}}{P_X(x)} & \boxed{P_X(x) > 0} \\ 0 & \text{otherwise} \end{cases}$$

Holds if and only if $x \in R_X$.

- The conditional PMF satisfies the basic PMF properties:

→ Non-negativity: $P_{X|Y}(x|y) \geq 0, \quad P_{Y|X}(y|x) \geq 0$

→ Normalization: $\sum_{x \in R_X} P_{X|Y}(x|y) = 1, \quad \sum_{y \in R_Y} P_{Y|X}(y|x) = 1$

→ Additivity: $\mathbb{P}[\{X \in B\} \mid \{Y = y\}] = \sum_{x \in B} P_{X|Y}(x|y)$

$$\mathbb{P}[\{Y \in B\} \mid \{X = x\}] = \sum_{y \in B} P_{Y|X}(y|x)$$

- Conditional probability techniques also apply:

→ Multiplication Rule: $P_{X,Y}(x,y) = P_{X|Y}(x|y)\, P_Y(y) = P_{Y|X}(y|x)\, P_X(x)$

→ Law of Total Probability: $P_X(x) = \sum_{y \in R_Y} P_{X|Y}(x|y)\, P_Y(y)$

$$P_Y(y) = \sum_{x \in R_X} P_{Y|X}(y|x)\, P_X(x)$$

→ Bayes' Rule: $P_{X|Y}(x|y) = \dfrac{P_{Y|X}(y|x)\, P_X(x)}{P_Y(y)} \qquad P_{Y|X}(y|x) = \dfrac{P_{X|Y}(x|y)\, P_Y(y)}{P_X(x)}$

Joint PMF

• **Example:** $P_{X,Y}(x,y)$

$P_{X,Y}(x,y)$	$x=1$	$x=2$	$x=3$
$y=1$	$\frac{1}{3}$	0	0
$y=2$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Sum over columns

Marginal PMF of Y

$$P_Y(y) = \begin{cases} \frac{1}{3} & y=1 \\ \frac{2}{3} & y=2 \end{cases}$$

Sum over rows

$$P_X(x) = \begin{cases} \frac{1}{2} & x=1 \\ \frac{1}{6} & x=2 \\ \frac{1}{3} & x=3 \end{cases}$$

Marginal PMF of X

Calculate $P_{X|Y}(x|y)$ and $P_{Y|X}(y|x)$.

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} \qquad P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$$

Unless denominator is 0, then conditional PMF is 0 too.

Normalization: Rows sum to 1.

$P_{X\|Y}(x\|y)$	$x=1$	$x=2$	$x=3$
$y=1$	$\frac{1/3}{1/3}=1$	$\frac{0}{1/3}=0$	$\frac{0}{1/3}=0$
$y=2$	$\frac{1/6}{2/3}=\frac{1}{4}$	$\frac{1/6}{2/3}=\frac{1}{4}$	$\frac{1/3}{2/3}=\frac{1}{2}$

Normalization: Columns sum to 1.

$P_{Y\|X}(y\|x)$	$x=1$	$x=2$	$x=3$
$y=1$	$\frac{1/3}{1/2}=\frac{2}{3}$	0	0
$y=2$	$\frac{1/6}{1/2}=\frac{1}{3}$	$\frac{1/6}{1/6}=1$	$\frac{1/3}{1/3}=1$

- The conditional PMF can be used to express hierarchical probability models. For instance, we can write $P_{Y|X}(y|x)$ using a family of random variables where the parameters are a function of x, which is generated using $P_X(x)$.

- <u>Example:</u> Model the number of photons Y observed at a detector as Poisson(λ) where $\lambda = g(x)$.

  → X is 0 if sample is absent, 1 if sample is present. Sample present with probability $\frac{1}{3}$. $\Rightarrow$ X is Bernoulli$(\frac{1}{3})$.

  → Average # photons is 2 when sample absent, 4 when present. Model by $\lambda = g(x) = \begin{cases} 2 & x=0 \\ 4 & x=1 \end{cases}$

  $\Rightarrow$ Y given $X=x$ is Poisson$(g(x))$.

$$P_{Y|X}(y|x) = \begin{cases} \frac{(g(x))^y}{y!} e^{-g(x)} & x=0,1 \\ & y=0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \boxed{\frac{2^y}{y!} e^{-2} \quad x=0,\ y=0,1,2,\ldots} \\ \boxed{\frac{4^y}{y!} e^{-4} \quad x=1,\ y=0,1,2,\ldots} \\ 0 \quad \text{otherwise} \end{cases}$$

Sample absent

Sample Present

• <u>Example</u>:

$$P_{Y|X}(y|x) = \begin{cases} \dfrac{2^y}{y!} e^{-2} & x = 0,\ y = 0, 1, 2, \ldots \\ \dfrac{4^y}{y!} e^{-4} & x = 1,\ y = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Sample absent

Sample Present

$$P_X(x) = \begin{cases} \frac{2}{3} & x = 0 \\ \frac{1}{3} & x = 1 \end{cases}$$

$$\mathbb{P}[Y = 3 \mid X = 1] = P_{Y|X}(3|1) = \frac{4^3}{3!} e^{-4} = \frac{64}{6} e^{-4} = \frac{32}{3} e^{-4}$$

$$\mathbb{P}[Y = 3, X = 1] = P_{X,Y}(1, 3) \underset{\text{Multiplication Rule}}{=} P_{Y|X}(3|1)\, P_X(1) = \frac{32}{3} e^{-4} \cdot \frac{1}{3} = \frac{32}{9} e^{-4}$$

comma means "and"

Multiplication Rule

$$\mathbb{P}[Y = 0] = P_Y(0) = \sum_{x \in R_X} P_{Y|X}(0|x)\, P_X(x)$$

Law of Total Probability

$$= P_{Y|X}(0|0)\, P_X(0) + P_{Y|X}(0|1)\, P_X(1)$$

$$= \frac{2^0}{0!} e^{-2} \cdot \frac{2}{3} + \frac{4^0}{0!} e^{-4} \cdot \frac{1}{3}$$

$$= \frac{2e^{-2} + e^{-4}}{3}$$