Pairs of Continuous Random Variables

- Recall the joint CDF $F_{X,Y}(x,y) = \mathbb{P}[\{X \le x\} \cap \{Y \le y\}]$.

- A pair of random variables X and Y is jointly continuous if their joint CDF is a continuous function and differentiable almost everywhere.

- The joint probability density function (PDF) $f_{X,Y}(x,y)$ of a pair of jointly continuous random variables is

$$\boxed{f_{X,Y}(x,y) = \begin{cases} \dfrac{\partial^2}{\partial x \, \partial y} F_{X,Y}(x,y) & \text{if } F_{X,Y}(x,y) \text{ differentiable at } (x,y) \\ \text{any value} & \text{otherwise} \end{cases}}$$

- The range $R_{X,Y}$ of a pair of jointly continuous random variables X and Y is $\boxed{R_{X,Y} = \{(x,y) \in \mathbb{R}^2 : f_{X,Y}(x,y) > 0\}}$

- **Joint PDF Properties:**

→ $f_{X,Y}(x,y) \geq 0$ (Non-negativity)

→ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$ (Normalization)

→ $\mathbb{P}[\{(X,Y) \in B\}] = \iint_B f_{X,Y}(x,y)\,dx\,dy$ (additivity)

→ $\int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv = F_{X,Y}(x,y)$ (PDF → CDF)

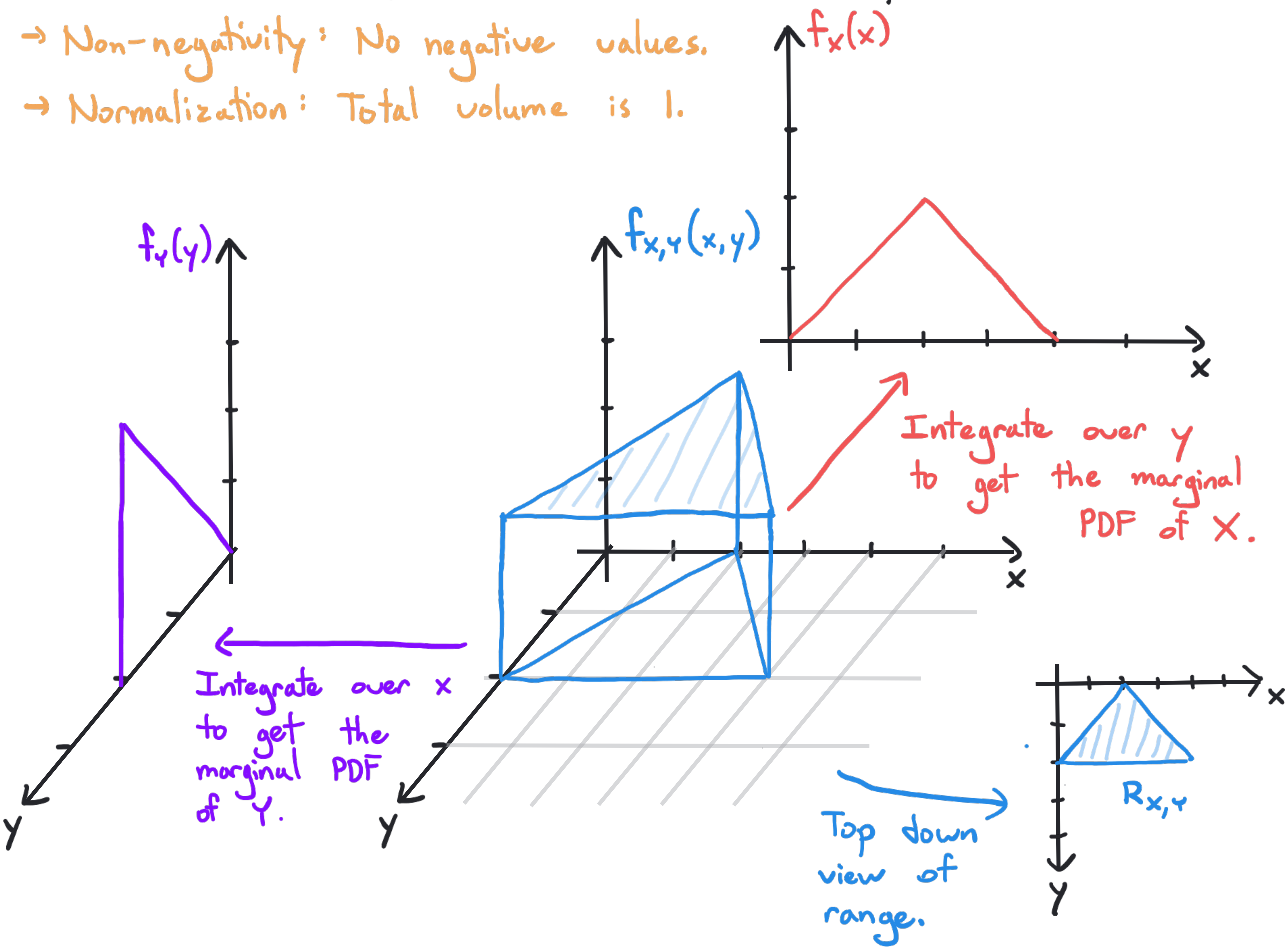
- The marginal PDFs $f_X(x)$ and $f_Y(y)$ are the PDFs for the individual random variables X and Y.

→ Obtain by integrating out undesired variable:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

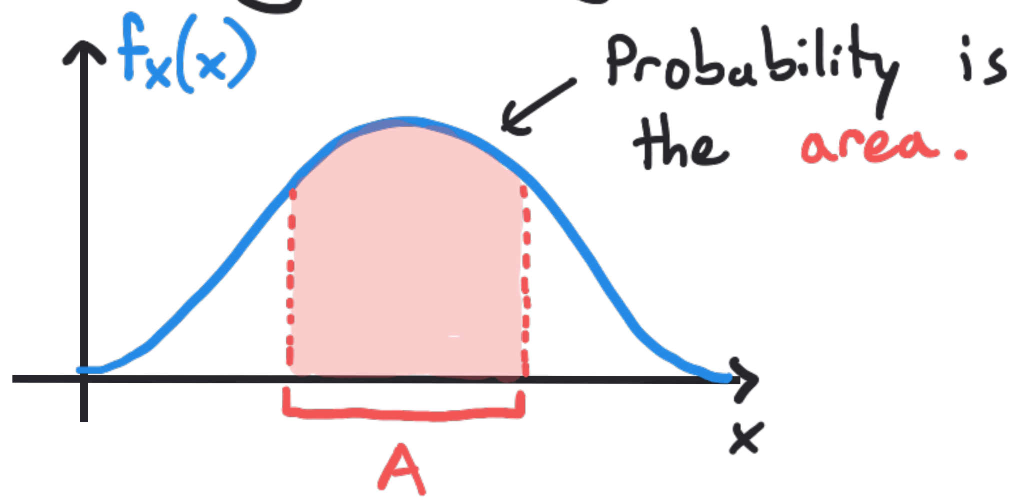
→ Marginals do not alone suffice to determine the joint PDF.

- Can visualize the joint PDF as a 3-d plot:
  - → Non-negativity: No negative values.
  - → Normalization: Total volume is 1.



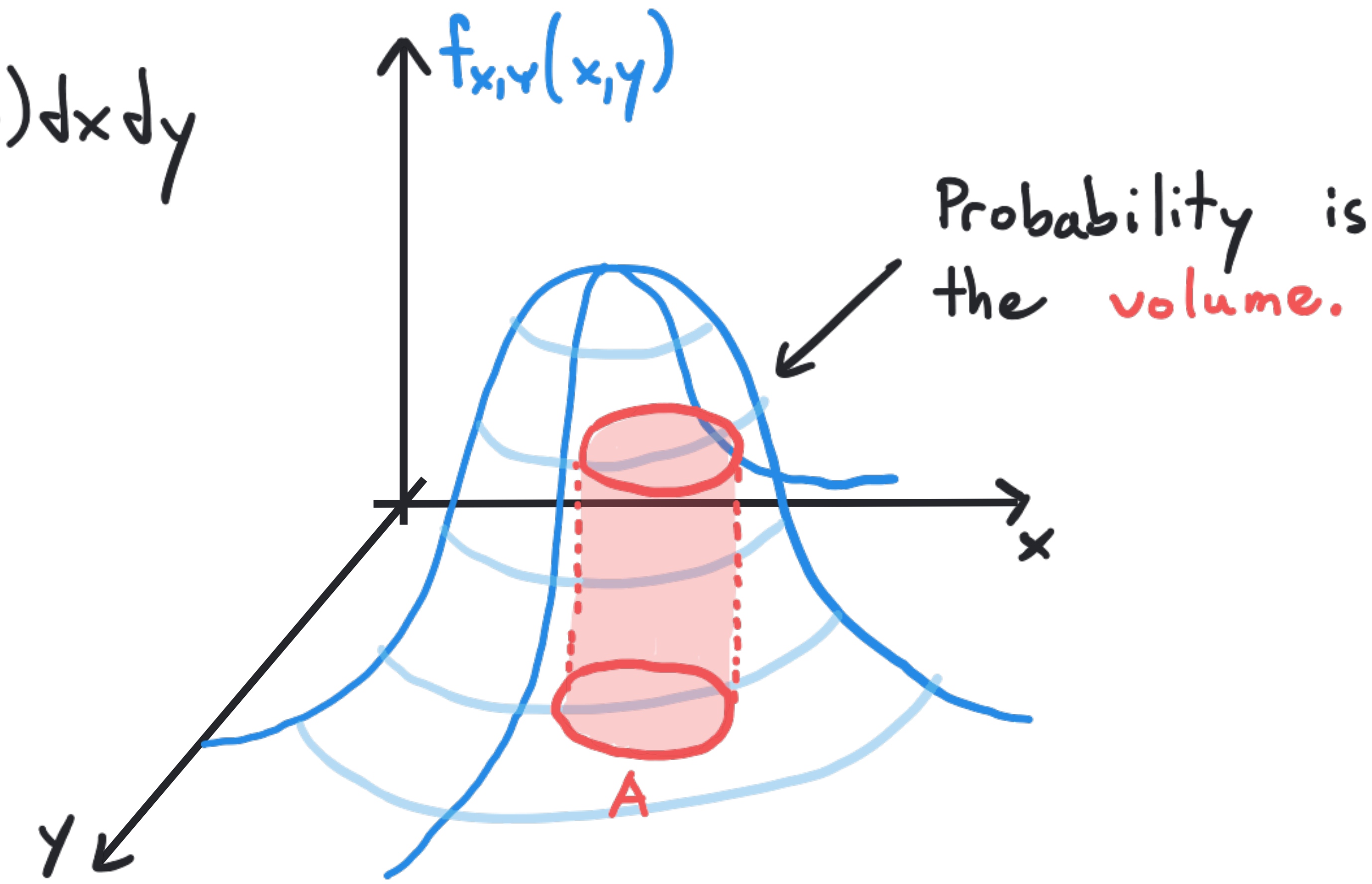
- For a single random variable, the probability of landing in a region is determined by a single integral:

$$\mathbb{P}[\{X \in A\}] = \int_A f_X(x)\,dx$$

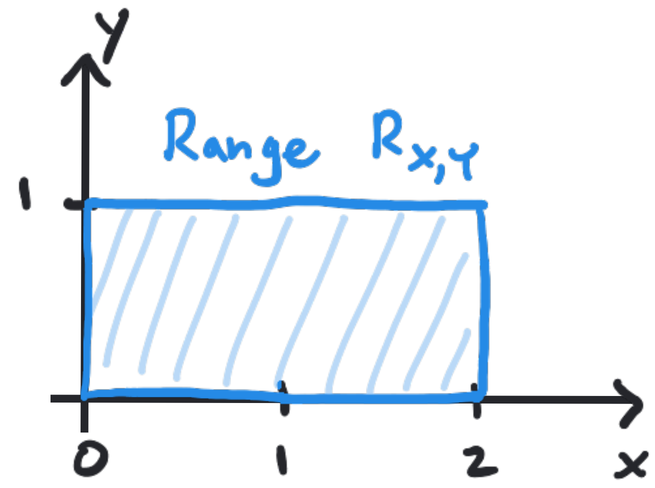


- For a pair of random variables, the probability of landing in a region is determined by a double integral.

$$\mathbb{P}[\{(X,Y) \in A\}] = \iint_A f_{X,Y}(x,y)\,dx\,dy$$



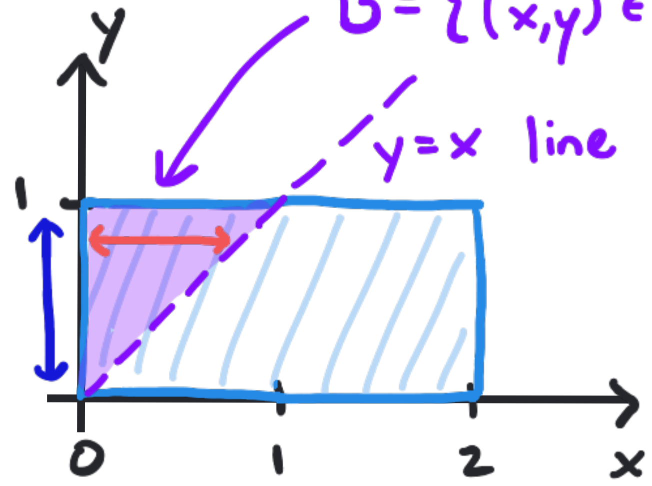
• **Example:** $f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \le x \le 2,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$



→ What is the probability that Y is greater than X?

$$\mathbb{P}[Y > X] = \mathbb{P}[\{(X,Y) \in B\}] = \iint_B f_{X,Y}(x,y)\,dx\,dy$$

→ Any probability question is implicitly asking about the probability of membership in a set. Sketching the range is a useful way to determine this set and the resulting integration region.



$$\mathbb{P}[Y > X] = \int_0^1 \int_0^y \tfrac{1}{3}(x+y)\,dx\,dy$$

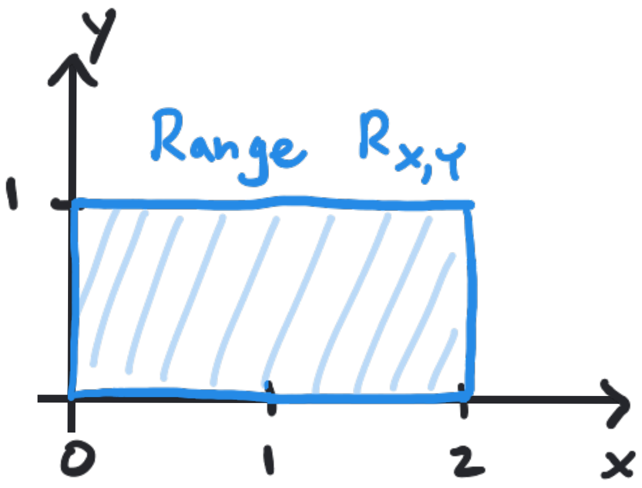
← Could also integrate over y first then x.

$$= \int_0^1 \left(\tfrac{1}{6}x^2 + \tfrac{1}{3}xy\right)\Big|_0^y\,dy$$

$$= \int_0^1 \left(\tfrac{1}{6}y^2 + \tfrac{1}{3}y^2\right)dy$$

$$= \left(\tfrac{1}{6}y^3\right)\Big|_0^1 = \tfrac{1}{6}$$

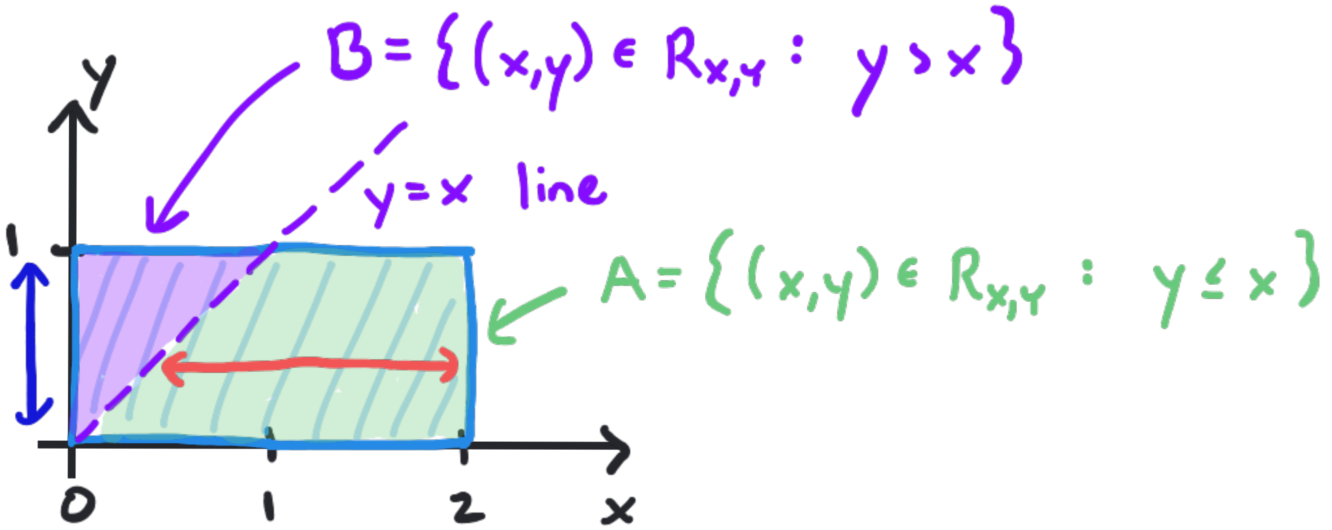
• **Example:** $f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \le x \le 2,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$



→ Calculate $\mathbb{P}[Y \le X]$. Two approaches.

① Reuse previous calculation.

$\mathbb{P}[Y \le X] = \mathbb{P}[\{(X,Y) \in A\}]$ where $A = \{(x,y) \in R_{X,Y} : y \le x\}$.



Since $A^c = B$, we can use the complement property:

$$\begin{aligned} \mathbb{P}[\{(X,Y) \in A\}] &= 1 - \mathbb{P}[\{(X,Y) \in A^c\}] \\ &= 1 - \mathbb{P}[\{(X,Y) \in B\}] \\ &= 1 - \frac{1}{6} = \frac{5}{6} \end{aligned}$$

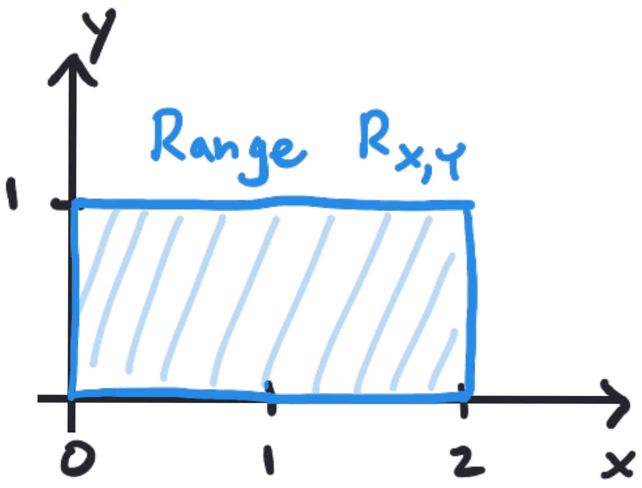
② Direct integration.

$$\mathbb{P}[Y \le X] = \iint_A f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_y^2 \frac{1}{3}(x+y)\,dx\,dy = \int_0^1 \left(\frac{1}{6}x^2 + \frac{1}{3}xy\right)\Big|_y^2 dy$$

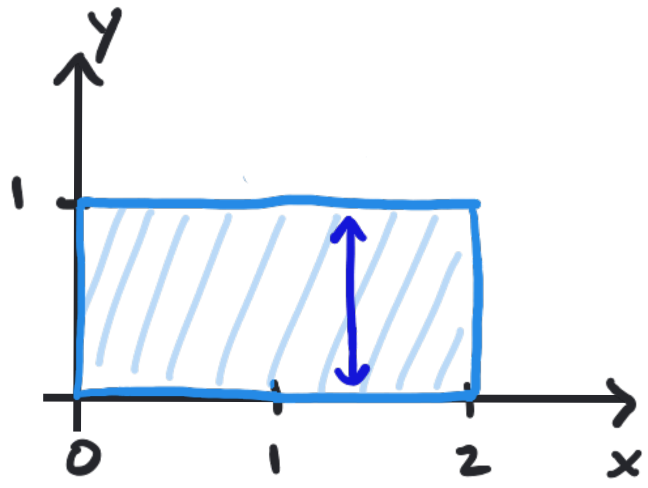
$$= \int_0^1 \left(\frac{4}{6} + \frac{2}{3}y - \frac{1}{6}y^2 - \frac{1}{3}y^2\right) dy = \int_0^1 \left(\frac{2}{3} + \frac{2}{3}y - \frac{1}{2}y^2\right) dy$$

$$= \left(\frac{2}{3}y + \frac{1}{3}y^2 - \frac{1}{6}y^3\right)\Big|_0^1 = \frac{2}{3} + \frac{1}{3} - \frac{1}{6} = \frac{5}{6}$$

• **Example:** $f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \le x \le 2,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

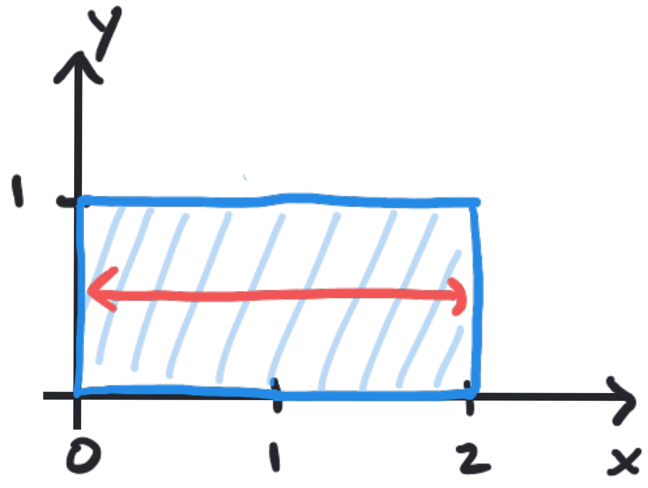


→ Calculate the marginal PDFs of X and Y.



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \int_0^1 \frac{1}{3}(x+y)\,dy & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \left(\frac{1}{3}xy + \frac{1}{6}y^2\right)\Big|_0^1 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{3}x + \frac{1}{6} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^2 \frac{1}{3}(x+y)\,dx$$

$$= \left(\frac{1}{6}x^2 + \frac{1}{3}xy\right)\Big|_0^2$$

← Deliberately left the range of Y out of these expressions to save space.

$$= \begin{cases} \frac{2}{3}y + \frac{2}{3} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

← Don't forget to include it in the last step!