- **Example:** Given H_0 occurs, Y is Binomial$(3, \frac{1}{2})$.
  Given H_1 occurs, Y is Binomial$(3, \frac{3}{4})$.

PMF for $X \sim$ Binomial(n,p): $P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$

$$P_{Y|H_0}(y) = \begin{cases} \binom{3}{y}\left(\frac{1}{2}\right)^y\left(\frac{1}{2}\right)^{3-y} & y = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{3}{0}\cdot\frac{1}{8} & y=0 \\ \binom{3}{1}\cdot\frac{1}{8} & y=1 \\ \binom{3}{2}\cdot\frac{1}{8} & y=2 \\ \binom{3}{3}\cdot\frac{1}{8} & y=3 \end{cases} = \begin{cases} \frac{1}{8} & y=0 \\ \frac{3}{8} & y=1 \\ \frac{3}{8} & y=2 \\ \frac{1}{8} & y=3 \end{cases}$$

$$P_{Y|H_1}(y) = \begin{cases} \binom{3}{y}\left(\frac{3}{4}\right)^y\left(\frac{1}{4}\right)^{3-y} & y = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{3}{0}\cdot\frac{1}{64} & y=0 \\ \binom{3}{1}\cdot\frac{3}{64} & y=1 \\ \binom{3}{2}\cdot\frac{9}{64} & y=2 \\ \binom{3}{3}\cdot\frac{27}{64} & y=3 \end{cases} = \begin{cases} \frac{1}{64} & y=0 \\ \frac{9}{64} & y=1 \\ \frac{27}{64} & y=2 \\ \frac{27}{64} & y=3 \end{cases}$$

ML Rule

If $P_{Y|H_1}(y) \geq P_{Y|H_0}(y)$, pick H_1. If not, pick H_0.

	y			
	0	1	2	3
$P_{Y\|H_0}(y)$	$\boxed{\frac{1}{8}}$	$\boxed{\frac{3}{8}}$	$\frac{3}{8}$	$\frac{1}{8}$
$P_{Y\|H_1}(y)$	$\frac{1}{64}$	$\frac{9}{64}$	$\boxed{\frac{27}{64}}$	$\boxed{\frac{27}{64}}$

$$D^{ML}(y) = \begin{cases} 1 & y = 2, 3 \\ 0 & y = 0, 1 \end{cases}$$

$A_0 = \{0, 1\}$ $\quad A_1 = \{2, 3\}$

- <u>Example</u>: Given H_0 occurs, Y is Binomial$(3, \frac{1}{2})$.
  Given H_1 occurs, Y is Binomial$(3, \frac{3}{4})$.

	y			
	0	1	2	3
$P_{Y\mid H_0}(y)$	$\boxed{\frac{1}{8}}$	$\boxed{\frac{3}{8}}$	$\boxed{\frac{3}{8}}$	$\boxed{\frac{1}{8}}$
$P_{Y\mid H_1}(y)$	$\boxed{\frac{1}{64}}$	$\boxed{\frac{9}{64}}$	$\boxed{\frac{27}{64}}$	$\boxed{\frac{27}{64}}$

$$\boxed{D^{ML}(y)} = \begin{cases} 1 & y = 2, 3 \\ 0 & y = 0, 1 \end{cases}$$

$A_0 = \{0, 1\}$ $\quad A_1 = \{2, 3\}$

→ What is the probability of error for the ML rule P_e^{ML}?

$$P_e^{ML} = P_{FA}\,\boxed{\mathbb{P}[H_0]} + P_{MD}\,\boxed{\mathbb{P}[H_1]}$$

Not specified in problem statement.

→ add to problem statement: $\mathbb{P}[H_0] = \frac{1}{5}$ $\quad \mathbb{P}[H_1] = \frac{4}{5}$

$$\boxed{P_{FA}} = \sum_{y \in A_1} P_{Y\mid H_0}(y) = P_{Y\mid H_0}(2) + P_{Y\mid H_0}(3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\boxed{P_{MD}} = \sum_{y \in A_0} P_{Y\mid H_1}(y) = \frac{1}{64} + \frac{9}{64} = \frac{5}{32}$$

$$P_e^{ML} = \frac{1}{2} \cdot \frac{1}{5} + \frac{5}{32} \cdot \frac{4}{5} = \frac{9}{40}$$

Can we do better?

- <u>Example</u>: Given H_0 occurs, Y is Binomial$(3, \frac{1}{2})$.
  Given H_1 occurs, Y is Binomial$(3, \frac{3}{4})$.

	y			
	0	1	2	3
$P_{Y\|H_0}(y)$	$\boxed{\frac{1}{8}}$	$\boxed{\frac{3}{8}}$	$\frac{3}{8}$	$\frac{1}{8}$
$P_{Y\|H_1}(y)$	$\frac{1}{64}$	$\frac{9}{64}$	$\boxed{\frac{27}{64}}$	$\boxed{\frac{27}{64}}$

$$\boxed{D^{ML}(y)} = \begin{cases} 1 & y = 2, 3 \\ 0 & y = 0, 1 \end{cases}$$

→ Assume $\mathbb{P}[H_0] = \frac{1}{5}$
$\mathbb{P}[H_1] = \frac{4}{5}$.

$P_e^{ML} = \frac{9}{40}$

→ Determine MAP rule and probability of error.

$$\boxed{D^{MAP}(y)} = \begin{cases} 1 & y = 1, 2, 3 \\ 0 & y = 0 \end{cases}$$

<u>MAP Rule</u>

If $P_{Y|H_1}(y)\,\mathbb{P}[H_1] \geq P_{Y|H_0}(y)\,\mathbb{P}[H_0]$, pick H_1. If not, pick H_0.

	y			
	0	1	2	3
$P_{Y\|H_0}(y)\,\mathbb{P}[H_0]$	$\frac{1}{8} \cdot \frac{1}{5} = \boxed{\frac{1}{40}}$	$\boxed{\frac{3}{8}} \cdot \frac{1}{5} = \frac{3}{40}$	$\boxed{\frac{3}{8}} \cdot \frac{1}{5} = \frac{3}{40}$	$\boxed{\frac{1}{8}} \cdot \frac{1}{5} = \frac{1}{40}$
$P_{Y\|H_1}(y)\,\mathbb{P}[H_1]$	$\boxed{\frac{1}{64}} \cdot \frac{4}{5} = \frac{1}{80}$	$\frac{9}{64} \cdot \frac{4}{5} = \boxed{\frac{9}{80}}$	$\frac{27}{64} \cdot \frac{4}{5} = \boxed{\frac{27}{80}}$	$\frac{27}{64} \cdot \frac{4}{5} = \boxed{\frac{27}{80}}$

$$P_e^{MAP} = P_{FA}\,\mathbb{P}[H_0] + P_{MD}\,\mathbb{P}[H_1]$$
$$= \frac{7}{8} \cdot \frac{1}{5} + \frac{1}{64} \cdot \frac{4}{5} = \frac{3}{16}$$

MAP outperforms ML.

$$\boxed{P_{MD}} = \sum_{y \in A_0} P_{Y|H_1}(y)$$
$$= P_{Y|H_1}(0) = \frac{1}{64}$$

$$\boxed{P_{FA}} = \sum_{y \in A_1} P_{Y|H_0}(y) = P_{Y|H_0}(1) + P_{Y|H_0}(2) + P_{Y|H_0}(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$