Conditional PDFs

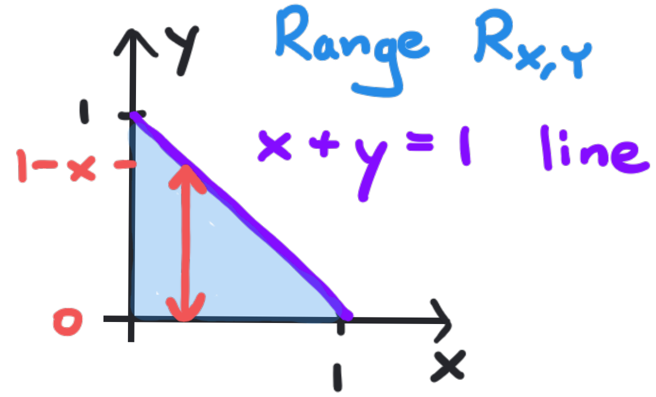
- Say we have a pair of continuous random variables X and Y described by a joint PDF $f_{X,Y}(x,y)$.
  - → Observe that $Y=y$.
  - → How can we update the joint PDF to include this?
- By conditioning on $\{Y=y\}$, we restrict the joint PDF to pairs where $Y=y$ and rescale by dividing by $f_Y(y)$.
- The conditional PDF $f_{X|Y}(x|y)$ of X given Y is

$$\boxed{f_{X|Y}(x|y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} & (x,y) \in R_{X,Y} \\ 0 & \text{otherwise} \end{cases}}$$

- Similarly, the conditional PDF $f_{Y|X}(y|x)$ of Y given X is

$$\boxed{f_{Y|X}(y|x) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_X(x)} & (x,y) \in R_{X,Y} \\ 0 & \text{otherwise} \end{cases}}$$

- **Example:** $f_{X,Y}(x,y) = \begin{cases} 2 & x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$



What is $f_{Y|X}(y|x)$? First, need marginal PDF.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{0}^{1-x} 2\,dy = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{1}{1-x} & x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- The conditional PDF satisfies the basic PDF properties:
  - → Non-negativity: $f_{X|Y}(x|y) \geq 0, \quad f_{Y|X}(y|x) \geq 0$
  - → Normalization: $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = 1, \quad \int_{-\infty}^{\infty} f_{Y|X}(y|x)\,dy = 1$
  - → Additivity: $\mathbb{P}[\{X \in B\} \mid \{Y = y\}] = \int_B f_{X|Y}(x|y)\,dx$

    $\mathbb{P}[\{Y \in B\} \mid \{X = x\}] = \int_B f_{Y|X}(y|x)\,dy$

- Conditional probability techniques also apply:
  - → Multiplication Rule: $f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y) = f_{Y|X}(y|x)\, f_X(x)$
  - → Law of Total Probability: $f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)\,dy$

    $f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\,dx$
  - → Bayes' Rule: $f_{X|Y}(x|y) = \dfrac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)} \qquad f_{Y|X}(y|x) = \dfrac{f_{X|Y}(x|y)\, f_Y(y)}{f_X(x)}$

- The conditional PDF can be used to express hierarchical probability models. For instance, we can write $f_{Y|X}(y|x)$ using a family of random variables where the parameters are a function of x, which is generated using $f_X(x)$.

- <u>Example</u>: Want to measure X, which is Uniform$(0,4)$.

  → Only get a noisy version Y, which given that $X=x$ is Gaussian$(x, 9)$. centered at x $\Rightarrow f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi \cdot 9}} \exp\left(-\frac{(y-x)^2}{2\cdot 9}\right)$

  → Calculate $\mathbb{P}[Y>1 \mid X=\frac{1}{2}]$.

  Given that $X=\frac{1}{2}$, Y is Gaussian$(\frac{1}{2}, 9)$.

$$
\begin{aligned}
\mathbb{P}[Y>1 \mid X=\tfrac{1}{2}] &= 1 - \mathbb{P}[Y \le 1 \mid X=\tfrac{1}{2}] \\
&= 1 - \Phi\left(\frac{1-\frac{1}{2}}{3}\right) \\
&= 1 - \Phi\left(-\tfrac{1}{6}\right) \\
&\approx 1 - 0.4338 \\
&= 0.5662
\end{aligned}
$$

Probability of an Interval for a Gaussian →

- **Example**: Want to measure X, which is Uniform$(0,4)$.
  - → Only get a noisy version Y, which given that $X=x$ is Gaussian$(x,9)$.

$$\Rightarrow f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi \cdot 9}} \exp\left(-\frac{(y-x)^2}{2\cdot 9}\right)$$

  - → Determine the marginal PDF of Y.

$$\Rightarrow f_X(x) = \begin{cases} \frac{1}{4} & \boxed{0 \le x \le 4} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\
&= \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, f_X(x)\,dx \\
&= \int_{0}^{4} \frac{1}{\sqrt{2\pi\cdot 9}} \exp\left(-\frac{(y-x)^2}{2\cdot 9}\right)\cdot \frac{1}{4}\,dx
\end{aligned}$$

$w = \frac{y-x}{3} \qquad dw = -\frac{dx}{3}$

$$\begin{aligned}
&= \frac{1}{4}\int_{\frac{y-4}{3}}^{\frac{y}{3}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right)dw \\
&= \frac{1}{4}\left(\Phi\left(\frac{y}{3}\right) - \Phi\left(\frac{y-4}{3}\right)\right)
\end{aligned}$$

Recall $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right)dw$