Likelihood Ratio

- Recall the framework of binary hypothesis testing:
  - → Two hypotheses H_0 and H_1 that partition Ω.
  - → An observation Y whose values are drawn according to

Discrete Case

$P_{Y|H_0}(y)$ if H_0 occurs

$P_{Y|H_1}(y)$ if H_1 occurs

Continuous Case

$f_{Y|H_0}(y)$ if H_0 occurs

$f_{Y|H_1}(y)$ if H_1 occurs

  - → A decision rule $D(Y)$ that outputs 0 or 1 as its guess for the hypothesis based only on Y.

- We will revisit the ML and MAP rules from the perspective of the likelihood ratio $L(y)$

$$\boxed{L(y) = \begin{cases} \dfrac{P_{Y|H_1}(y)}{P_{Y|H_0}(y)} & Y \text{ is discrete} \\[2ex] \dfrac{f_{Y|H_1}(y)}{f_{Y|H_0}(y)} & Y \text{ is continuous} \end{cases}}$$

- **Maximum Likelihood (ML) Rule:**

$$\boxed{D^{ML}(y) = \begin{cases} 1, & L(y) \geq 1 \\ 0, & L(y) < 1 \end{cases} = \begin{cases} 1, & \ln(L(y)) \geq 0 \\ 0, & \ln(L(y)) < 0 \end{cases}}$$

Discrete Case

$$L(y) = \frac{P_{Y|H_1}(y)}{P_{Y|H_0}(y)}$$

Continuous Case

$$L(y) = \frac{f_{Y|H_1}(y)}{f_{Y|H_0}(y)}$$

→ $\ln(L(y))$ is called the log-likelihood ratio and can sometimes help simplify the form of the decision rule.

- **Maximum a Posteriori (MAP) Rule:**

$$\boxed{D^{MAP}(y) = \begin{cases} 1, & L(y) \geq \frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]} \\ 0, & L(y) < \frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]} \end{cases} = \begin{cases} 1, & \ln(L(y)) \geq \ln\left(\frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \\ 0, & \ln(L(y)) < \ln\left(\frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \end{cases}}$$

→ Why? $P_{Y|H_1}(y)\,\mathbb{P}[H_1] \geq P_{Y|H_0}(y)\,\mathbb{P}[H_0] \Leftrightarrow \underbrace{\boxed{\frac{P_{Y|H_1}(y)}{P_{Y|H_0}(y)}}}_{L(y)} \geq \frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}$

- <u>Example</u>: Given H_0 occurs, Y is Binomial(n, p_0).
  Given H_1 occurs, Y is Binomial(n, p_1).

$$L(y) = \frac{P_{Y|H_1}(y)}{P_{Y|H_0}(y)} = \frac{\binom{n}{y} p_1^y (1-p_1)^{n-y}}{\binom{n}{y} p_0^y (1-p_0)^{n-y}} = \left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)^y \frac{(1-p_1)^n}{(1-p_0)^n}$$

$$D^{ML}(y) = \begin{cases} 1, & L(y) \geq 1 \\ 0, & L(y) < 1 \end{cases} = \begin{cases} 1, & \left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)^y \frac{(1-p_1)^n}{(1-p_0)^n} \geq 1 \\ 0, & \left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)^y \frac{(1-p_1)^n}{(1-p_0)^n} < 1 \end{cases}$$

Can simplify using the log-likelihood ratio.

$$\ln(L(y)) = \ln\left(\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)^y \frac{(1-p_1)^n}{(1-p_0)^n}\right) = y \ln\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right) + n \ln\left(\frac{1-p_1}{1-p_0}\right)$$

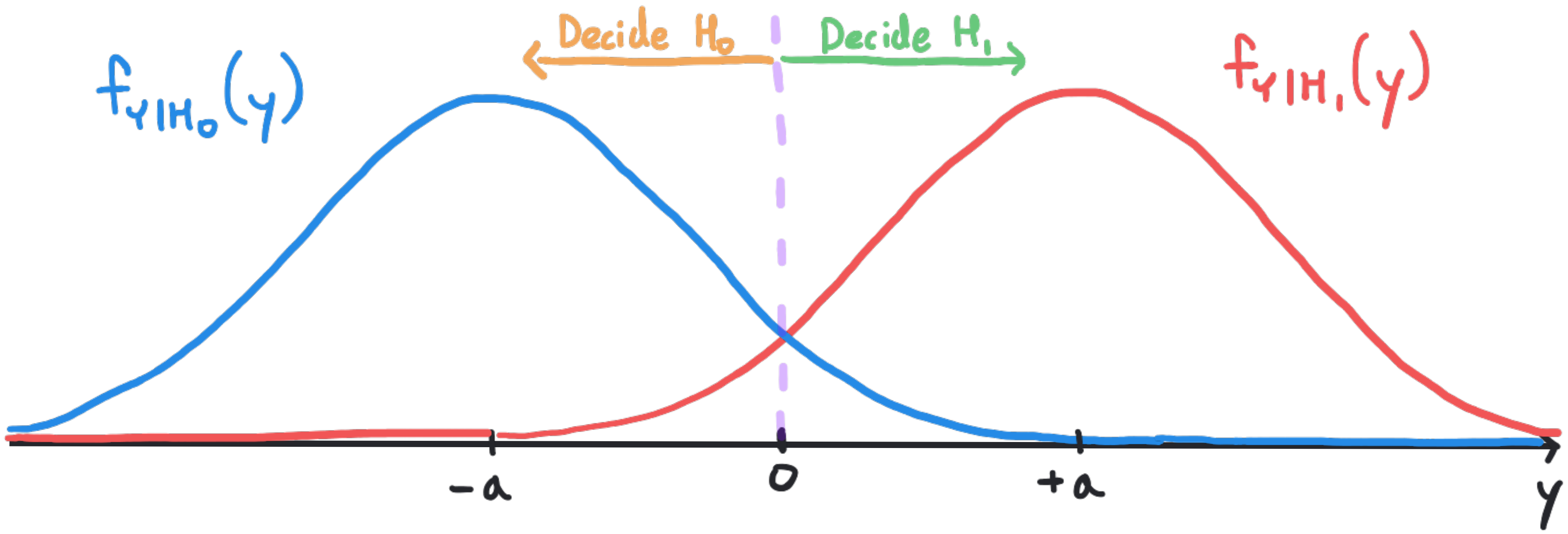
$$D^{ML}(y) = \begin{cases} 1, & \ln(L(y)) \geq 0 \\ 0, & \ln(L(y)) < 0 \end{cases} = \begin{cases} 1, & y \ln\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right) \geq -n \ln\left(\frac{1-p_1}{1-p_0}\right) \\ 0, & y \ln\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right) < -n \ln\left(\frac{1-p_1}{1-p_0}\right) \end{cases}$$

In the previous video, $n = 3$, $p_0 = \frac{1}{2}$, $p_1 = \frac{3}{4}$.

$$y \ln\left(\frac{\frac{3}{4}(1-\frac{1}{2})}{\frac{1}{2}(1-\frac{3}{4})}\right) \geq -3 \ln\left(\frac{1-\frac{3}{4}}{1-\frac{1}{2}}\right) \Rightarrow y \ln(3) \geq -3 \ln\left(\frac{1}{2}\right) \Rightarrow y \geq 1.89 \Rightarrow y \in \{2, 3\}$$

Same decision rule!

- **Example:** Given H_0 occurs, Y is Gaussian$(-a, \sigma^2)$. Assume $a > 0$.
  Given H_1 occurs, Y is Gaussian$(+a, \sigma^2)$.



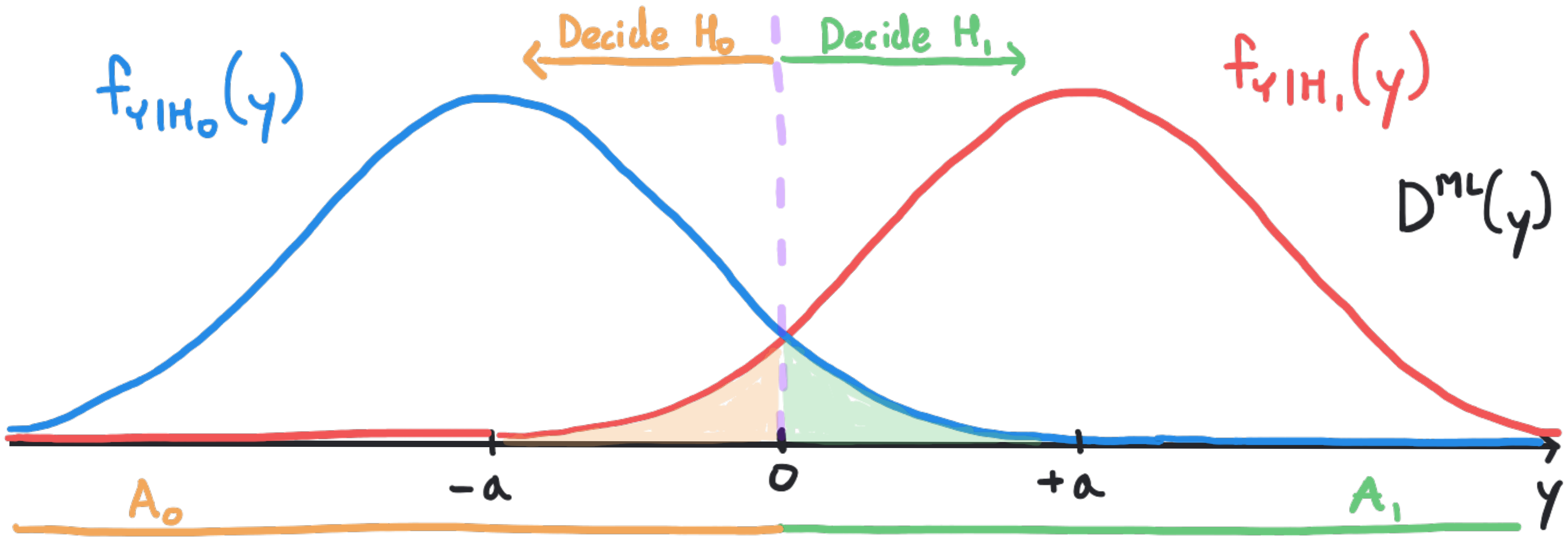
→ From the plot, we can guess that the ML rule decides H_1 when $y \geq 0$ and H_0 when $y < 0$. Check using log-likelihood ratio.

$$\ln(L(y)) = \ln\left(\frac{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left(-\frac{(y-a)^2}{2\sigma^2}\right)}{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left(-\frac{(y+a)^2}{2\sigma^2}\right)} \right)$$

$$= \ln\left(\exp\left(-\frac{1}{2\sigma^2}\left(\cancel{y^2} - 2ay + \cancel{a^2}\right) + \frac{1}{2\sigma^2}\left(\cancel{y^2} + 2ay + \cancel{a^2}\right)\right)\right) = \frac{2ay}{\sigma^2}$$

$$D^{ML}(y) = \begin{cases} 1, & \ln(L(y)) \geq 0 \\ 0, & \ln(L(y)) < 0 \end{cases} = \begin{cases} 1, & \frac{2ay}{\sigma^2} \geq 0 \\ 0, & \frac{2ay}{\sigma^2} < 0 \end{cases} = \begin{cases} 1, & y \geq 0 \\ 0, & y < 0 \end{cases} \checkmark$$

- **Example:** Given H_0 occurs, Y is Gaussian$(-a, \sigma^2)$. Assume $a > 0$.
  Given H_1 occurs, Y is Gaussian$(+a, \sigma^2)$.



$$D^{ML}(y) = \begin{cases} 1, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

→ What is the probability of error for the ML rule?

$$P_e = P_{FA}\,\mathbb{P}[H_0] + P_{MD}\,\mathbb{P}[H_1]$$

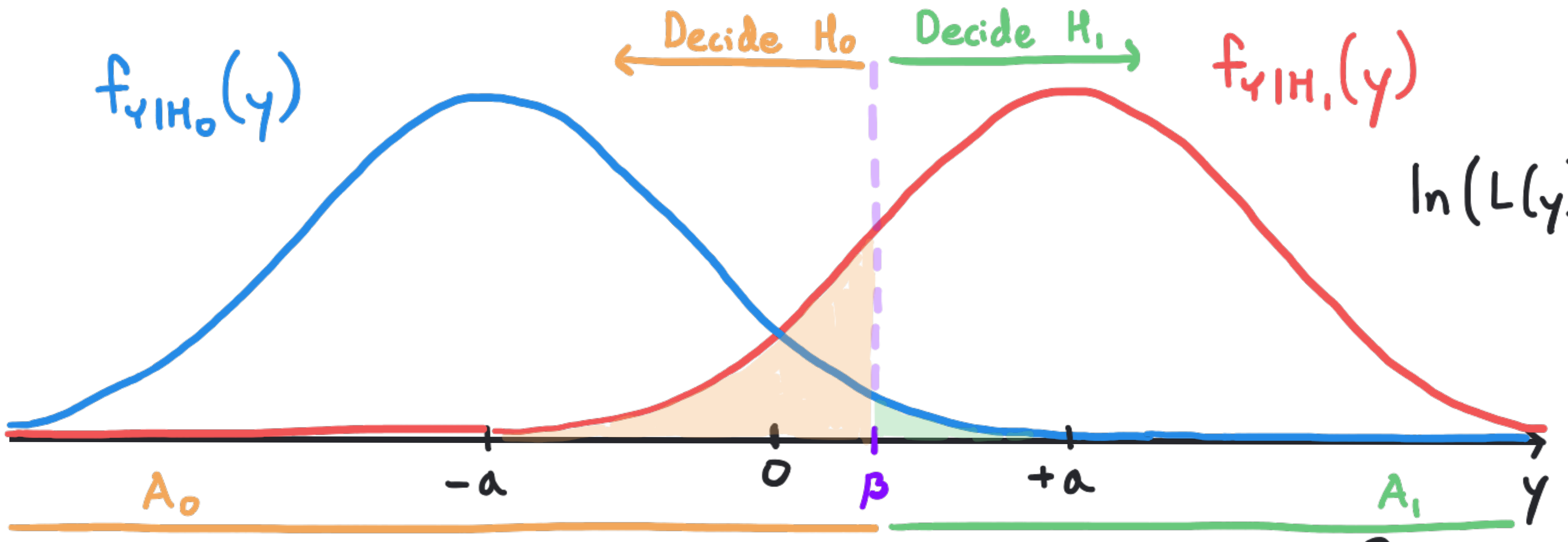
$$P_{FA} = \mathbb{P}[\{Y \in A_1\} \mid H_0] = \mathbb{P}[Y \geq 0 \mid H_0] = \int_0^{\infty} f_{Y|H_0}(y)\,dy = Q\left(\frac{0-(-a)}{\sigma}\right) = Q\left(\frac{a}{\sigma}\right)$$

$$P_{MD} = \mathbb{P}[\{Y \in A_0\} \mid H_1] = \mathbb{P}[Y < 0 \mid H_1] = \int_{-\infty}^{0} f_{Y|H_1}(y)\,dy = \Phi\left(\frac{0-(a)}{\sigma}\right) = Q\left(\frac{a}{\sigma}\right)$$

$$P_e = Q\left(\frac{a}{\sigma}\right)\mathbb{P}[H_0] + Q\left(\frac{a}{\sigma}\right)\mathbb{P}[H_1] = Q\left(\frac{a}{\sigma}\right)$$

In this special case, we do not need $\mathbb{P}[H_0]$, $\mathbb{P}[H_1]$ explicitly.

- **Example**: Given H_0 occurs, Y is Gaussian$(-a, \sigma^2)$. Assume $a > 0$.
  Given H_1 occurs, Y is Gaussian$(+a, \sigma^2)$.



$\ln(L(y)) = \frac{2ay}{\sigma^2}$

→ What is the MAP rule and its probability of error?

$$D^{MAP}(y) = \begin{cases} 1, & \ln(L(y)) \geq \ln\left(\frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \\ 0, & \ln(L(y)) < \ln\left(\frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \end{cases} = \begin{cases} 1, & y \geq \frac{\sigma^2}{2a}\ln\left(\frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \\ 0, & y < \frac{\sigma^2}{2a}\ln\left(\frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \end{cases}$$

$$\beta = \frac{\sigma^2}{2a}\ln\left(\frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right)$$

$$P_e = P_{FA}\,\mathbb{P}[H_0] + P_{MD}\,\mathbb{P}[H_1]$$

$$P_{FA} = \mathbb{P}[\{Y \in A_1\} \mid H_0] = \mathbb{P}[Y \geq \beta \mid H_0] = \int_{\beta}^{\infty} f_{Y|H_0}(y)\,dy = Q\left(\frac{\beta - (-a)}{\sigma}\right) = Q\left(\frac{a + \beta}{\sigma}\right)$$

$$P_{MD} = \mathbb{P}[\{Y \in A_0\} \mid H_1] = \mathbb{P}[Y < \beta \mid H_1] = \int_{-\infty}^{\beta} f_{Y|H_1}(y)\,dy = \Phi\left(\frac{\beta - a}{\sigma}\right) = Q\left(\frac{a - \beta}{\sigma}\right)$$