Expected Value of a Function of Two Random Variables

- While the joint PMF or PDF provides a full characterization of a pair of random variables X and Y, we are sometimes more interested in the average (or expected) value of a function $W = g(X,Y)$.

- One approach is to first determine the distribution of $W = g(X,Y)$ and then its expected value $\mathbb{E}[W]$. However, determining the PMF (or PDF) of W can be quite challenging and is actually unnecessary.

  See lecture notes for examples.

- The expected value $\mathbb{E}[g(X,Y)]$ of a function $g(X,Y)$ is

$$\boxed{\begin{aligned} &\text{Discrete:} \quad \mathbb{E}[g(X,Y)] = \sum_{x \in R_X} \sum_{y \in R_Y} g(x,y)\, P_{X,Y}(x,y) \\ &\text{Continuous:} \quad \mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy \end{aligned}}$$

- **Linearity of Expectation:** For any functions $g_1(x,y), \dots, g_n(x,y)$ and constants $a_1, \dots, a_n$,

$$\boxed{\mathbb{E}[a_1 g_1(X,Y) + \cdots + a_n g_n(X,Y)] = a_1 \mathbb{E}[g_1(X,Y)] + \cdots + a_n \mathbb{E}[g_n(X,Y)]}$$

→ Why?

$$\sum_{x \in R_X} \sum_{y \in R_Y} (a_1 g_1(x,y) + \cdots + a_n g_n(x,y)) P_{X,Y}(x,y) = a_1 \sum_{x \in R_X} \sum_{y \in R_Y} g_1(x,y) P_{X,Y}(x,y) + \cdots + a_n \sum_{x \in R_X} \sum_{y \in R_Y} g_n(x,y) P_{X,Y}(x,y)$$

Same argument for continuous case.

- Special Cases: $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$$\mathbb{E}[aX + bY + c] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] + c$$

- Linearity of Expectation always holds!

Does not require independence of X and Y!

- **Expectation of Products:** If X and Y are independent, then for any functions $g(x)$ and $h(y)$

$$\boxed{\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]}$$

→ Why?

$$\sum_{x \in R_X} \sum_{y \in R_Y} g(x)h(y)\, P_{X,Y}(x,y) \overset{\text{independence}}{=} \sum_{x \in R_X} \sum_{y \in R_Y} g(x)h(y)\, P_X(x) P_Y(y)$$

$$= \sum_{x \in R_X} g(x) P_X(x) \sum_{y \in R_Y} h(y) P_Y(y)$$

Change sums to integrals for continuous case.

- Caveats: → Does not hold in general for dependent X, Y.

→ But even if $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]$ for a particular example, X and Y might be dependent!

• **Example:** $p_{X,Y}(x,y)$

$p_{X,Y}(x,y)$		$x = -1$	$x = +1$	$x = +2$
y	-1	$\frac{1}{3}$	0	$\frac{1}{12}$
	$+2$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$

add up each row

Marginal PMF of Y

$$p_Y(y) = \begin{cases} \frac{5}{12} & y = -1 \\ \frac{7}{12} & y = +2 \end{cases}$$

→ Calculate $\mathbb{E}[Y^2]$. Can work with the joint PMF directly.

$$\mathbb{E}[Y^2] = \sum_{x \in R_X} \sum_{y \in R_Y} y^2 \, p_{X,Y}(x,y)$$

$$= (-1)^2 \big(p_{X,Y}(-1,-1) + p_{X,Y}(+1,-1) + p_{X,Y}(+2,-1) \big)$$

$$+ (+2)^2 \big(p_{X,Y}(-1,+2) + p_{X,Y}(+1,+2) + p_{X,Y}(+2,+2) \big)$$

$$= \frac{1}{3} + 0 + \frac{1}{12} + 4 \cdot \left(\frac{1}{6} + \frac{1}{4} + \frac{1}{6} \right) = \frac{4 + 1 + 8 + 12 + 8}{12} = \frac{33}{12} = \frac{11}{4}$$

→ Double check calculation using marginal.

$$\mathbb{E}[Y^2] = \sum_{y \in R_Y} y^2 \, p_Y(y) = (-1)^2 \frac{5}{12} + (+2)^2 \frac{7}{12} = \frac{5}{12} + \frac{28}{12} = \frac{33}{12} = \frac{11}{4} \checkmark$$

• Example: $P_{X,Y}(x,y)$

		x		
		-1	$+1$	$+2$
y	-1	$\frac{1}{3}$	0	$\frac{1}{12}$
	$+2$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$

→ Calculate $\mathbb{E}[X^2Y]$.

$$\mathbb{E}[X^2Y] = \sum_{x \in R_X} \sum_{y \in R_Y} x^2 y \, P_{X,Y}(x,y)$$

$$= (-1)^2 \cdot (-1) P_{X,Y}(-1,-1) + (+1)^2 \cdot (-1) P_{X,Y}(+1,-1) + (+2)^2 \cdot (-1) P_{X,Y}(+2,-1)$$

$$+ (-1)^2 \cdot (+2) P_{X,Y}(-1,+2) + (+1)^2 \cdot (+2) P_{X,Y}(+1,+2) + (+2)^2 \cdot (+2) P_{X,Y}(+2,+2)$$

$$= (-1) \cdot \frac{1}{3} + (-1) \cdot 0 + (-4) \cdot \frac{1}{12} + (+2) \cdot \frac{1}{6} + (+2) \cdot \frac{1}{4} + (+8) \cdot \frac{1}{6}$$

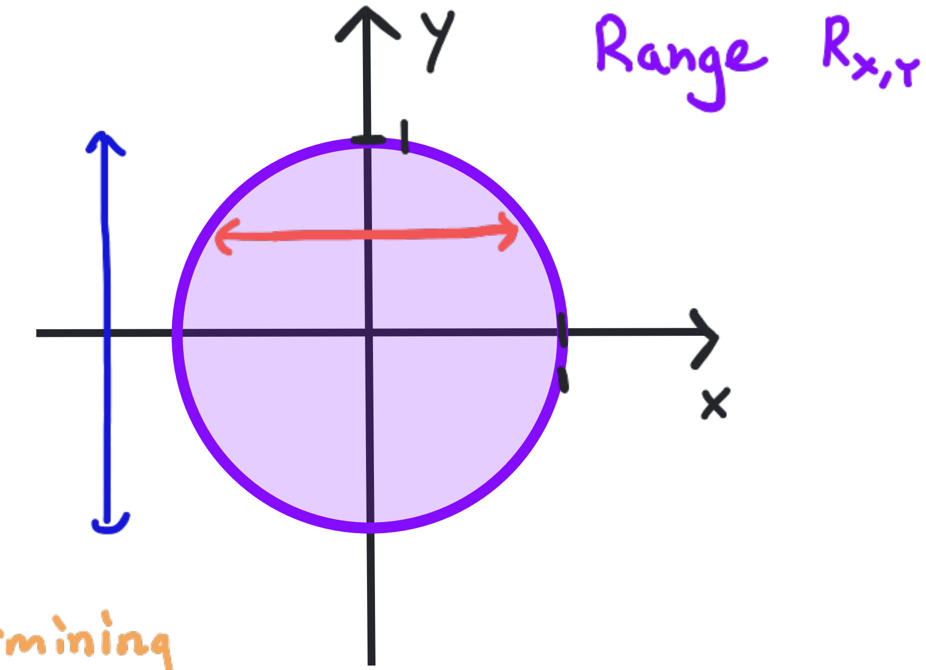
$$= \frac{-4 - 4 + 4 + 6 + 16}{12} = \frac{18}{12} = \frac{3}{2}$$

→ Calculate $\mathbb{E}[4X^2Y - Y^2]$.

$$\mathbb{E}[4X^2Y - Y^2] = 4\,\mathbb{E}[X^2Y] - \mathbb{E}[Y^2] = 4 \cdot \frac{3}{2} - \frac{11}{4} = \frac{24 - 11}{4} = \frac{13}{4}$$

Linearity of Expectation

• **Example:** $f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$



→ Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

$$\mathbb{E}[X] = \iint_{R_{X,Y}} x\, f_{X,Y}(x,y)\,dx\,dy$$

← Direct calculation instead of determining $f_X(x)$ first.

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,\frac{1}{\pi}\,dx\,dy$$

$$= \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1}{2}x^2\right)\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{2}(1-y^2) - \frac{1}{2}(1-y^2)\,dy = 0 \overset{\text{symmetry}}{=} \mathbb{E}[Y]$$

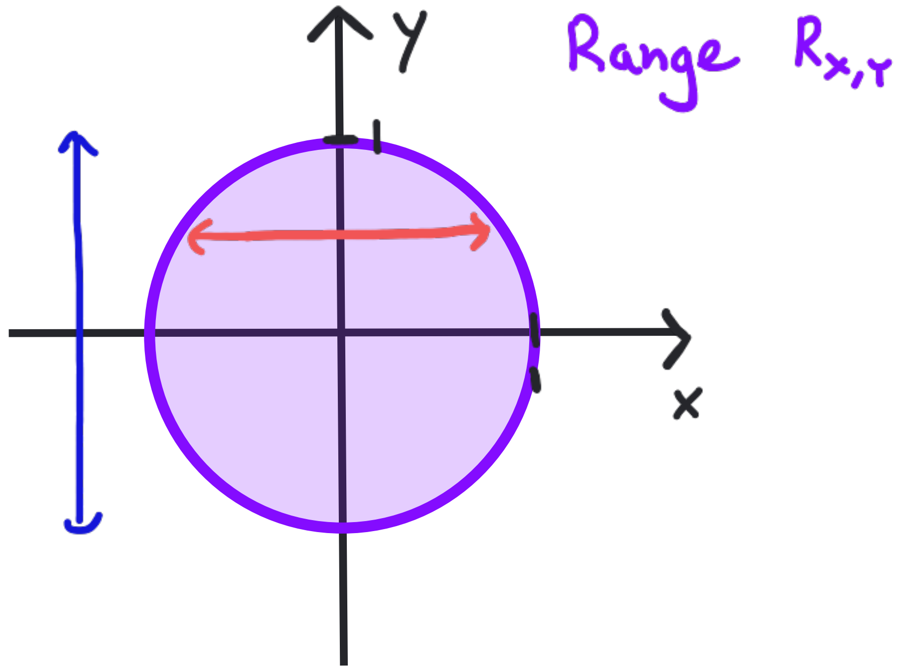
→ Calculate $\mathbb{E}[XY]$. From range, X and Y are dependent.

$$\mathbb{E}[XY] = \iint_{R_{X,Y}} xy\, f_{X,Y}(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xy\,\frac{1}{\pi}\,dx\,dy = \int_{-1}^{1}\left(\frac{1}{2}x^2\right)\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\,\frac{1}{\pi}\,dy$$

$$= \int_{-1}^{1}\left(\frac{1}{2}(1-y^2) - \frac{1}{2}(1-y^2)\right) y\,\frac{1}{\pi}\,dy = 0 = \mathbb{E}[X]\,\mathbb{E}[Y]$$

But X, Y dependent!

- **Example:** $f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$



→ Calculate $\mathbb{E}[X^2]$ and $\mathbb{E}[Y^2]$.

$$\mathbb{E}[X^2] = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 \frac{1}{\pi}\,dx\,dy$$

$$= \int_{-1}^{1} \left(\frac{1}{3}x^3\right)\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\,dy = \int_{-1}^{1}\left(\frac{1}{3}(1-y^2)^{\frac{3}{2}} - (-1)\frac{1}{3}(1-y^2)^{\frac{3}{2}}\right)\frac{1}{\pi}\,dy$$

$$= \int_{-1}^{1} \frac{2}{3\pi}(1-y^2)^{\frac{3}{2}}\,dy \overset{\text{Computer}}{=} \frac{1}{4} \overset{\text{Symmetry}}{=} \mathbb{E}[Y^2]$$

→ Calculate $\mathbb{E}[X^2Y^2]$.

$$\mathbb{E}[X^2Y^2] = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2y^2\frac{1}{\pi}\,dx\,dy \overset{\text{Computer}}{=} \frac{1}{24} \neq \frac{1}{4}\cdot\frac{1}{4} = \mathbb{E}[X^2]\,\mathbb{E}[Y^2]$$

For this choice of function, there was no factorization.