Gaussian Vectors

- A standard Gaussian random vector is a random vector whose entries are independent Gaussian random variables with mean 0 and variance 1.

$$\underline{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_m \end{bmatrix}$$ where $Z_1, \dots, Z_m$ are independent and Z_i is Gaussian$(0,1)$, $i = 1, \dots, m$

Shorthand notation

$$\underline{Z} \sim N(\underline{0}, \mathbf{I})$$

- A (jointly) Gaussian random vector is a random vector that can be expressed as a linear transformation of a standard Gaussian random vector.

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$ where $\underline{X} = \mathbf{A}\underline{Z} + \underline{b}$ for some standard Gaussian random vector $\underline{Z}$, matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$, and vector $\underline{b} \in \mathbb{R}^n$.

→ Fully specified by its mean vector $\underline{\mu}_{\underline{X}} = \mathbb{E}[\underline{X}]$ and covariance matrix $\Sigma_{\underline{X}} = \mathbb{E}[(\underline{X} - \underline{\mu}_{\underline{X}})(\underline{X} - \underline{\mu}_{\underline{X}})^T]$.

- Some equivalent definitions: $\underline{X}$ is a (jointly) Gaussian random vector if,
  - → for any choice of vector $\underline{a} \in \mathbb{R}^n$, $\underline{a}^T \underline{X}$ is a scalar Gaussian random variable.
  - → assuming that $\Sigma_{\underline{X}}$ is invertible, the joint PDF is

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma_{\underline{X}})}} \exp\left(-\frac{1}{2}(\underline{x} - \underline{\mu}_{\underline{X}})^T \Sigma_{\underline{X}}^{-1} (\underline{x} - \underline{\mu}_{\underline{X}})\right)$$

- Shorthand Notation: $\underline{X} \sim N(\underline{\mu}_{\underline{X}}, \Sigma_{\underline{X}})$

- Linear transformations of Gaussian random vectors are themselves Gaussian random vectors.
  - → If $\underline{X} \sim N(\underline{\mu}_{\underline{X}}, \Sigma_{\underline{X}})$ and $\underline{Y} = B\underline{X} + \underline{c}$,
    then $\underline{Y} \sim N(B\underline{\mu}_{\underline{X}} + \underline{c},\ B\Sigma_{\underline{X}}B^T)$.

- Recall that jointly Gaussian random variables X and Y are independent if and only if $\text{Cov}[X, Y] = 0$.
- Similarly, the entries $X_1, \ldots, X_n$ of a jointly Gaussian vector $\underline{X}$ are independent if and only if $\text{Cov}[X_i, X_j] = 0$ for all pairs $i \neq j$. This condition is equivalent to the requirement that the covariance matrix is diagonal

$$\Sigma_{\underline{X}} = \begin{bmatrix} \text{Var}[X_1] & 0 & \cdots & 0 \\ 0 & \text{Var}[X_2] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \text{Var}[X_n] \end{bmatrix}$$

- One can also call $X_1, \ldots, X_n$ jointly Gaussian random variables if they satisfy the definition of a jointly Gaussian random vector when grouped into a vector.

- Example: $\underline{X}$ is a Gaussian random vector with mean vector $\mu_{\underline{X}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and covariance matrix

$$\Sigma_{\underline{X}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

→ Let $\underline{Y} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \underline{X} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Note that $\underline{Y}$ is also a Gaussian random vector.

→ Determine the mean vector and covariance matrix of $\underline{Y}$.

Linearity of Expectation: $\mu_{\underline{Y}} = A \mu_{\underline{X}} + \underline{b}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

skipped calculation

Covariance Matrix of a Linear Transformation: $\Sigma_{\underline{Y}} = A \Sigma_{\underline{X}} A^T$

$$\Sigma_{\underline{Y}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

skipped calculation