- **Example**: X and Y are jointly Gaussian with parameters

$$W = X - 4Y + 1, \quad Z = 2X + 4Y$$

$\mu_X = 3$

$\mu_Y = -1$

$\sigma_X^2 = 4$

$\sigma_Y^2 = 1$

$\rho_{X,Y} = -\frac{1}{4}$

→ Determine the means of W and Z.

Linearity of Expectation: $\mathbb{E}[aX + bY + c] = a\mu_X + b\mu_Y + c$

$$\mu_W = \mathbb{E}[X - 4Y + 1] = 1\cdot\mu_X + (-4)\cdot\mu_Y + 1 = 1\cdot 3 + (-4)\cdot(-1) + 1 = 8$$

$$\mu_Z = \mathbb{E}[2X + 4Y] = 2\cdot\mu_X + 4\cdot\mu_Y = 2\cdot 3 + 4\cdot(-1) = 2$$

→ Determine the variances of W and Z.

Variance of Linear Functions: $\mathrm{Var}[aX + bY + c] = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\overbrace{\rho_{X,Y}\sigma_X\sigma_Y}^{\mathrm{Cov}[X,Y]}$

$$\sigma_W^2 = \mathrm{Var}[X - 4Y + 1] = 1^2\cdot\sigma_X^2 + (-4)^2\cdot\sigma_Y^2 + 2\cdot 1\cdot(-4)\cdot\rho_{X,Y}\sigma_X\sigma_Y$$

$$= 1\cdot 4 + 16\cdot 1 - 8\cdot\left(-\tfrac{1}{4}\right)\cdot 2\cdot 1 = 24$$

$$\sigma_Z^2 = \mathrm{Var}[2X + 4Y] = 2^2\cdot\sigma_X^2 + 4^2\cdot\sigma_Y^2 + 2\cdot 2\cdot 4\cdot\rho_{X,Y}\sigma_X\sigma_Y$$

$$= 4\cdot 4 + 16\cdot 1 + 16\cdot\left(-\tfrac{1}{4}\right)\cdot 2\cdot 1 = 24$$

• **Example**: X and Y are jointly Gaussian with parameters

$\mu_X = 3$
$\mu_Y = -1$
$\sigma_X^2 = 4$
$\sigma_Y^2 = 1$
$\rho_{X,Y} = -\frac{1}{4}$

$W = X - 4Y + 1, \quad Z = 2X + 4Y$

Just showed: $\mu_W = 8, \ \mu_Z = 2, \ \sigma_W^2 = \sigma_Z^2 = 24$

→ Determine the correlation coefficient of W and Z.

$$\rho_{W,Z} = \frac{\text{Cov}[W,Z]}{\sigma_W \sigma_Z}$$

* Covariance of Linear Functions:

$$\text{Cov}[aX + bY + c, \ dX + eY + f] = ad\,\sigma_X^2 + (ae + bd)\overbrace{\rho_{X,Y}\,\sigma_X \sigma_Y}^{\text{Cov}[X,Y]} + be\,\sigma_Y^2$$

$$\begin{aligned}\text{Cov}[W,Z] &= \text{Cov}[X - 4Y + 1, \ 2X + 4Y] \\ &= 1 \cdot 2 \cdot \sigma_X^2 + (1 \cdot 4 + (-4) \cdot 2) \cdot \rho_{X,Y}\,\sigma_X \sigma_Y + (-4) \cdot 4 \cdot \sigma_Y^2 \\ &= 2 \cdot 4 - 4 \cdot (-\tfrac{1}{4}) \cdot 2 \cdot 1 - 16 \cdot 1 = -6\end{aligned}$$

$$\rho_{W,Z} = \frac{-6}{\sqrt{24} \cdot \sqrt{24}} = -\frac{6}{24} = -\frac{1}{4}$$

• <u>Example</u>: X and Y are jointly Gaussian with parameters $\mu_X = 3$, $\mu_Y = -1$, $\sigma_X^2 = 4$, $\sigma_Y^2 = 1$, $\rho_{X,Y} = -\frac{1}{4}$

$W = X - 4Y + 1, \quad Z = 2X + 4Y$

Just showed: $\mu_W = 8$, $\mu_Z = 2$, $\sigma_W^2 = \sigma_Z^2 = 24$, $\rho_{W,Z} = -\frac{1}{4}$

→ Determine the conditional PDF of W given that $Z = z$.

Conditional PDF is Gaussian with mean $\mathbb{E}[W|Z=z] = \mu_W + \rho_{W,Z}\frac{\sigma_W}{\sigma_Z}(z - \mu_Z)$ and variance $\text{Var}[W|Z=z] = (1-\rho_{W,Z}^2)\,\sigma_W^2$.

$$\mathbb{E}[W|Z=z] = 8 + \left(-\frac{1}{4}\right)\cdot\frac{\sqrt{24}}{\sqrt{24}}(z-2) = -\frac{z}{4} + \frac{17}{2}$$

$$\text{Var}[W|Z=z] = \left(1 - \left(-\frac{1}{4}\right)^2\right)\cdot 24 = \frac{15}{16}\cdot 24 = \frac{45}{2}$$

W given $Z = z$ is Gaussian$\left(-\frac{z}{4} + \frac{17}{2}, \frac{45}{2}\right)$.

→ Given that $Z = 6$, what is the probability that $W < 7$?

$$\mathbb{P}[\{W<7\}|\{Z=6\}] = \Phi\left(\frac{7-\left(-\frac{6}{4}+\frac{17}{2}\right)}{\sqrt{\frac{45}{2}}}\right) = \Phi\left(\frac{7-7}{\sqrt{\frac{45}{2}}}\right) = \Phi(0) = \frac{1}{2}$$

W given $Z=6$ is Gaussian$\left(-\frac{6}{4}+\frac{17}{2}, \frac{45}{2}\right)$

symmetry of standard Gaussian